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A PIECEWISE EXPONENTIAL MODEL FOR RELIABILITY GROWTH AND ASSOCIATED INFERENCES

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Abstract

Nonhomogeneous Poisson process (NHPP) with Weibull intensity has been widely used in modeling reliability growth (RG), and some elegant results are available for statistical inferences. However, a key feature of this model, namely, the intensity changing continuously over time regardless of the failure history, is not physically meaningful if fixes or design changes for a system improvement take place only after the observation of failures. For a test-fix-retest setting, we propose a simple stochastic model of RG, called a piecewise exponential (PEXP) model, which assumes that, after the $(i-1)$st failure, a homogeneous Poisson process with intensity $\lambda_i = (\mu/\delta)i^{1-\delta}$, $1 < \delta$, $0 < \mu$, governs the event of the next failure. The step intensity with this particular parameterization provides an alternative to the NHPP as a stochastic formulation of the Duane plot. By an analogy with the NHPP, closed-form estimators of the model parameters are constructed and are compared to the maximum likelihood estimators (MLE) in terms of asymptotic efficiency as well as finite sample simulation. The development of asymptotic properties of the MLE's involves some modifications of the standard arguments due to the singularity of the covariance matrix. The proposed model is applied to two data sets which were previously analyzed using the NHPP model.

1 Introduction

Modeling reliability growth has received considerable attention in the statistical and engineering literature over the past three decades. At the initial stage of any production involving complex systems, prototypes are put into life test under a development testing program, corrective actions or design changes are made when failures occur, and the modified system is tested again. As this test-redesign-retest sequence contributes to an improvement in the system performance, failure data become increasingly sparse at the later stages of testing making it more difficult to assess the current reliability. A reliability growth (RG) model provides a structure through which the failure data from the current as well as previous stages of testing could be analyzed in an integrated way in order to make inferences on the current system reliability.

A major thrust to RG modeling rose from certain empirical findings of Duane(1964) from examination of the failure data of a variety of systems such as complex hydromechanical devices, aircraft generators and jet engines in the course of their development. When plotted on a log-log scale, the cumulative number of failures was typically found to produce a linear pattern of relationship with the cumulative operating time. This phenomenon, later came to be known as the "Duane postulate", was given a concrete stochastic basis by Crow(1974) who assumed that the failures during the development stage of a new system follow a nonhomogeneous Poisson process (NHPP) with an intensity function $\lambda(t)$ of the form $\mu\delta t^{\delta-1}$. The corresponding cumulative failure rate $\Lambda(t) = \mu t^{\delta}$ is linear on a log-log scale. Incidentally, an NHPP formulation was proposed by Ascher(1968) in modeling the reliability change of a *bad-as-old system*, and was later used by Bassin(1969, 1973) with the Weibull intensity to obtain optimal overhaul intervals for various machines. A large body of literature has evolved thereafter in the areas of statistical inferences as well as applications of the NHPP model in RG analysis (see, for instance, Crow(1974), Bell and Midouski(1976), Finkelstein(1976), Lee and Lee(1978), Bain and Engelhardt(1980, 1982), Durr(1980), Lee(1980), Higgins and Tsokos(1981), Crow(1982), Rigdon and Basu(1988), Guida, Calabria and Pulcini(1989)).

For single-mission systems such as missiles or torpedos, the test results are binary in nature as opposed to time-to-failure in a continuous-time framework. The NHPP model in the continuous-time case has yielded a natural counterpart for the discrete case, which is called a Logarithmic Growth model or a nonhomogeneous binomial (NHB) model. Some estimation procedures were suggested by Finkelstein(1983), and asymptotic properties were studied by Bhattacharyya, Fries and Johnson(1987) and Bhattacharyya and Ghosh(1988).

In the continuous-time case, the NHPP has gained vast popularity due to its empirical fit to a variety of data vis-a-vis its conformity to the Duane learning curve, and availability of elegant distributional results concerning statistical inferences. While empirical fit, nice mathematical properties and tractability of statistical inferences are very important aspects of a stochastic model, a clear physical interpretation is also of paramount importance. Duane(1964) indicates that a learning curve is used to monitor developmental progress and plan for reliability improvement. One point of concern with the NHPP model is its continually changing intensity function irrespective of the failure history, which is in direct conflict with the concept of the Duane learning curve as well as the conceptual framework of a test-redesign-retest course of development testing. If system improvement is assumed to be effected only after a failure is observed, a realistic model should be flexible enough to incorporate a change in the failure rate at the occurrences of the failures. Thompson(1988) expresses the same concern by saying that "··· some provision needs to be present for altering the process of failures when modifications or corrective actions are applied to the system".

This is however, not to imply that the NHPP is inappropriate in all cases. Much depends on whether failure occurrences and fixes or design changes are synchronized in the real operational setting. Even if they are, an NHPP can be thought of as an approximation or "idealization" of a step-intensity model, as mentioned in Benton and Crow(1989). Our goal here is to formulate a model that avoids the approximation and also to make a comparative study of the two approaches of modeling. The discrete model which assumes that the probability of a failure decreases from stage to stage irrespective of the outcome of a trial, also needs to be modified for an operational setting where no design changes are made until a failure is observed. The continuous-time model described in this paper yields a discrete analog which handles this concern in a physically meaningful manner.

A continuous-time RG model, called Piecewise Exponential (PEXP), is formulated in the next section as a stochastic version of the Duane learning curve. In Section 3 we describe estimation procedures for our model and derive large sample inference results. In Section 4 some simulation results are reported along with a comparative study of the different estimation procedures. Finally, in the same section, we fit the PEXP model to two data sets which were previously analyzed under the assumption of NHPP, and compare the results.

2 The PEXP and related models

We incorporate the philosophy of learning curve into building a simple model for reliability growth. At the initial stage of testing, we consider a unit to have a constant failure intensity λ_1. At the first failure, fixes or design changes are made, thereby decreasing the rate of failure to a constant λ_2, and the modified unit is tested again. In this process we consider observations until n failures (failure truncated scheme). Then the data would consist of n ordered failure times $0 < T_1 < T_2 < \ldots < T_n$. A constant rate of failure prevailing at each stage of testing and the rate decreasing at each failure (after the corrective actions are taken) would amount to the assumption that the inter-failure times $T_i - T_{i-1}$ are independent exponential random variables with means $1/\lambda_i$, $i = 1, \ldots, n$. As for the pattern of change of failure rate in the successive stages, we assume the parameterization

$$\frac{1}{\lambda_i} = (\delta/\mu) i^{\delta-1}, \quad \mu > 0, \ \delta > 1 \tag{2.1}$$

and call the resulting model *Piecewise Exponential*, abbreviated as PEXP. Here λ_i is parameterized in a way that makes the model a stochastic version of the *Duane postulate*. To this end, we recall that the NHPP intensity $\lambda(t) = \mu\delta t^{\delta-1}$ was already formulated as a reflection of the Duane curve. Therefore, it would be appropriate to have a structure for λ_i that would bring it in line with the NHPP intensity.

To pursue this idea of "parameter matching" we observe that the cumulative failure rate at the nth failure time, $\int_0^{T_n} \lambda(t)\, dt$, equals μT_n^δ and $\sum_{i=1}^n \lambda_i (T_i - T_{i-1})$ for NHPP and PEXP, respectively. Since both quantities equal n in expectation, they form a common basis for matching the two models. Equating μT_n^δ to its expectation n amounts to setting the correspondence of T_n with $(n/\mu)^{1/\delta} = n^{\delta'}/\mu'$, where $\mu' = \mu^{1/\delta}$and $\delta' = 1/\delta$. Consequently, $T_i - T_{i-1}$ would correspond to $[i^{\delta'} - (i-1)^{\delta'}]/\mu'$. Finally, we replace the random variable $T_i - T_{i-1}$ by its expectation under PEXP model, and arrive at the relation

$$\frac{1}{\lambda_i} = \frac{i^{\delta'} - (i-1)^{\delta'}}{\mu'}$$

For large i, the right hand side can be approximated by $(\delta'/\mu') i^{\delta'-1}$ which yields (2.1). Alternatively, we can directly motivate (2.1) as a model of "logarithmic growth" of the failure rate. Henceforth, for clarity and notational convenience, we shall drop the primes attached to the parameters.

A continuous-time Markov chain with a step-intensity is termed as a "Pure Birth Process" in the stochastic processes literature. Two special "Pure Birth" models, namely, the Yule Process for which $\lambda_i = i\lambda$, and an epidemic model for which $\lambda_i = (m-i)i\lambda$, $i \leq m-1$ have received considerable attention. Jelinsky and Moranda(1972) formulated a step-intensity model which assumes that the inter-failure times $T_i - T_{i-1}$ of a software are independent exponential with parameters $\lambda_i = (N-i+1)\phi$, where N denotes the unknown number of faults in the system and ϕ, the rate of occurrence of the faults.

Discrete Analog of PEXP

Concerning the test-redesign-retest cycle of development program for single-mission systems, a discrete analog of the PEXP is readily apparent. At each stage (configuration), independent trials are repeated until a failure is observed. Fixes or design changes are then made, and the modified system is tested again according to the same inverse sampling scheme. Then N_i, the *number of trials to the first failure under the ith configuration*, can be modeled as a geometric (q_i) random variable where q_i denotes the system failure probability. As for a reasonable structure for q_i that mimics the Duane postulate, we consider each trial to take a unit amount of time. Then the number of trials N_i between consecutive failures in the discrete case would correspond to the *inter-failure times* in the continuous case for which the PEXP model is appropriate. Thus taking q_i to be of the same form as the λ_i for PEXP, we arrive at a *Piecewise Geometric* model with logarithmic growth. Specifically, the failure probability at the *ith* configuration is given by

$$q_i = (\mu/\delta)i^{1-\delta}, \quad 0 < \mu < 1, \ \delta > 1 \quad i = 1, \ldots, n.$$

An alternative parameterization, namely, $q_i = \mu\delta^i$, yields the discrete RG model due to Dubman and Sherman(1969).

