Department of Statistics
University of Wisconsin

Madison, Wisconsin 53706

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USE OF SIMULTANEOUS AUTOREGRESSIVE MODELS IN BALANCED INCOMPLETE BLOCK AND LATTICE SQUARE DESIGNS

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N. R. Draper and D. Faraggi

University of Wisconsin

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0. SUMMARY

Simultaneous autoregressive models are applied to balanced incomplete block and lattice square designs. Relationships between the resulting maximum likelihood estimators and standard analyses are explored and clarified. Also, it is shown that the Papadakis estimator, as applied to these designs, has the characteristics of the standard analyses, but simply involves fewer plots in the adjustment procedure for treatment averages. A worked example illustrates the differences in treatment estimates in a lattice square design, for standard inter-block analysis and for simultaneous autoregressive assumptions.

KEYWORDS: Balanced incomplete block design, Lattice square design, Maximum likelihood estimators, Papadakis estimators, Simultaneous autoregressive models.

1. INTRODUCTION

The analysis of field trials when observations in nearby or neighbouring plots are correlated has received extensive study over a number of years. An early reference point is the work of Papadakis (1937). The relationship of his suggested estimator to the maximum likelihood estimators in certain one- and two-dimensional designs under certain simultaneous autoregressive models has been explored by Draper and Faraggi (1984).

It has been suggested by Kempton and Howes (1981, p. 65 and p. 69) that the Papadakis model may be used quite generally for field trials and specifically, for example, for lattice squares (LS's). On the other hand, a standard analysis for such designs under the "usual $N(0, I\sigma^2)$" type of error assumption is due to Yates (1936, 1939, 1940). Under Yates' analysis, adjustments are made to the treatment averages for the incompleteness of the blocks.

In this paper, we investigate lattice squares under the assumption that a simultaneous autoregressive model which assumes correlation within rows and columns is appropriate. This is done with different correlations, ρ_r for rows and ρ_c for columns. We then obtain maximum likelihood estimators for treatments and correlations and discuss the relationships and differences between the three possible treatments estimators namely maximum likelihood estimators, Yates', and Papadakis. It will be shown that, for these designs, the Yates estimator is very close to the first order term in an expansion, where valid, of the maximum likelihood estimator. Moreover, the Yates' and Papadakis corrections to the treatment means have similar forms. However, neither is actually needed in designs completely balanced for rows and columns because an explicit solution can be obtained. An example is presented for illustration in Section 5.

The lattice square is a two-dimensional design. As a preliminary, we investigate what can be regarded as a one-dimensional equivalent, namely the balanced incomplete block design. Results and conclusions which closely parallel those of the lattice square are obtained.

2. BALANCED INCOMPLETE BLOCK DESIGNS

Suppose we wish to examine t treatments in b blocks of size $k < t$. Suppose, further, that each treatment occurs the same number, m, of times and appears λ times with every other treatment. Then the design is said to be a balanced incomplete block design (BIBD). Note that there are $n = bk = tm$ plots, and that $\lambda(t-1) = m(k-1)$. For the standard details of analysis reproduced below see, for example, Cochran and Cox (1957), or Yates (1940). The usual model considered is

$$y_{js} = \mu + \alpha_s^* + \beta_j + \varepsilon_{js} \tag{2.1}$$

if "replicates" are not a factor; if they are, some minor differences occur in some formulas. Here y_{js} is the observation on treatment s in block j, μ is an overall mean, α_s^* is the effect of the sth treatment, β_j is the effect of the jth block and ε_{js} is an error such that ε = (column vector of ε_{js} in a defined order) $\sim N(0, I\sigma^2)$. Two alternative assumptions on the β_j are commonly made.

(a) β_j is a fixed effect. This leads to the so-called intra-block analysis.

(b) $\beta = (\beta_1, \ldots, \beta_b)' \sim N(0, I\sigma_\beta^2)$ independently of the ε_{js}. This leads to the analysis with the recovery of inter-block information. As $\sigma_\beta^2 \to \infty$, case (b) $\to$ case (a).

We set

B_s = total of all response observations y from blocks in which treatment s appears.

T_s = total of all observations on treatment s.

$G = \Sigma_s T_s$ = grand total $(= \Sigma_s B_s)$.

