Dept Copy

Department of Statistics

University of Wisconsin-Madison

PhD Qualifying Exam Part I

January 13, 2009

12:30–4:30pm, Room 133 SMI

- There are a total of FOUR (4) problems in this exam. Please do a total of THREE (3) problems.
- Each problem must be done in a separate exam book.
- Please turn in THREE (3) exam books.
- Please write your code name and **NOT** your real name on each exam book.

2. Let $\boldsymbol{Y} = (\boldsymbol{Y}_1', \ldots, \boldsymbol{Y}_n')'$ denote an $nk \times 1$ random vector, where $\boldsymbol{Y}_i$ is a $k \times 1$ random vector, for $i = 1, \ldots, n$. Consider the model,

$$\boldsymbol{Y}_i = \boldsymbol{X}\boldsymbol{\beta}_i + \boldsymbol{\epsilon}_i, \qquad i = 1, \ldots, n,$$

where $\boldsymbol{X}$ is a known $k \times p$ matrix of rank p. Further, $\boldsymbol{\beta}_i$ is a $p \times 1$ random vector with,

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \Sigma),$$

where $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters and Σ is an unknown $p \times p$ positive definite matrix. Finally, $\boldsymbol{\epsilon}_i$ is a $k \times 1$ random vector with,

$$\boldsymbol{\epsilon}_i \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}),$$

where $\boldsymbol{0}$ is a $k \times 1$ vector of zeros, $\sigma^2 > 0$ is an unknown parameter, and $\boldsymbol{I}$ is a $k \times k$ identity matrix. Assume that $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n, \boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n$ are independently distributed.

(a) Provide matrices $\boldsymbol{Z}$ and $\boldsymbol{V}$ such that the marginal distribution of the random vector $\boldsymbol{Y}$ follows $N(\boldsymbol{Z}\boldsymbol{\beta}, \boldsymbol{V})$.

(b) Let $\bar{\boldsymbol{Y}} = n^{-1}\sum_{i=1}^{n} \boldsymbol{Y}_i$. Provide a matrix $\boldsymbol{A}$ such that $\widehat{\boldsymbol{\beta}} = \boldsymbol{A}\bar{\boldsymbol{Y}}$ is the ordinary least squares (OLS) estimator of $\boldsymbol{\beta}$.

(c) Determine whether,

$$\frac{1}{n(n-1)}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\sum_{i=1}^{n}(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})(\boldsymbol{Y}_i - \bar{\boldsymbol{Y}})'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1},$$

is an unbiased estimator of the variance-covariance matrix of $\widehat{\boldsymbol{\beta}}$, the OLS estimator of $\boldsymbol{\beta}$.

(d) For a known $p \times 1$ vector $\boldsymbol{\lambda}$, show that a $100(1-\alpha)\%$ confidence interval for $\boldsymbol{\lambda}'\boldsymbol{\beta}$ is of the form,

$$\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}} \pm t_{\alpha/2,d}\sqrt{\boldsymbol{\lambda}'\boldsymbol{B}\boldsymbol{\lambda}},$$

where $\boldsymbol{\lambda}'\widehat{\boldsymbol{\beta}}$ is the OLS estimator of $\boldsymbol{\lambda}'\boldsymbol{\beta}$ and $t_{\alpha/2,d}$ is the $100(1-\alpha/2)^{th}$ percentile of the Student t-distribution with d degrees of freedom. Find the constant d and the matrix $\boldsymbol{B}$.

(e) Construct a $100(1-\alpha)\%$ confidence interval for σ^2.

4. Let $X_1, ..., X_n$ be i.i.d. observations with finite mean 0 and variance 1. Let $\overline{X}$ be the sample mean, S^2 be the sample variance, and Φ be the standard normal distribution function.

(a) Define

$$T_n = \begin{cases} \overline{X}, & \text{if } \sqrt{n}|\overline{X}| \geq z_\alpha S, \\ 0, & \text{if } \sqrt{n}|\overline{X}| < z_\alpha S, \end{cases}$$

where $z_\alpha = \Phi^{-1}(1 - \alpha)$, $0 < \alpha < 0.5$. Let

$$H_n(t) = P(\sqrt{n}T_n \leq t), \quad -\infty < t < \infty.$$

Find $\lim_{n \to \infty} H_n(t)$.

(b) Assume that $g(x)$ is a continuous function such that the derivative $g'(x)$ exists and is continuous for all $x \neq 0$ and

$$\lim_{x>0, x \to 0} \frac{g(x) - g(0)}{x} = g'_+ > 0 \quad \text{and} \quad \lim_{x<0, x \to 0} \frac{g(x) - g(0)}{x} = g'_- < 0.$$

(An example of such a function is $g(x) = |x|$.) Let

$$G_n(t) = P(\sqrt{n}[g(\overline{X}) - g(0)] \leq t), \quad 0 < t < \infty.$$

Find $\lim_{n \to \infty} G_n(t)$.