A Generalization of the PEXP

The PEXP model assumes that the failure rate remains constant between failures. A natural generalization of the model would be to incorporate into the intensity function another component that changes continually with time. The role of the second component would be to account for such factors as wear out or other contributors to failure that are not affected by design changes. Essentially, we assume that the failure process has two components : one relates to the step intensity indicating reliability growth following an intervention, while the other pertains to reliability changes (i.e. degradation) not associated with the intervention.

To formalize this idea in a concrete physical setting, we consider observing the failure time of a two-component series system where the components are subject to two kinds of failures. As for the mechanism of the test-analyze-fix program, we assume that every time the system fails, fixes or design changes are made to component A while component B is replaced by a good-as-new unit (see Ascher(1968)). Consequently, component A undergoes reliability growth, while the clock for the failure process of B is reset to zero at each system failure. If T_{1i} and T_{2i} denote the lifetimes of A and B, respectively, at the i-th stage of the development program, then the system inter-failure time $T_i - T_{i-1}$ equals $min(T_{1i}, T_{2i})$, $i = 1, \ldots, n$. We further assume the following :

1. For all i, T_{1i} and T_{2i} are independent (independence of the components).

2. T_{1i}'s are independent exponential random variables with parameters λ_i (step-changing pattern) while T_{2i}'s are i.i.d. copies from a Weibull distribution W(λ, β).

These amount to the assumption that the successive failures arise from a composite intensity function

$$\lambda(t) = \lambda_{N(t)} + \lambda\beta t^{\beta-1},$$

where N(t) stands for the number of system failures in the time interval [0,t). Bain(1978, pp 421-426) discusses a model where the failure rate is a polynomial in t. Note that our formulation is a generalization of the polynomial hazard function model in that it allows the constant term to be a function of the cumulative number of failures, and the power of t to be a positive real number. This model yields the PEXP as a limiting case when $\lambda \longrightarrow 0$ (or when only component A is present). In the rest of this paper we confine our investigation to the PEXP which itself is physically meaningful and serves as a simple alternative to NHPP for reliability growth modeling.

3 Parameter Estimation for PEXP

In this section we develop estimation procedures for the parameters of PEXP. We first study maximum likelihood estimation (MLE) which requires iterative solutions and also gives rise to an interesting non-standard situation of asymptotic theory. This will be followed by the development of an alternative simple estimation procedure which is motivated from the connection between the PEXP and the NHPP models.

3.1 The MLE and its Asymptotics

The likelihood of the failure times $T_1 < T_2 < \ldots < T_n$ under a failure-truncated sampling scheme can be written by using the fact that $Y_i \equiv T_i - T_{i-1}$ are independent exponential random variables with means $1/\lambda_i = (\delta/\mu)i^{\delta-1}$, $i = 1, \ldots n$. For simplicity of exposition, we write the mean in the log-linear form ($\exp(\boldsymbol{\beta}'\mathbf{x_i})$) where $\boldsymbol{\beta}' = (\beta_1, \beta_2)$, $\beta_1 = log(\delta/\mu)$, $\beta_2 = \delta - 1$ and $\mathbf{x}_i' = (1, logi)$. The log-likelihood is then:

$$logL = -\boldsymbol{\beta}'\sum_{i=1}^{n}\mathbf{x}_i - \sum_{i=1}^{n} Y_i\, exp(-\boldsymbol{\beta}'\mathbf{x}_i)$$

and we have

$$\boldsymbol{\psi}_n(\boldsymbol{\beta}) \equiv \frac{\partial logL}{\partial \boldsymbol{\beta}} = -\sum_{i=1}^{n}\mathbf{x}_i + \sum_{i=1}^{n} Y_i\, exp(-\boldsymbol{\beta}'\mathbf{x}_i)\mathbf{x}_i \tag{3.1}$$

$$\mathbf{A}_n(\boldsymbol{\beta}) \equiv -\frac{\partial^2 logL}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} = \sum_{i=1}^{n} Y_i\, exp(-\boldsymbol{\beta}'\mathbf{x}_i)\mathbf{x}_i\mathbf{x}_i' \tag{3.2}$$

Expression(3.2) shows that $\mathbf{A}_n(\boldsymbol{\beta})$ is positive definite so $\boldsymbol{\psi}_n(\boldsymbol{\beta})$ is strictly concave. However, the likelihood equations $\boldsymbol{\psi}_n(\boldsymbol{\beta}) = \mathbf{0}$ do not seem to have a closed form solution. Numerical solutions can be obtained through standard iterative methods such as the Newton-Raphson or the scoring method. We proceed to derive the large sample properties of the maximum likelihood estimates. In our subsequent discussion all limits will be taken as $n \longrightarrow \infty$ unless otherwise mentioned. Also, the symbol $\sim$ placed between two functions of n will indicate that the ratio of the two tends to 1 as $n \longrightarrow \infty$.

Denoting the true parameter point by $\boldsymbol{\beta}_0$, we define

$$\boldsymbol{\psi}_n = \boldsymbol{\psi}_n(\boldsymbol{\beta}_0), \quad \mathbf{A}_n = \mathbf{A}_n(\boldsymbol{\beta}_0)$$

$$Y_{i0} = Y_i\, exp(-\boldsymbol{\beta}_0'\mathbf{x}_i), \quad e_i = Y_{i0} - 1, \quad i = 1, 2, \ldots, n$$

and note that e_i's are i.i.d random variables with zero mean and unit variance. Using (3.1) and (3.2) we can then express $\boldsymbol{\psi}_n$ and $\mathbf{A}_n$ in terms of e_i's as :

$$\boldsymbol{\psi}_n \equiv (l_{1n}, l_{2n})' = \sum_{i=1}^{n} e_i\mathbf{x}_i$$

$$\mathbf{A}_n = \sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i' + \sum_{i=1}^{n} e_i\mathbf{x}_i\mathbf{x}_i'$$

These in our special case, have the components

$$l_{1n} = \sum_{i=1}^{n} e_i, \quad l_{2n} = \sum_{i=1}^{n} e_i logi$$
$$a_{11} = \sum_{i=1}^{n} e_i + n, \quad a_{12} = \sum_{i=1}^{n} e_i logi + \sum_{i=1}^{n} logi$$
$$a_{22} = \sum_{i=1}^{n} e_i (logi)^2 + \sum_{i=1}^{n} (logi)^2 \tag{3.3}$$

We state a general asymptotic result for the sum of powers of $logi$ which will be repeatedly used in the sequel. The proof is easy and hence omitted.

Lemma 3.1 *For all fixed nonnegative integer k,* $n^{-1}(logn)^{-k}\sum_{i=1}^{n}(logi)^k = 1 - k(logn)^{-1}\epsilon_{kn}$ *where* ϵ_{kn} *converges to 1 as* $n \longrightarrow \infty$.

Let

$$U_k = n^{-1/2}(logn)^{-k}\sum_{i=1}^{n} e_i (logi)^k \quad k = 0, 1, 2 \tag{3.4}$$
$$\mathbf{Z}_n = (Z_{1n}, Z_{2n})' = (n^{-1/2} l_{1n}, n^{-1/2}(logn)^{-1} l_{2n})' \tag{3.5}$$

Lemma 3.2 *Asymptotically* $\mathbf{Z}_n$ *is bivariate normal* $N_2(\mathbf{0}, \mathbf{\Sigma})$, *where* $\mathbf{\Sigma} = \mathbf{11}'$ *is singular.*

Proof : From (3.4) and (3.5) identify Z_{1n} and Z_{2n} to be U_0 and U_1, respectively. By the central limit theorem the asymptotic distribution of U_0 is standard normal. Also, $U_0 - U_1 = o_p(1)$ because $E[U_0 - U_1] = 0$ and $Var[U_0 - U_1]$ converges to zero by virtue of Lemma 3.1. The stated result then follows. //

The singularity of $\mathbf{\Sigma}$ poses a problem in doing the usual Taylor series expansion proof for the MLE's. This situation is very similar to one encountered by Bhattacharyya and Ghosh(1988) in the context of a nonhomogeneous binomial model. In order to use their line of arguments we will show that although the scaled matrix of second derivatives of the log likelihood is asymptotically singular, the probability that it is positive definite tends to 1. To this end let us denote

$$\mathbf{C}_n(\boldsymbol{\beta}) = n^{-1}\begin{bmatrix} a_{11}(\boldsymbol{\beta}) & a_{12}(\boldsymbol{\beta})/(logn) \\ a_{12}(\boldsymbol{\beta})/(logn) & a_{22}(\boldsymbol{\beta})/(logn)^2 \end{bmatrix}$$

$$\mathbf{C}_n = \mathbf{C}_n(\boldsymbol{\beta}_0) = (c_{ij}), \quad d_n = |\mathbf{C}_n| \tag{3.6}$$

Lemma 3.3 *(i)* $\mathbf{C}_n \xrightarrow{P} \Sigma$, *(ii)* $(logn)^2 d_n \xrightarrow{P} 1$.

Proof : (i) From (3.3) and (3.6) we obtain,

$$\begin{aligned} c_{11} &= n^{-1/2}U_0 + 1 \\ c_{12} &= n^{-1/2}U_1 + (nlogn)^{-1}\sum_{i=1}^{n} logi \\ c_{22} &= n^{-1/2}U_2 + n^{-1}(logn)^{-2}\sum_{i=1}^{n}(logi)^2 \end{aligned} \tag{3.7}$$

Hence (i) follows from Lemma 3.1 and the fact that $U_k = O_p(1)$.