Then, from Cochran and Cox (1957, pp. 445-446) we can estimate the effect of the sth treatment by

$$\hat{\alpha}_s^* = m^{-1}\{T_s+\hat{\theta}[(t-k)T_s-(t-1)B_s+(k-1)G]\} \tag{2.2}$$

where $\hat{\theta}$ is an adjustment factor taking the value

$$\hat{\theta} = \{t(k-1)\}^{-1} \tag{2.3}$$

in case (a) when inter-block information is <u>not</u> recovered, and the value

$$\hat{\theta} = (b-1)(E_b-E_e)/\{t(k-1)(b-1)E_b+(t-k)(b-t)E_e\} \tag{2.4}$$

where it is recovered. The quantities E_b and E_e are the mean squares due to blocks adjusted for treatments, and residual, respectively. Define the t by $bk = n$ matrix T as follows. When the blocked design is listed in order, block by block, $T_{si} = 1$ if the sth treatment is associated with the ith observation, and $T_{si} = 0$ otherwise. Let

$$D_k = 11' - I_k \tag{2.5}$$

where $1' = (1,1,\ldots,1)$ of dimension 1 by k. Then define

$$D = I_b \otimes D_k \tag{2.6}$$

where $\otimes$ denotes the Kronecker product. Then, in matrix form, with $\alpha^* = (\alpha_1^*,\ldots,\alpha_s^*,\ldots,\alpha_t^*)'$, (2.2) becomes

$$\hat{\alpha}^* = m^{-1}\{TY-\hat{\theta}(t-1)TD[I-m^{-1}T'T]Y\} \tag{2.7}$$

where Y is the vector of observations, recorded in the same pattern as the block by block enumeration.

We now consider, instead of case (a) or case (b) above, an alternative model $y_i = \alpha_s + x_i$, where

$$x_i = \rho \sum_{j \neq i} n_{ij} x_j + \varepsilon_i \tag{2.8}$$

and where $n_{ij} = 1$ if y_i and y_j occur in the same block, and $n_{ij} = 0$ otherwise. We assume $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)' \sim N(0, I\sigma^2)$ independently of the x's. Following Besag's (1974, Eq. (4.13) with $\mu = 0$) simultaneous autoregressive model, we obtain the likelihood

$$(2\pi\sigma^2)^{-n/2}|B|\exp\{-(2\sigma^2)^{-1}X'B'BX\}, \tag{2.9}$$

where $B = I - \rho D$ and $X = Y - T'\alpha$. The maximum likelihood estimator $\hat{\alpha}$ is of form (Ripley, 1981, (5.34))

$$(TV^{-1}T')^{-1}TV^{-1}Y, \tag{2.10}$$

where $TV^{-1}T' = T(I-\hat{\rho}D)^2T' = m\{I-m^{-1}T(2\hat{\rho}D-\hat{\rho}^2D^2)T'\}$ using the fact that $TT' = mI$. The matrix $(TV^{-1}T')^{-1}$ can be expanded as a convergent series if and only if the eigenvalues of $Q_B = m^{-1}T(2\hat{\rho}D-\hat{\rho}^2D^2)T'$ are all less than one in absolute value. This is true if

$$(1-2^{1/2})/(k-1) < \hat{\rho} < \begin{cases} \min\{(1+2^{1/2}), t^{1/2}-1\}, & k = 2, \\ \min\{\frac{1}{2}(1+2^{1/2}), \frac{1}{4}(t-1)\}, & k = 3, \\ (1+2^{1/2})/(k-1), & k \geq 4. \end{cases} \tag{2.11}$$

