Department of Statistics
University of Wisconsin, Madison
PhD Qualifying Exam Part I
Tuesday, January 22, 2013
12:30-4:30pm, Room 133 SMI

- There are a total of FOUR (4) problems in this exam. Please do a total of THREE (3) problems.
- Each problem must be done in a separate exam book.
- Please turn in THREE (3) exam books.
- Please write your code name and **NOT** your real name on each exam book.

1. Let $U_1, \ldots, U_n$ be independent and have a uniform distribution on $(0, 1)$, and $U_{(1)} \leq \cdots \leq U_{(n)}$ be their order statistics.

   (a) Show that $\left(\frac{U_{(1)}}{U_{(n)}}, \cdots, \frac{U_{(n-1)}}{U_{(n)}}\right)$ is independent of $U_{(n)}$.

   (b) Find the joint distribution of $\left(\frac{U_{(1)}}{U_{(n)}}, \cdots, \frac{U_{(n-1)}}{U_{(n)}}\right)$.

   (c) For $n = 1, 2, \ldots$, derive

$$P\left(\bigcap_{j=1}^{n} \left\{U_{(j)} > \frac{\alpha j}{n}\right\}\right),$$

   where $0 \leq \alpha \leq 1$.

2. A system involves mutually independent and Poisson-distributed random variables X_i, Z_i, for $i = 1, 2, \ldots, n$. For a parameter of interest $\theta > 0$ and positive nuisance parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, we know that $\mathrm{E}(X_i) = \lambda_i$ and $\mathrm{E}(Z_i) = \theta\lambda_i$. The measurements available for inference on θ are incomplete; for each $i = 1, 2, \ldots, n$, we observe the pair (X_i, Y_i), where

$$Y_i = I(X_i > 0)\, I(Z_i > 0),$$

and where $I(\cdot)$ is the indicator function. Define $R_i = 2Y_i - 1$, and $T_n = \sum_{i=1}^n R_i X_i$.

(a) Compute the mean $\mathrm{E}(T_n)$ and show that it is increasing in θ.

(b) Compute the variance $\sigma_n^2 = \mathrm{var}(T_n)$ and show that on the null hypothesis $H_0 : \theta = 1$,

$$\sum_{i=1}^n \lambda_i \le \sigma_n^2 \le \sum_{i=1}^n \left(\lambda_i + \lambda_i^2\right).$$

(c) In the special case that all λ_i equal a common value λ, derive the maximum likelihood estimator of (θ, λ).

(d) In another special case, the λ_i's are not arbitrary but are viewed as independent and identically distributed realizations of a standard exponential distribution. Find a consistent estimator for θ in this case.

3. Suppose there are two independent Bernoulli random variables X and Y. Assume $P(X = 1) = p_1$ and $P(Y = 1) = p_2$. For each random variable, we have n independent observations: $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$, where X_i are identically distributed as X, Y_j are identically distributed as Y, and $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ are independent.

We consider the following hypothesis testing problem:

$$H_0 : p_1 = p_2 \ vs. \ H_1 : p_1 \neq p_2.$$

Let $\hat{p}_1 = \sum_i X_i/n$ and $\hat{p}_2 = \sum_i Y_i/n$. Define

$$T = \frac{\hat{p}_2 - \hat{p}_1}{\sqrt{2\hat{p}\hat{q}/n}},$$

where $\hat{p} = (\hat{p}_1 + \hat{p}_2)/2$ and $\hat{q} = 1 - \hat{p}$.

We will reject H_0 if $|T| > z_{\alpha/2}$, where α is the type-I error, $z_{\alpha/2}$ satisfies $P(Z > z_{\alpha/2}) = \alpha/2$, and Z is a standard normal random variable.

(a) Define

$$\Pi_n = P\left(\frac{\hat{p}_2 - \hat{p}_1}{\sqrt{2\hat{p}\hat{q}/n}} < -z_{\alpha/2}\right).$$

When $p_1 = p_2$, show that Π_n converges to $\alpha/2$ as $n \to \infty$. When $p_2 > p_1$, show that $\lim_{n\to\infty} \Pi_n < \alpha/2$.

(b) Show that the power of the test tends to 1 as $n \to \infty$.

4. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density

$$f_{a,b}(x) = \frac{1}{b} \exp^{-\frac{1}{b}(x-a)} I(x \geq a), \tag{1}$$

where $-\infty < a < \infty$ and $b > 0$. Define $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ to be the order statistics of the $X_1, \cdots, X_n$. It can be shown that the joint density of $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ for $r \leq n$ is

$$f(x_{(1)}, \ldots, x_{(r)}) = \frac{1}{b^r} \frac{n!}{(n-r)!} \exp\left[-\frac{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)} - na}{b} \right] I(x_{(1)} \geq a). \tag{2}$$

(a) Let $Z_1 = 2n[X_{(1)} - a]/b$. For $i = 2, \ldots, n$, define

$$Z_i = \frac{(n-i+1)(X_{(i)} - X_{(i-1)})}{b/2}.$$

Give an explicit form of the joint density of $(Z_1, \ldots, Z_n)$. Provide details.

(b) For $r = 1, \ldots, n$, show that

$$\frac{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)} - na}{b/2}$$

follows the χ^2 distribution with $2r$ degrees of freedom. Give details.

(c) Assuming b is known, determine an explicit form of the UMP test at level α of $H_0 : a = a_0$ versus $H_a : a = a_1 < a_0$. Give details.

(d) When b is unknown, for testing $H_0 : a = a_0$ versus $H_a : a \neq a_0$, consider the following test statistic:

$$T = \frac{X_{(1)} - a_0}{\sum_{i=1}^{n} [X_i - X_{(1)}]}. \tag{3}$$

Derive the distribution of T under H_0. Provide details.