(ii) Note that $d_n = (nlogn)^{-2}(a_{11}a_{22} - a_{12}^2)$, where (using (3.3) and (3.4)) :

$$\begin{aligned} a_{11} &= n(1 + n^{-1/2}U_0) \\ a_{12} &= n(logn)[(nlogn)^{-1}\sum_{i=1}^{n} logi + n^{-1/2}U_1] \\ a_{22} &= n(logn)^2[n^{-1}(logn)^{-2}\sum_{i=1}^{n}(logi)^2 + n^{-1/2}U_2] \end{aligned}$$

Using these expressions, $(logn)^2 d_n$ equals

$$\begin{aligned} &\left(1 + n^{-1/2}U_0\right)\left[n^{-1}\sum_{i=1}^{n}(logi)^2 + n^{-1/2}(logn)^2 U_2\right] - \left[n^{-1}\sum_{i=1}^{n}(logi) + n^{-1/2}(logn)U_1\right]^2 \\ &= \left\{n^{-1}\sum_{i=1}^{n}(logi)^2 - n^{-2}\left(\sum_{i=1}^{n} logi\right)^2\right\} + n^{-1/2}U_0\left[n^{-1}\sum_{i=1}^{n}(logi)^2\right] + n^{-1/2}(logn)^2 U_2 \\ &\quad - 2n^{-1/2}(logn)U_1[n^{-1}\sum_{i=1}^{n} logi] + n^{-1}(logn)^2 U_0 U_2 - n^{-1}(logn)^2 U_1^2 \\ &= \left\{n^{-1}\sum_{i=1}^{n}(logi)^2 - \left(n^{-1}\sum_{i=1}^{n} logi\right)^2\right\} + o_p(1) \end{aligned} \tag{3.8}$$

The last equality follows by observing that $\sum_{i=1}^{n}(logi)^k \sim n(logn)^k$ and $n^{-1/2}(logn)^k U_j = o_p(1)$. Using the identity $n^{-1}\sum_{i=1}^{n} h_i^2 - \bar{h}^2 = n^{-1}\sum_{i=1}^{n} h_i^{*2} - \bar{h}^{*2}$ with $h_i^* = h_i - h_n$, the nonrandom term in the braces in (3.8) can be written as

$$\frac{1}{n}\sum_{i=1}^{n}[log(i/n)]^2 - \left[\frac{1}{n}\sum_{i=1}^{n}log(i/n)\right]^2$$

which converges to $\int_0^1 (logu)^2\, du - \left[\int_0^1 (logu)\, du\right]^2 = 1.$ //

Part(ii) of Lemma 3.3 yields the crucial result that $P[d_n > 0] = P[(logn)^2 d_n > 0] \longrightarrow 1.$

Therefore, defining the set $G_n = \{d_n \neq 0\}$, we form the perturbed inverse of $\mathbf{C}_n$ as

$$\mathbf{F}_n = \mathbf{C}_n^{-1} I(G_n) + \mathbf{I}_2 I(G_n^c) \tag{3.9}$$

where I denotes the the indicator function and $\mathbf{I}_2$ the 2×2 identity matrix. Assume that $\psi_n(\beta) = \mathbf{0}$ has a solution $\widehat{\beta_n} = (\widehat{\beta}_{1n}, \widehat{\beta}_{2n})$. The appropriate neighborhood of β_0 in which the solution exists is specified in Lemma 3.5. A Taylor expansion of $\psi_n(\widehat{\beta_n}) = \mathbf{0}$ around β_0 yields

$$\psi_n(\beta_0) = \mathbf{A}_n(\zeta_n)(\widehat{\beta_n} - \beta_0) \tag{3.10}$$

where ζ_n is on the line segment joining $\widehat{\beta_n}$ and β_0.

Defining,

$$W_{1n} = n^{1/2}(logn)^{-1}(\widehat{\beta}_{1n} - \beta_{10}), \quad W_{2n} = n^{1/2}(\widehat{\beta}_{2n} - \beta_{20}), \quad \mathbf{W_n} = (W_{1n}, W_{2n})'$$

we observe from (3.5) and (3.10) that

$$\mathbf{Z}_n = (logn)\mathbf{C}_n(\zeta_n)\mathbf{W}_n \tag{3.11}$$

$$\mathbf{K}_n \equiv (logn)^{-1}\mathbf{F}_n\mathbf{Z}_n = \mathbf{F}_n\mathbf{C}_n(\zeta_n)\mathbf{W}_n \tag{3.12}$$

Lemma 3.4 *Asymptotically,* $\mathbf{K}_n$ *is bivariate* (singular) *normal* $N_2(\mathbf{0}, \Sigma_1)$ *with*

$$\Sigma_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Proof : We observe that on the set G_n

$$\mathbf{F}_n\mathbf{Z}_n = \frac{1}{d_n}\begin{bmatrix} c_{22} & -c_{12} \\ -c_{12} & c_{11} \end{bmatrix}\begin{pmatrix} U_0 \\ U_1 \end{pmatrix}$$

where, using (3.7) and Lemma 3.1, we write

$$\begin{aligned} c_{11} &= n^{-1/2}U_0 + 1 \\ c_{12} &= n^{-1/2}U_1 + 1 - (logn)^{-1}\epsilon_{1n} \\ c_{22} &= n^{-1/2}U_2 + 1 - 2(logn)^{-1}\epsilon_{2n} \end{aligned}$$

Denoting $\mathbf{K}_n^* = d_n(logn)^2\mathbf{K}_n$, we have

$$\begin{aligned} K_{1n}^* &= (logn)(c_{22}U_0 - c_{12}U_1) \\ &= (logn)\left[\left(n^{-1/2}U_2 + 1 - 2(logn)^{-1}\epsilon_{2n}\right)U_0 - \left(n^{-1/2}U_1 + 1 - (logn)^{-1}\epsilon_{1n}\right)U_1\right] \\ &= (logn)(U_0 - U_1) - U_0 + o_p(1) \end{aligned}$$

The last equality follows from the fact that (U_0, U_1) is asymptotically distributed as $N(\mathbf{0}, \mathbf{1}\mathbf{1}')$. By similar steps, it follows that

$$\begin{aligned} K_{2n}^* &= (logn)(U_1 - U_0) + U_0 + o_p(1) \\ &= -K_{1n}^* + o_p(1) \end{aligned}$$

By characteristic function argument for exponential random variables it can be shown that $(logn)(U_0 - U_1) - U_0$ has an asymptotic standard normal distribution. The stated result then follows by observing that $(logn)^2 d_n$ converges to 1 in probability (by part(ii) of Lemma 3.3) and the fact that on G_n^c, $\mathbf{K}_n = (logn)^{-1}\mathbf{Z}_n = o_p(1)$ (by Lemma 3.2). //

Lemma 3.5 *Define a sequence of neighborhoods of* β_0 *by :*

$$M_n(\beta_0) = \{(\beta_1, \beta_2) : \beta_1 = \beta_{10} + \tau_1(logn)n^{-\gamma}, \quad \beta_2 = \beta_{20} + \tau_2 n^{-\gamma}, \ ||\tau|| \leq h\},$$

where γ *and* h *are fixed numbers,* $0 < \gamma < 1/2$, $0 < h < \infty$, *and* τ_1, τ_2 *are real. Then* $\mathbf{F}_n[\mathbf{C}_n(\beta) - \mathbf{C}_n] \xrightarrow{P} \mathbf{0}$ *uniformly in* $\beta \in M_n(\beta_0)$.

The proof rests on showing the uniform convergence of certain exponential functions that arise in the expressions for the mean of the elements in $\mathbf{C}_n(\beta)$. The details are outlined in the Appendix.

The main results concerning the existence and asymptotic normality of a consistent sequence of roots of the likelihood equations are stated in the next two theorems.

Theorem 3.1 (Existence) *With probability tending to 1 as* $n \longrightarrow \infty$, *there exists a sequence of roots* $\widehat{\beta}_n \in M_n(\beta_0)$ *of the likelihood equations. Furthermore, such* $\widehat{\beta}_n$*'s correspond to local maxima of the likelihood function.*

Theorem 3.2 (Asymptotic Normality) $\mathbf{W_n}$ *is asymptotically bivariate* (singular) *normal* $N_2(\mathbf{0}, \Sigma_1)$.