If we retain only terms of orders zero and one in Q_B, we obtain

$$\hat{\alpha}_1 = m^{-1}[TY-\{2\hat{\rho}-\hat{\rho}^2(k-2)\}TD(I-m^{-1}T'T)Y] \tag{2.12}$$

where we have used the facts that $D^2 = (k-1)I_n + (k-2)D$, and $T(I-m^{-1}T'T) = 0$. Comparing (2.12) with (2.7) we notice the estimators are identical except that $\hat{\theta}(t-1)$ in (2.7) replaces $\{2\hat{\rho}-\hat{\rho}^2(k-2)\}$ in (2.12). Because of the validity of the expansion, $\hat{\alpha}_1$ will converge to the maximum likelihood estimator $\hat{\alpha}$, for fixed $\hat{\rho}$, if the former is applied iteratively in the following sense:

$$\hat{\alpha}_{j+1} = m^{-1}[TY-\{2\hat{\rho}-\hat{\rho}^2(k-2)\}TD(Y-T'\hat{\alpha}_j)] \tag{2.13}$$

where $\hat{\alpha}_o = m^{-1}TY$. In fact, iteration is not needed here because the balanced incomplete block design produces a $TV^{-1}T'$ of patterned form which can be explicitly inverted (Rao, 1973, p. 67) to give $\hat{\alpha}$ in terms of $\hat{\rho}$ as

$$\hat{\alpha} = a(I+bTDT')T(I-\hat{\phi}D)Y \tag{2.14}$$

where $a = \{m-(t-2)\hat{\phi}\}/[\{m-(t-1)\hat{\phi}\}\{m+\hat{\phi}\}]$, $b = \hat{\phi}/\{m-(t-2)\hat{\phi}\}$ and $\hat{\phi} = \{2\hat{\rho}-(k-2)\hat{\rho}^2\}/\{1+(k-1)\hat{\rho}^2\}$. The maximum likelihood $\hat{\rho}$ is obtained as follows. Setting the first derivative of the log likelihood with respect to ρ equal to zero provides

$$\frac{(Y-T'\hat{\alpha})'(D-\hat{\rho}D^2)(Y-T'\hat{\alpha})}{(Y-T'\hat{\alpha})(I-2\hat{\rho}D+\hat{\rho}^2D^2)(Y-T'\hat{\alpha})} = \frac{(k-1)\hat{\rho}}{\{1+(n-1)\hat{\rho}\}\{1+\hat{\rho}\}} \tag{2.15}$$

The $\hat{\alpha}$ may be substituted from (2.14) to provide an equation in $\hat{\rho}$ which can be solved, for example, via the Newton-Raphson method. From this, $\hat{\alpha}$ is then obtained. In practice, it is easier to replace α in the likelihood by $\hat{\alpha}$ from (2.14) and then to maximize the likelihood numerically by a search over $\hat{\rho}$.

For least squares estimation, the term on the right of (2.15) is replaced by zero, so that

$$\hat{\rho}_{LS} = (Y-T\hat{\alpha}_{LS})'D(Y-T\hat{\alpha}_{LS})/\{(Y-T\hat{\alpha}_{LS})'D^2(Y-T\hat{\alpha}_{LS})\}, \tag{2.16}$$

where $\hat{\alpha}_{LS}$ has the same form as $\hat{\alpha}$ but with $\hat{\rho}$ replaced by $\hat{\rho}_{LS}$. Again numerical maximization is simplest.

3. LATTICE SQUARES

Suppose we wish to examine t treatments in a set of m k by k lattice squares. Typically $m = \delta(k+1)$ where $\delta = \frac{1}{2}, 1, 2, \ldots$. The different values of δ correspond to various association relationships among the treatments. When $\delta = \frac{1}{2}$, any treatment s will appear with any other treatment w once in either a row or a column but not both. When $\delta = 1$, s and w appear together both in a row and a column, once each, and so on. The usual model considered is

$$y_i = \mu + \alpha_s^* + \pi_q + \beta_r + \gamma_c + \varepsilon_i, \tag{3.1}$$

where the set of subscripts (q,r,c) representing the position in the rth row and the cth column within the qth replicate have been replaced by a single subscript $i = 1,2,\ldots,n$ which denotes the plot's position in a row by row enumeration of the plots, replicate by replicate. Two alternative assumptions on the β's and the γ's are commonly made

(a) β_r, γ_c are fixed effects. This leads to an intra-block analysis.

(b) $\beta_r \sim N(0,\sigma_\beta^2)$ and $\gamma_c \sim N(0,\sigma_\gamma^2)$ and all β's and γ's are independent.

Typically it is assumed that $\sigma_\gamma^2 = \sigma_\beta^2$, although this is not necessary. As $\sigma_\beta^2 \to \infty$ and $\sigma_\gamma^2 \to \infty$, case (b) $\to$ case (a).