The proof of Theorem 3.1 follows along the same lines of the proof of a corresponding result in Bhattacharyya and Ghosh(1991) and is outlined in the Appendix. As for the proof of Theorem 3.2 note that

$$\mathbf{F}_n\mathbf{C}_n = \mathbf{I}_2 I(G_n) + \mathbf{C}_n(\beta_0) I(G_n^c)$$

By Lemma 3.3 we have, $P(G_n) \longrightarrow 1$ and $\mathbf{C}_n \xrightarrow{P} \Sigma$. We can then conclude

$$\mathbf{F}_n\mathbf{C}_n \xrightarrow{P} I_2 \tag{3.13}$$

Equation (3.12) gives

$$\mathbf{K}_n = \mathbf{F}_n \mathbf{C}_n(\zeta_n) \mathbf{W}_n = \mathbf{F}_n[\mathbf{C}_n(\zeta_n) - \mathbf{C}_n(\beta_0) + \mathbf{C}_n(\beta_0)]\mathbf{W}_n$$

For $\zeta_n \in M_n(\beta_0)$, Lemmas 3.4-3.5 along with equation(3.13) then yields the result. //

Consequences

(i) Noting the parameter relations $\beta_1 = log(\delta/\mu)$, $\beta_2 = \delta - 1$, we can translate the results of Theorem 3.2 (via delta method and Slutsky's theorem) in terms of the original parameters as :

$$\sqrt{n}(\hat{\delta} - \delta_0) \xrightarrow{d} N(0,1), \quad \sqrt{n}(logn)^{-1}(\hat{\mu} - \mu_0) \xrightarrow{d} N(0, \mu_0^2)$$

(ii) The *current system reliability* under the PEXP model is a 1-1 function of the current intensity of failure. The intensity λ_n at the n-th failure can be expressed in terms of the parameters β_1 and β_2 as :

$$-log\ \lambda_n = \beta_1 + \beta_2 logn$$

An estimate of this can be obtained by replacing the parameters by their MLE's. By virtue of Lemma 3.5 and Lemma 3.3 we have $\mathbf{C}_n(\zeta_n) \xrightarrow{P} \Sigma$ uniformly in the neighborhood $M_n(\beta_0)$. Also, Z_{1n} has an asymptotic standard normal distribution by Lemma 3.2. Using these two facts in (3.11), we deduce

$$\sqrt{n}(log\ \hat{\lambda}_n - log\ \lambda_n) = -(logn)(W_{1n} + W_{2n}) \xrightarrow{d} N(0,1)$$

3.2 Simple Estimators — Weibull Process Analog

We have noted in Section 3.1 that the MLE's for the parameters in the PEXP are not available in closed form. Also, the simulations described in Section 4.1 indicate that for small sample sizes, the MLE of μ often falls far off the true parameter value. Here we construct an alternative set of estimators by exploiting the link between the PEXP and NHPP models. To this end, we note the following correspondence between parameters of the two models :

NHPP	PEXP
$\mu^{1/\delta}$	μ
$1/\delta$	δ

The fact that the MLE's under the NHPP model are given by $\hat{\delta} = n/\sum_{i=1}^{n} log(T_n/T_i)$ and $\hat{\mu} = n/T_n^{\hat{\delta}}$ then motivates the following set of estimators for the PEXP :

$$\delta^* = \frac{1}{n}\sum_{i=1}^{n} log(T_n/T_i), \qquad \mu^* = \frac{n^{\delta^*}}{T_n} \tag{3.14}$$

Since these estimators stem from the NHPP model with Weibull intensity, we call them Weibull process analog estimators (WPAE). The main advantage of (3.14) over the MLE's is the simple closed form expressions of the estimators. Simulations demonstrate that μ^* behaves better than the MLE $\hat{\mu}$ for small sample sizes.

If the true model is PEXP and one wrongly assumes the model to be NHPP, then the WPAE's constitute a set of estimators for the "misspecified" model. From this point of view of misspecificaton, it is worth comparing the properties of the WPAE's with those of the MLE's.
We denote the true parameter values by μ_0 and δ_0 and define,

$$W_{1n}^* = n^{1/2}(logn)^{-1}(\mu^* - \mu_0), \quad W_{2n}^* = n^{1/2}(\delta^* - \delta_0), \quad \mathbf{W}_n^* = (W_{1n}^*, W_{2n}^*)' \tag{3.15}$$

The following theorem shows that the WPAE's are consistent and asymptotically normal (CAN) estimators of the parameters under PEXP.

Theorem 3.3 *Asymptotically,* $\mathbf{W}_n^*$ *is bivariate normal* $N(\mathbf{0}, \Sigma_1^*)$ *where*

$$\Sigma_1^* = \delta_0^2/(2\delta_0 - 1)\begin{bmatrix} \mu_0^2 & \mu_0 \\ \mu_0 & 1 \end{bmatrix}$$

To prepare the groundwork for proving the theorem we define

$$X_i = \frac{\mu_0 \, T_i}{i^{\delta_0}}, \quad i = 1, \ldots, n$$

and express the WPAE's in terms of these random variables as :

$$\begin{aligned} \delta^* &= log(T_n/n^{\delta_0}) - n^{-1}\sum_{i=1}^{n} log(T_i/i^{\delta_0}) + \delta_0 n^{-1}\sum_{i=1}^{n} log(n/i) \\ &= n^{-1}\sum_{i=1}^{n} log(X_n/X_i) + \delta_0[logn - (logn!)/n] \end{aligned} \tag{3.16}$$

$$\begin{aligned} log\mu^* - log\mu_0 &= log(n^{\delta_0}/\mu_0 T_n) + (logn)(\delta^* - \delta_0) \\ &= (logn)(\delta^* - \delta_0) - logX_n \end{aligned} \tag{3.17}$$

The next lemma provides some results concerning the random variables X_i which will be used in proving our main results.

Lemma 3.6 $(a) n^{1/2}(X_n - 1) \xrightarrow{d} N(0, \delta_0^2/(2\delta_0 - 1))$
$(b) n^{1/2}(X_n - \overline{X_n}) \xrightarrow{d} N(0, \delta_0^2/(2\delta_0 - 1))$
$(c) n^{-1/2}\sum_{i=1}^{n} log(X_n/X_i) = n^{1/2}(X_n - \overline{X_n}) + o_p(1)$.

Proof : (a) Noting that X_n is a linear function of the independent exponential random variables $Y_i \equiv T_i - T_{i-1}$, $i = 1, \ldots, n$, the result follows from an application of the Lindeberg-Feller central limit theorem.

(b) We first write $\overline{X_n} - X_n$ as a linear function of $Y_i's$. Using the fact : $\sum_{i=1}^{n} \sum_{j=1}^{i} a_i b_j = \sum_{i=1}^{n} \sum_{j=i}^{n} b_i a_j$, we write

$$\sum_{i=1}^{n} X_i = \mu_0 \sum_{i=1}^{n} i^{-\delta_0} \sum_{j=1}^{i} Y_j$$
$$= \mu_0 \sum_{i=1}^{n} Y_i \left(\sum_{j=i}^{n} j^{-\delta_0} \right).$$

Also, $nX_n = \mu_0 n^{1-\delta_0} \sum_{i=1}^{n} Y_i = \mu_0 n^{1-\delta_0} \sum_{i=1}^{n} Y_i \sum_{j=i}^{n} (n-i+1)^{-1}$

Thus, we have

$$S_n \equiv n^{-1/2} \sum_{i=1}^{n} (X_i - X_n)$$
$$= \mu_0 n^{-1/2} \sum_{i=1}^{n} Y_i \, d_{in}$$

where

$$d_{in} = \sum_{j=i}^{n} \left(\frac{1}{j^{\delta_0}} - \frac{n}{n^{\delta_0}(n-i+1)} \right)$$

By using the relations

$$\delta (j+1)^{\delta-1} \geq (j+1)^{\delta} - j^{\delta} \geq \delta j^{\delta-1} \tag{3.18}$$

we obtain bounds for $E(Y_j)$ and correspondingly for $E(S_n)$ as :

$$-\delta_0 n^{-1/2} \sum_{i=1}^{n} i^{-\delta_0} (i+1)^{\delta_0 - 1} \leq E(S_n) \leq \delta_0 n^{1/2-\delta_0} (n+1)^{\delta_0-1}$$

Since the lower bound is $O(n^{-1/2} logn)$ and the upper bound is $O(n^{-1/2})$, we have $E(S_n) \longrightarrow 0$. Also, from the expression of S_n we readily obtain

$$s_n \equiv Var(S_n) = \delta_0^2 n^{2\delta_0 - 3} \sum_{i=1}^{n} (i/n)^{2\delta_0 - 2} \, d_{in}^2$$

$$\mu_0^4 n^{-2} \sum_{i=1}^{n} E(Y_i)^4 \, d_{in}^4 = 24 \delta_0^4 n^{4\delta_0 - 6} \sum_{i=1}^{n} (i/n)^{4\delta_0 - 4} \, d_{in}^4$$

Setting a correspondence of d_{in} with a Riemann sum, we observed that as $n \longrightarrow \infty$,

$$s_n \longrightarrow \delta_0^2 \int_0^1 u^{2\delta_0 - 2} \left\{ \int_u^1 \left(\frac{1}{v^{\delta_0}} - \frac{1}{1-u} \right) dv \right\}^2 du = \delta_0^2 / (2\delta_0 - 1)$$

$$\mu_0^4 n^{-2} \sum_{i=1}^{n} E(Y_i)^4 \, d_{in}^4 \sim n^{-1} 24 \delta_0^4 \int_0^1 u^{4\delta_0 - 4} \left\{ \int_u^1 \left(\frac{1}{v^{\delta_0}} - \frac{1}{1-u} \right) dv \right\}^4 du \longrightarrow 0$$

These facts in conjunction with the result that $E(S_n) \longrightarrow 0$, enable us to use the central limit theorem to conclude part(b) of the lemma.

(c) Let

$$U_{in} \equiv log(X_n/X_i) - (X_n - X_i)$$

Since $\quad (x-1)/x \leq logx \leq x-1$ for $x > 0$, we have

$$\left(\frac{1}{X_n} - 1\right) n^{-1/2}\sum_{i=1}^{n}(X_n - X_i) \leq n^{-1/2}\sum_{i=1}^{n}U_{in} \leq n^{-1/2}\sum_{i=1}^{n}(X_n - X_i)\left(\frac{1}{X_i} - 1\right) \quad (3.19)$$

We use Slutsky's theorem in conjunction with the results in parts (a) and (b) to conclude that the lower bound in equation (3.19) is $o_p(1)$. The upper bound equals

$$n^{-1/2}\sum_{i=1}^{n}[(X_n - 1) - (X_i - 1)]\left(\frac{1}{X_i} - 1\right)$$

$$= n^{1/2}(X_n - 1)\left\{n^{-1}\sum_{i=1}^{n}\left(\frac{1}{X_i} - 1\right)\right\} - n^{-1/2}\sum_{i=1}^{n}(X_i - 1)\left(\frac{1}{X_i} - 1\right)$$

Denote the first and second terms on the right hand side by B_1 and B_2, respectively. By part(a) of the lemma, we have $B_1 = o_p(1)$. The proof is completed once we establish that $B_2 = o_p(1)$. Now, by the Cauchy-Schwarz inequality,

$$B_2^2 \leq \sum_{i=1}^{n}(X_i - 1)^2\left(n^{-1}\sum_{i=1}^{n}\left(\frac{1}{X_i} - 1\right)^2\right)$$

Again part(a) of the lemma implies that $n^{-1}\sum_{i=1}^{n}(1/X_i - 1)^2 = o_p(1)$. To show that $\sum_{i=1}^{n}(X_i - 1)^2 = O_p(1)$ we note that

$$\begin{aligned}
\sum_{i=1}^{n}Var(X_i) &= \delta_0^2\sum_{i=1}^{n}(1/i^2)\sum_{j=1}^{i}(j/i)^{2\delta_0-2} \\
&= \delta_0^2 n^{-1}\sum_{i=1}^{n}\frac{1}{(i/n)}\frac{1}{i}\sum_{j=1}^{i}(j/i)^{2\delta_0-2} \\
&\longrightarrow \delta_0^2\int_0^1\frac{1}{u}\int_0^u v^{2\delta_0-2}\,dv\,du = \delta_0^2/(2\delta_0-1)^2
\end{aligned}$$

Using equation (3.18) we also have,

$$0 \leq \sum_{i=1}^{n}\{E(X_i) - 1\}^2 \leq \sum_{i=1}^{n}i^{-2\delta_0}\{\delta_0^2(i+1)^{2\delta_0-2} - 2\delta_0(i+1)^{\delta_0-1} + 1\}$$

Since $\delta_0 > 1$, all the terms in the sum on the right hand side converge to finite numbers. Thus, $\sum_{i=1}^{n}(X_i - 1)^2$ is bounded in expectation and hence is $O_p(1)$ which implies $B_2 = o_p(1)$. //

Proof of Theorem 3.3

From (3.15) and (3.16), we have

$$\begin{aligned} W_{2n}^* &= n^{-1/2}\sum_{i=1}^{n} log(X_n/X_i) + n^{1/2}\delta_0 [logn - (logn!)/n - 1] \\ &= n^{1/2}(X_n - \overline{X_n}) + n^{1/2}\delta_0 [logn - (logn!)/n - 1] + o_p(1) \\ &= n^{1/2}(X_n - \overline{X_n}) + o_p(1) \end{aligned}$$

The second equality follows from lemma 3.6(c) and the last equality obtains from Stirling's formula,

$$logn - (logn!)/n - 1 = -(logn)/2n + O(1/n)$$

Then Lemma 3.6(b) entails that W_{2n}^* is asymptotically distributed as $N(0,\ \delta_0^2/(2\delta_0 - 1))$. Finally, from (3.17) we have,

$$n^{1/2}(logn)^{-1}(log\mu^* - log\mu_0) = n^{1/2}(\delta^* - \delta_0) - n^{1/2}(logn)^{-1}\ logX_n$$

Since Lemma 3.6(a) along with an application of the delta method yields the fact that $n^{1/2}(logn)^{-1}\ logX_n$ converges to zero in probability, we obtain

$$W_{1n}^* = \mu_0 W_{2n}^* + o_p(1).$$

which completes the proof. //

Calculation of ARE's

In view of the asymptotic results derived in this section, we summarize the comparative features between the MLE's and the WPAE's.

- The rates of convergence for both sets of estimators $(\hat{\mu}, \mu^*)$ and $(\hat{\delta}, \delta^*)$ are identical.
- Both sets of estimators (properly scaled) have an asymptotically singular normal distribution.
- The WPAE's incur a loss of asymptotic efficiency compared to the MLE's. In fact, for both μ and δ the

$$\text{ARE (WPAE : MLE)} = \frac{2\delta_0 - 1}{\delta_0^2} < 1.$$

  The ARE decreases with an increase in δ, becomes close to 1 as δ gets close to 1. For large δ the ARE tends to 0. Also note that the ARE does not depend on the parameter μ.

4 Simulation and applications

Summary of simulation results

Monte Carlo simulation techniques were employed to study the performances of the maximum likelihood and the Weibull process analog estimators in both small and large samples. Three pairs of (μ, δ) values, (0.5,1.5), (1.0,2.0) and (2.5,2.5) were used for the study, and for each case 100 realizations of the MLE's and the WPAE's were obtained with the sample sizes n=10, 25, 50 and 100.

Exponential random variables were generated using the inverse cdf transformation on uniform(0,1) random numbers obtained from the *Uniform Random Number Generator* (UNI) residing in the Fortran Library **CMLIB**. The MLE $\hat{\delta}$ was obtained through a single-variable Newton-Raphson iteration procedure using the WPAE δ^* as the initial value. The MLE $\hat{\mu}$ is then calculated from the relation

$$\hat{\mu} \;=\; n\hat{\delta}/\textstyle\sum_{i=1}^{n} i^{1-\hat{\delta}} Y_i$$

where Y_i's are the generated values of the inter-failure times which are independent exponential random variables under the PEXP model. Table 1 gives the estimated bias and mean squared error of the MLE's as well as the WPAE's.

The MLE for μ has a tendency to overestimate as is evidenced from positive bias in all cases in Table 1. Also, it shows a substantial variability especially for small sample sizes (*e.g.* n*=10*). By contrast, the MLE for δ appears to be quite stable. Although it exhibits a positive bias in most cases, the magnitudes of the bias as well as the MSE's are substantially smaller compared to those for $\hat{\mu}$. The WPAE's for both the parameters show a tendency of underestimation in almost all cases. With respect to the MSE's the performances of the estimates of δ are comparable (with the MLE behaving slightly better for larger δ values), while the WPAE of μ performs better than the corresponding MLE in all cases. Even for n as large as 50 or 100, the finite-sample efficiency of μ^* relative to $\hat{\mu}$ as measured by the ratio (estimated) MSE($\hat{\mu}$)/MSE(μ^*) is quite different from the value of the asymptotic relative efficiency $(2\delta - 1)/\delta^2$. Specifically, the ARE values are 0.88, 0.75, and 0.64 for the three cases $\delta \;=\; 1.5$, 2.0, and 2.5 respectively, while the corresponding finite-sample relative efficiencies are found to be 2.04, 3.068, and 2.89 for n=50, and 2.10, 1.49, and 1.62 for n=100.

For practical applications of the asymptotic results, it is important to examine how the normal approximation improves with increasing sample sizes. An investigation in that direction is made through the normal scores plots of the estimates. Plots for both $\hat{\delta}$ and δ^* (Figure 1) indicate a fairly linear pattern in all cases of n, small or large. However, the corresponding plots for $\hat{\mu}$ and μ^* (Figure 2) show a substantial departure from a straight line pattern, which persists even for n as large as 50. This appears to be due to a considerable fluctuation in the estimated μ values and also the slow rate ($\sqrt{n}/logn$) of convergence. However, logarithmic transformation on the estimates of μ is found to stabilize their variations substantially, and the agreement with the normal scores is also considerably improved. Figure 2 exhibits these features for the case n=25. This indicates that when setting large-sample confidence interval for μ, one should first construct a confidence interval for $log\mu$ using the asymptotic normality result and then transform the result to μ. Specifically, a $100(1 - \alpha)\%$ confidence interval for μ constructed in this process would be of the form

$$\hat{\mu}\left[n^{\pm z_{\alpha/2}/\sqrt{n}}\right] \tag{4.1}$$

In the context of *reliability growth*, underestimation (or overestimation) of the parameters may have serious implications to the physical interpretation of an assumed model. For instance, an estimate of δ less than (greater than) 1 will indicate a reliability decay (growth), while the true parameter might demonstrate otherwise. Table 2 gives a summary of the proportion of times this type of error happens for n=10, 20 and 25 for both the MLE's and WPAE's in the following two situations: PEXP with $\delta = 1.5$ (a case of reliability growth), and PEXP with $\delta = 0.5$ (a case of reliability decay), with μ in both the cases taken to be 0.5. The proportion is calculated by dividing the number of such "unwanted" occurrences by the total number of realizations (100 in our case). The results indicate that for the reliability growth case, the performance of the MLE is better in this respect, and the relative proportion of "error" decreases fast with increase in the sample size. In the reliability decay case, the performance of WPAE is better than the MLE for small sample sizes.

Applications

Here we fit the PEXP and apply the inference results of Section 3 to two data sets which were previously analyzed under the NHPP model. We also employ some graphical checks for model adequacy, and compare the inference results between the two models, especially with regard to estimating the current system reliability.

Example 1 **Tank failure data** For a tank system, the number of miles accumulated was recorded for the first 25 failures : 1, 57, 252, 310, 485, 693, 720, 727, 779, 1028, 1561, 1766, 1793, 1938, 2030, 2065, 2289, 2423, 2560, 3086, 3458, 3626, 4252 and 4582 (Source : Military Handbook 189 (1981, pp 111)). The objective of the study was to assess the extent to which parts improvement and other design changes reduced the intensity of failure. From the PEXP fit, the maximum likelihood estimators for the parameters are found to be $\hat{\mu} = 0.04699$ and $\hat{\delta} = 1.6614$. A large sample 95% confidence interval for δ is given by $\hat{\delta} \pm 1.96/\sqrt{n} = [1.27, 2.05]$, and it indicates reliability growth. Using (4.1), a 95% confidence nterval for μ is found to be $[0.17, 1.73]$. The corresponding WPAE's (or the MLE's for the misspecified model) are found to be $\mu^* = 0.03027$, $\delta^* = 1.5323$, with the associated large sample 95% confidence intervals $[1.11, 1.95]$ and $[.008, .116]$ for δ and μ, respectively.

To develop a graphical model checking procedure we define the residuals

$$\hat{e}_i = \frac{\hat{\mu} Y_i}{\hat{\delta} i^{\hat{\delta}-1}}, \quad i = 1, \ldots, 25$$

where Y_i denotes the miles between the ith and $(i-1)st$ failure. Note that these residuals correspond to the standard exponential variates $e_i = (\mu/\delta)Y_i/(i^{\delta-1})$. Figure 3(a) shows a plot of of the points $(\hat{e}_{(i)}, \ \alpha_i)$ where $\alpha_i = \sum_{i=1}^{25}(25 - j + 1)^{-1}$ is the expected value of the ith standard exponential order statistic in a sample of size 25 and $\hat{e}_{(i)}$ denotes the ordered residuals. The points lie roughly along a straight line with unit slope – a pattern that supports the assumption of exponentiality and hence the PEXP model. Note that $\hat{e}_i$ is of the scale free form $a_i(\hat{\delta})/\overline{a}(\hat{\delta})$, where $a_i(\hat{\delta}) = Y_i i^{1-\hat{\delta}}$. For a corresponding graphical check for the NHPP fit, we define the residuals to be equal to $\hat{\delta} log(t_n/t_i)$, $i = 1, \ldots, n-1$, t_i being the accumulated mileage at the ith failure and $\hat{\delta}$ being the maximum likelihood estimate of δ under the NHPP model. Note that these residuals correspond to a set of order statistics of size $n-1$ from the standard exponential and are also of the scale free form $b_i/\bar{b}$, where $b_i = log(t_n/t_i)$. Figure 3(b) shows a plot of $(\tilde{e}_{(i)}, \ \tilde{\alpha}_i)$, where $\tilde{e}_{(i)}$ denotes the ordered residuals and $\tilde{\alpha}_i$ are the exponential scores in a sample of size 24.

As mentioned in Section 3, one aspect of practical importance in RG analysis is the estimation of the current reliability of the system, as measured by the reciprocal of the current value of the intensity. For our situation, this is also the common ground for comparing the two models. For the present data "time" is identified with "miles" and the current "mean time between failures" (MTBF), defined as the reciprocal of the current intensity of failure, is 297.21 and 292.78 for the

PEXP and the NHPP fits respectively. An associated approximate 95% confidence interval for the MTBF for the PEXP is provided by the formula $\hat{\lambda}_n[1 \pm 1.96/\sqrt{n}]^{-1}$ and is calculated as : [213.51, 488.83]. The corresponding large sample confidence interval for the NHPP is computed as [188.36, 657]. Therefore, if PEXP were indeed the true model, the current MTBF can be estimated with more precision in this case.

Example 2 <u>**Mine explosion data**</u> Maguire, Pearson and Wynn (1952) provide the data of the number of days between mine explosions in Great Britain involving more than 10 men killed between December, 1875 and May, 1951. We want to see to what extent the safety regulations and other necessary precautions decrease the intensity of the accidents. For a graphical check for the PEXP fit, the residuals are plotted against the exponential scores (Figure 4(a)). Most of the points lie around the straight line with unit slope thereby ensuring a reasonable fit. By contrast, the corresponding residual plot for the NHPP clearly shows more departure from the line. It is, however, evident from both the plots that a different model may be needed for the latter part of the data. The MLE's for the parameters are $\hat{\mu} = 0.08885$ and $\hat{\delta} = 1.6512$ with associated approximate confidence intervals [0.036, 0.214] and [1.46, 1.84] for μ and δ, respectively, thereby indicating reliability growth. The WPAE's are $\mu^* = 0.02719$ and $\delta^* = 1.40068$. The associated 95% large sample confidence intervals are [0.036, 0.22] and [1.20, 1.60] respectively. The estimate of the current MTBF is 394.38 days with an associated approximate 95% confidence interval [332.04, 485.53]. For the NHPP fit which is demonstrated by Crow (1974), the current MTBF is estimated as 337.51 days with the associated large sample 95% confidence interval [266.71, 459.51].

5 Appendix

Proof of Lemma 3.5 : Observe that $\mathbf{F_n}$ involves the fixed point $\boldsymbol{\beta}_0$ and is $O_p((logn)^2)$. By virtue of expressions (3.4) and (3.7), it then suffices to show that

$$T_{n,k} = (logn)^2 n^{-1}(logn)^{-k}\sum_{i=1}^{n}(logi)^k[e_i - e_i(\boldsymbol{\beta}_0)]$$

converges in probability to 0 uniformly in $\boldsymbol{\beta} \in M_n(\boldsymbol{\beta}_0)$ for k = 0,1,2 where

$$e_i(\boldsymbol{\beta}) = Y_i exp(-\boldsymbol{\beta}'\mathbf{x}_i) - 1, \quad e_i = e_i(\boldsymbol{\beta}_0), \qquad i = 1, 2, \ldots, n$$

The result would follow by an application of the Markov inequality once we show that

$E_{\boldsymbol{\beta}_0}(|\ T_{n,k}\ |) \longrightarrow 0$ uniformly in $\boldsymbol{\beta} \in M_n(\boldsymbol{\beta}_0)$. By Triangle inequality, we have

$$E_{\boldsymbol{\beta}_0}(|\ T_{n,k}\ |) \leq (logn)^2 n^{-1}(logn)^{-k}\sum_{i=1}^{n}(logi)^k \,|\, 1 - exp\{-(\boldsymbol{\beta} - \boldsymbol{\beta}_0)'\mathbf{x}_i\} \,|$$

For any real number x, we note .

$$\begin{aligned} |\, 1 - exp(-x)\,| &\leq x, \qquad for\ x > 0 \\ &\leq |\, x\,|\, exp(-x), \qquad for\ x < 0 \end{aligned}$$

For $\boldsymbol{\beta}\ in\ M_n(\boldsymbol{\beta}_0)$, we have

$$|\,(\boldsymbol{\beta} - \boldsymbol{\beta}_0)'\mathbf{x}_i\,| \leq 2hn^{-\gamma}(logn + logi)$$

Thus, we arrive at the inequality

$$E_{\boldsymbol{\beta}_0}(|\ T_{n,k}\ |) \leq 4hn^{-\gamma}(logn)^3 \exp(4hn^{-\gamma}logn)$$

Since $\gamma > 0$ and $(logn)^m n^{-\gamma} \longrightarrow 0$ for any fixed nonnegative integer m, we have the required uniform convergence for $E_{\boldsymbol{\beta}_0}(|\ T_{n,k}\ |)$. //

Proof of Theorem 3.1 : For $\boldsymbol{\beta} \in M_n(\boldsymbol{\beta}_0)$ we have $\boldsymbol{\beta} - \boldsymbol{\beta}_0 = n^{-\gamma}(\tau_1 logn,\ \tau_2)'$. Viewing $\boldsymbol{\beta}$ as a function of $\boldsymbol{\tau}$, a taylor expansion of $\boldsymbol{\psi}_n(\boldsymbol{\beta})$ around $\boldsymbol{\beta}_0$ yields

$$\lambda_{1n}(\boldsymbol{\tau}) \equiv l_{1n}(\boldsymbol{\beta}) = l_{1n} - n^{-\gamma}(a_{11}(\boldsymbol{\zeta})\tau_1 logn + a_{12}(\boldsymbol{\zeta})\tau_2)$$

$$\lambda_{2n}(\boldsymbol{\tau}) \equiv (logn)^{-1}l_{2n}(\boldsymbol{\beta}) = logn^{-1}l_{2n} - n^{-\gamma}(a_{21}(\boldsymbol{\zeta})\tau_1 + a_{22}(\boldsymbol{\zeta})(logn)^{-1}\tau_2)$$

where $(l_{1n},\ l_{2n})'$ and $a_{ij}(\boldsymbol{\zeta})$ are as defined in (3.3) and (3.2) respectively, with $\boldsymbol{\zeta}$ being a point on the line segment joining $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_0$. Denoting $\boldsymbol{\lambda}_n(\boldsymbol{\tau}) = (\lambda_{1n}(\boldsymbol{\tau}), \lambda_{2n}(\boldsymbol{\tau}))$, and referring to the definitions of $\mathbf{Z}_n$ and $\mathbf{C}_n(\boldsymbol{\beta})$ in (3.5) and (3.6), it follows that

$$\boldsymbol{\lambda}_n(\boldsymbol{\tau}) = n^{1/2}\mathbf{Z}_n - (logn)n^{1-\gamma}\mathbf{C}_n(\boldsymbol{\zeta})\boldsymbol{\tau} \qquad (5.1)$$

Define,

$$\mathbf{g}_n(\boldsymbol{\tau}) = (n^{1-\gamma}logn)^{-1}\mathbf{F}_n\boldsymbol{\lambda}_n(\boldsymbol{\tau}), \qquad \mathbf{g}_{n0} = (n^{1/2-\gamma}logn)^{-1}\mathbf{F}_n\mathbf{Z}_n$$

Premultiplying both sides of (5.1) by $\boldsymbol{\tau}'\mathbf{F}_n(n^{1/2-\gamma}logn)^{-1}$, we obtain the relation

$$\begin{aligned} \boldsymbol{\tau}'\mathbf{g}_n(\boldsymbol{\tau}) &= -\boldsymbol{\tau}'\boldsymbol{\tau} + \boldsymbol{\tau}'\mathbf{g}_{n0} + \boldsymbol{\tau}'[\mathbf{I}_2 - \mathbf{F}_n\mathbf{C}_n(\boldsymbol{\tau})]\boldsymbol{\tau} \\ &= -\boldsymbol{\tau}'\boldsymbol{\tau} + o_p(1) \end{aligned} \qquad (5.2)$$

where the last equality follows from Lemma 3.5, the fact that $\gamma < 1/2$, and $(logn)^{-1}\mathbf{F}_n\mathbf{Z}_n = O_p(1)$ by Lemma 3.4.

Result (5.2) implies that, given an $\epsilon > 0$, there exists $n_0 = n_0(\epsilon, h)$ such that

$$P\left(\sup_{\|\tau\|=h} \tau' \mathrm{g}_n(\tau) < 0\right) \geq 1 - \epsilon \qquad \forall\, n > n_0$$

According to a version of Brouwer's fixed point theorem (see Smith(1985)), we have that $\mathrm{g}_n(\hat{\tau}) = 0$ for some $\hat{\tau} < h$. Thus, for all $n > n_0$, the probability is atleast $1-\epsilon$ that a $\hat{\tau}_n$ exists that satisfies $\mathrm{g}_n(\hat{\tau}_n) = 0 \;\; and \;\; \|\hat{\tau}_n\| < h$. The corresponding $\hat{\beta}_n = \beta_0 + n^{-\gamma}(\hat{\tau}_{1n} logn, \hat{\tau}_{2n})$ meets the requirements of the theorem. //

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Table 1 : Comparative Study of MLE and WPAE

Sample Size (n)	$\mu_0 = 0.5,\ \delta_0 = 1.5$							
	Maximum Likelihood				Weibull Process Analog			
	Bias($\hat{\mu}$)	MSE($\hat{\mu}$)	Bias($\hat{\delta}$)	MSE($\hat{\delta}$)	Bias(μ^*)	MSE(μ^*)	Bias(δ^*)	MSE(δ^*)
10	0.500	8.718	0.147	0.242	−0.049	0.341	−0.233	0.162
25	0.107	0.218	−0.029	0.056	−0.171	0.067	−0.185	0.071
50	0.127	0.167	0.010	0.028	−0.084	0.082	−0.089	0.032
100	0.076	0.082	0.008	0.011	−0.075	0.039	−0.053	0.012
	$\mu_0 = 1.0,\ \delta_0 = 2.0$							
10	2.143	64.438	0.160	0.250	−0.476	0.505	−0.396	0.271
25	0.567	4.1006	0.006	0.073	−0.324	1.033	−0.234	0.115
50	0.328	1.031	0.011	0.034	−0.282	0.336	−0.135	0.049
100	0.175	0.345	0.013	0.013	−0.196	0.232	−0.070	0.018
	$\mu_0 = 2.5,\ \delta_0 = 2.5$							
10	1.970	38.785	0.067	0.180	−1.721	3.589	−0.583	0.453
25	0.83	6.391	0.005	0.059	−1.356	2.481	−0.306	0.147
50	0.789	5.698	0.019	0.030	−0.955	1.971	−0.172	0.062
100	0.406	2.181	0.005	0.012	−0.619	1.345	−0.091	0.023

Table 2 : Relative proportion of misclassification

Sample Size (n)	Proportion of times estimate of δ less than 1			
	$\mu_0 = 0.5,\ \delta_0 = 1.5$		$\mu_0 = 0.5,\ \delta_0 = 0.5$	
	Maximum Likelihood	WP Analog	Maximum Likelihood	WP Analog
10	0.08	0.22	0.76	0.88
20	0.03	0.05	0.93	0.94
25	0.04	0.04	0.96	0.98

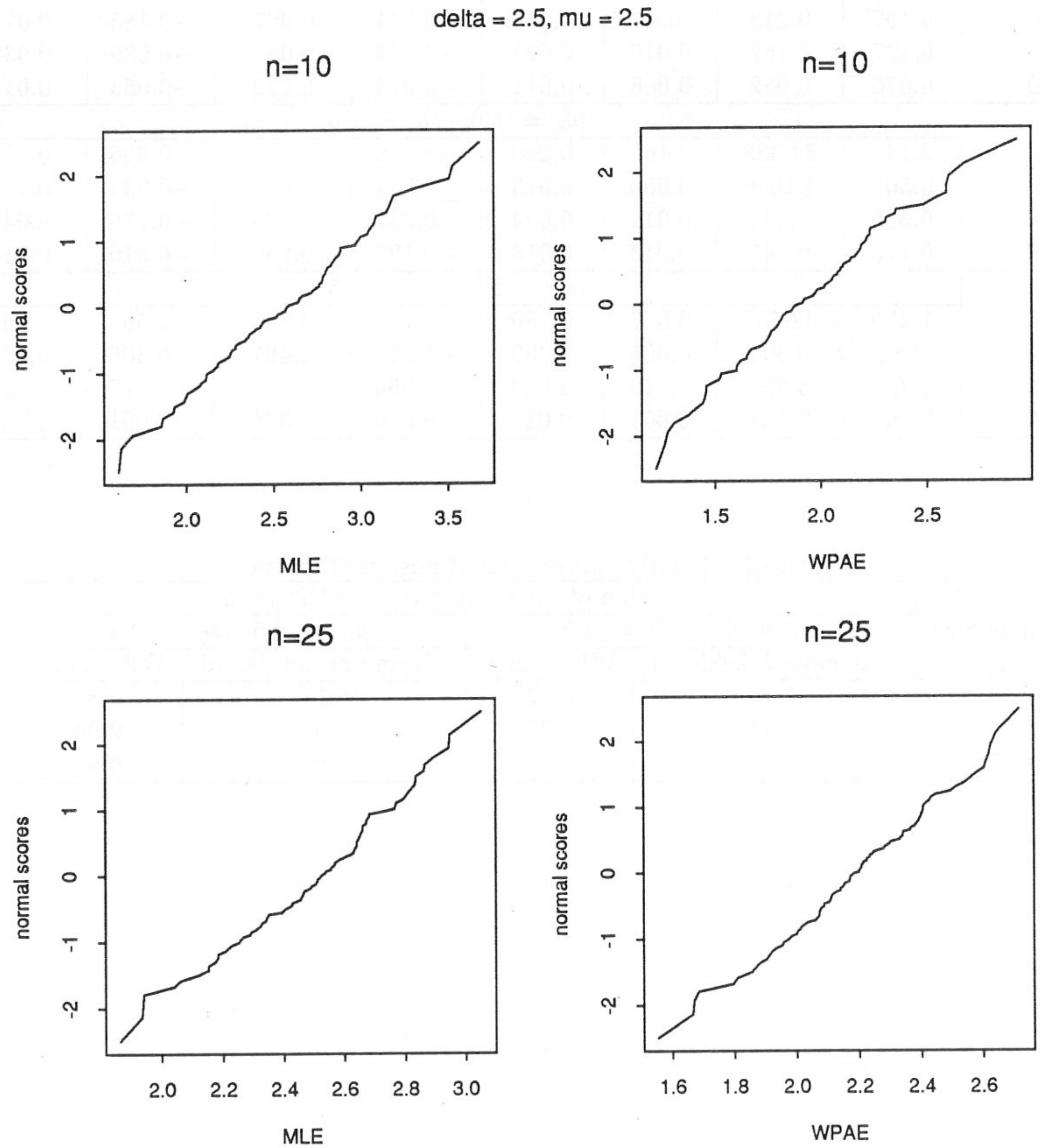

Figure 1: Normal Scores plots for $\hat{\delta}$ and δ^*

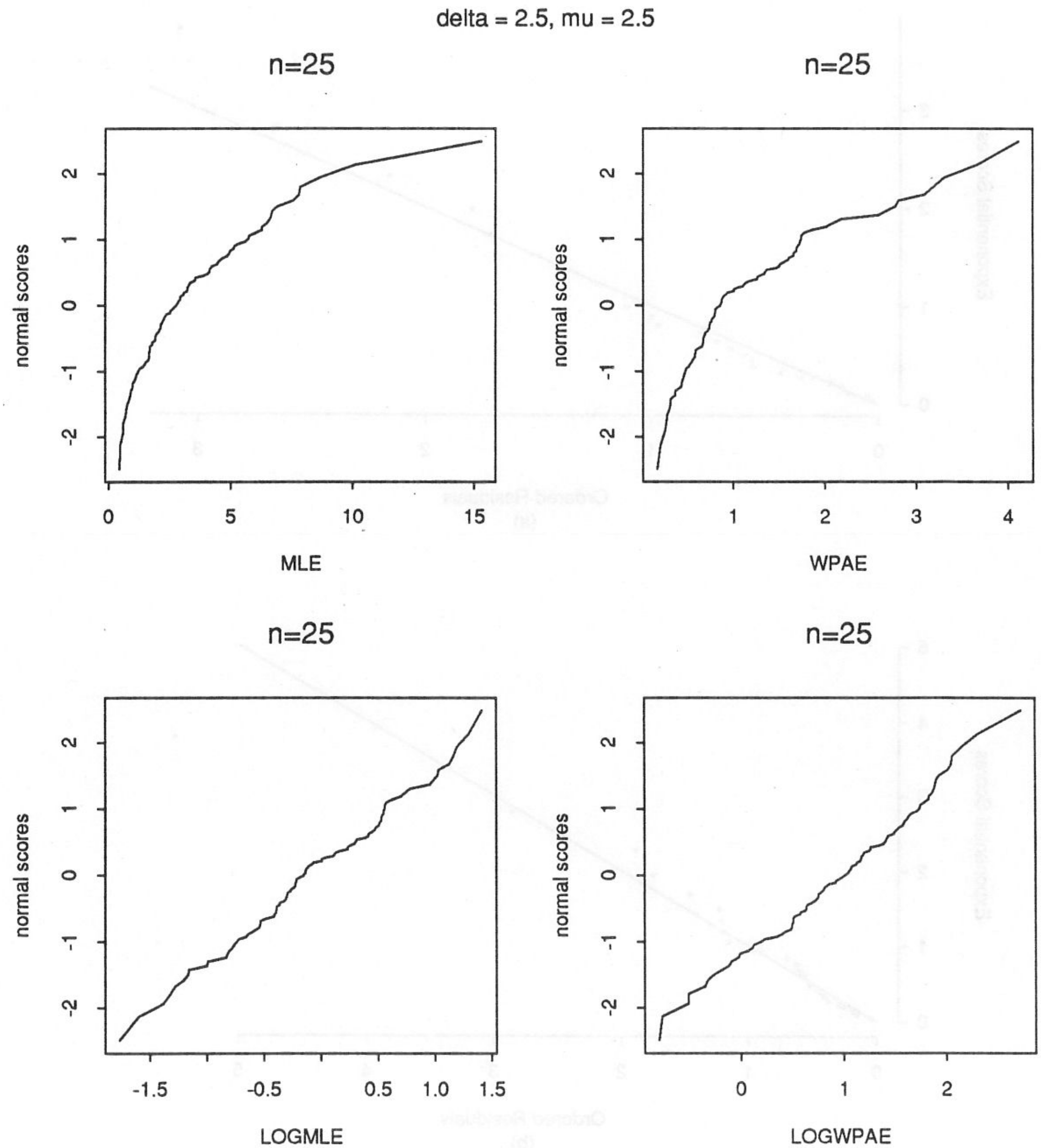

Figure 2: Normal Scores plots for the estimates of μ and their logarithms

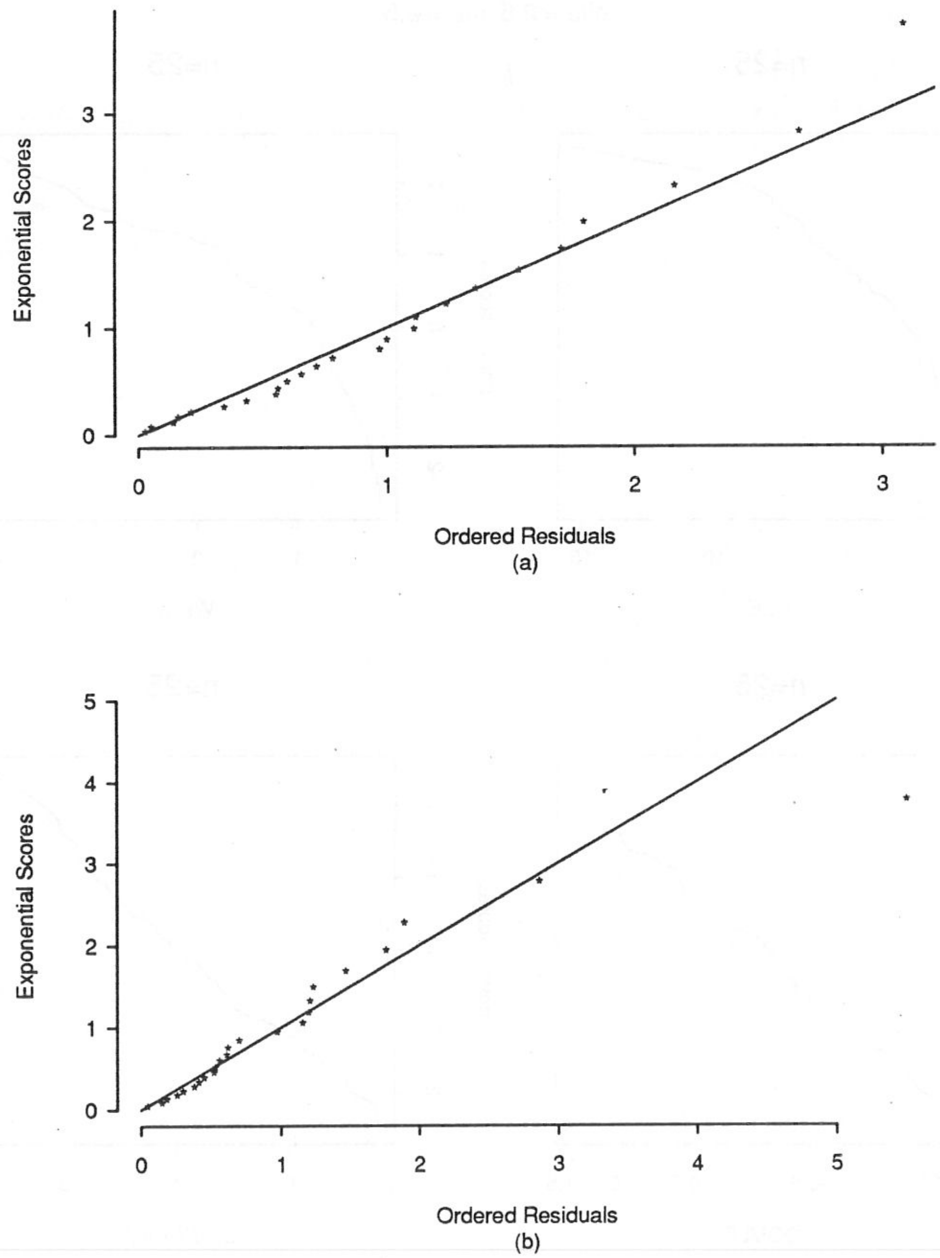

Figure 3: Residual checks for the tank data with (a) PEXP fit, (b) NHPP fit

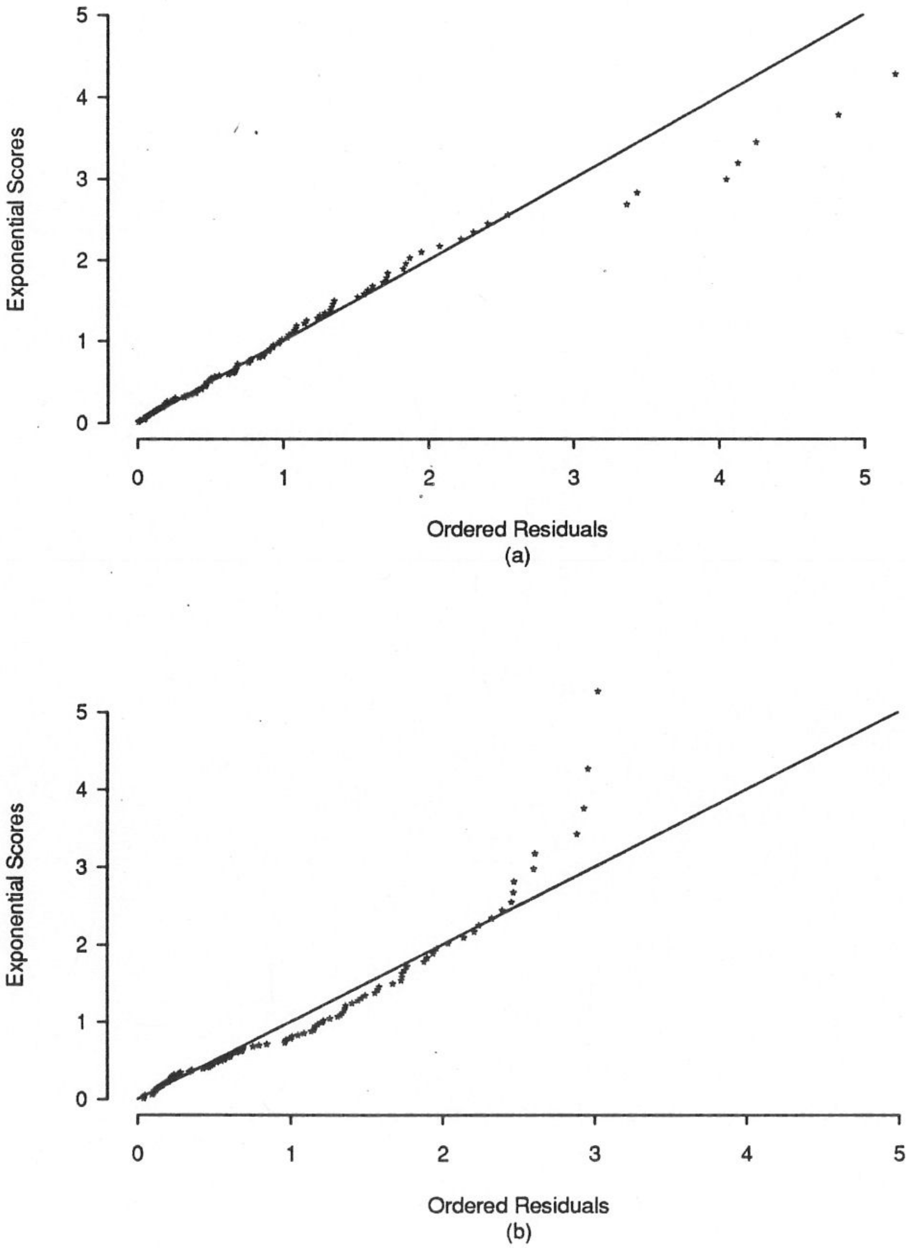

Figure 4: Residual checks for the mine data with (a) PEXP fit, (b) NHPP fit