We set, for $\delta \geq 1$,

T_s = total of all response observations y from all plots i which receive treatment s,

R_s = total of all y's in rows in which treatment s appears,

C_s = total of all y's in columns in which treatment s appears,

G = grand total of all y's,

$L_s = (m-1)T_s - mR_s + G$,

$M_s = (m-1)T_s - mC_s + G$.

Then, from Cochran and Cox (1957, p. 493), we can estimate the effect of the sth treatment by

$$\hat{\alpha}_s^* = m^{-1}\{T_s+\lambda_r L_s+\lambda_c M_s\}, \tag{3.2}$$

where λ_r and λ_c are coefficients for the adjustments needed for the incompleteness of the rows and columns. The values of λ_r and λ_c depend on the replication selected and which case, (a) or (b) is assumed. Examples are in Appendix A.

For $\delta = \frac{1}{2}$, the formula for $\hat{\alpha}_s^*$ is of the same form as (3.2) but with redefined symbols. We do not discuss this because our matrix formulation does not contain this ambiguity.

Define the t by n matrix T as follows. When the blocked design is written out row by row within replicate and then replicate by replicate, $T_{si} = 1$ if the sth treatment is associated with the ith observation, and $T_{si} = 0$ otherwise. Let $D_k = 11' - I_k$, where $1' = (1,1,\ldots,1)$ of dimension

1 by k, and define

$$Z_r = I_m \otimes I_k \otimes D_k,$$
$$Z_c = I_m \otimes D_k \otimes I_k, \qquad (3.3)$$
$$Z = Z_r + Z_c,$$

where $\otimes$ denotes the Kronecker product. Then, in matrix form, with $\alpha^L = (\alpha_1^L, \alpha_2^L, \ldots, \alpha_t^L)'$, the L denoting lattice, (3.2) becomes

$$\hat{\alpha}^L = m^{-1}\{TY - mT(\lambda_r Z_r + \lambda_c Z_c)(I - m^{-1}T'T)Y\} \qquad (3.4)$$

where Y is the vector of observations, recorded in the same pattern as described earlier.

We now consider, instead of case (a) or case (b), an alternative model $y_i = \alpha_s + x_i$, $i = 1,\ldots,n$, $s = 1,\ldots,t$, where

$$x_i = \rho_r \sum_{j \neq i} n_{ij}^{(r)} x_j + \rho_c \sum_{j \neq i} n_{ij}^{(c)} x_j \qquad (3.5)$$

and where $n_{ij}^{(r)} = 1$ if y_i and y_j occur in the same row and replicate and is zero otherwise, while $n_{ij}^{(c)} = 1$ if y_i and y_j occur in the same column and replicate and is zero otherwise. We assume $\varepsilon \sim (\varepsilon_1, \ldots, \varepsilon_n)' \sim N(0, I\sigma^2)$ independently of the x's. Following Besag's (1974, Eq. (4.13) with $\mu=0$) simultaneous autogressive model, we obtain the likelihood $(2\pi\sigma^2)^{-n/2}|B|\exp\{-(2\sigma^2)^{-1}X'B'BX\}$ where $B = I - \rho_r Z_r - \rho_c Z_c$,

$X = Y - T'\alpha$. The maximum likelihood estimator $\hat{\alpha}$ is of form $(TV^{-1}T')^{-1}TV^{-1}Y$, where $TV^{-1}T' = T(I-\hat{\rho}_r Z_r - \hat{\rho}_c Z_c)^2 T' = m(I-Q_L)$ and where

$$Q_L = m^{-1}T(2\hat{\rho}_r Z_r + 2\hat{\rho}_c Z_c - \hat{\rho}_r^2 Z_r^2 - \hat{\rho}_c^2 Z_c^2 - 2\hat{\rho}_r \hat{\rho}_c Z_r Z_c)T'. \tag{3.6}$$

The matrix $(TV^{-1}T')^{-1}$ can be expanded as a convergent series if and only if the eigenvalues of Q_L are all less then one in absolute value. This is true if

$$-1 < q_1 + (k^2-1)q_2 < 1,$$
$$-1 < q_1 - q_2 < 1,$$

where

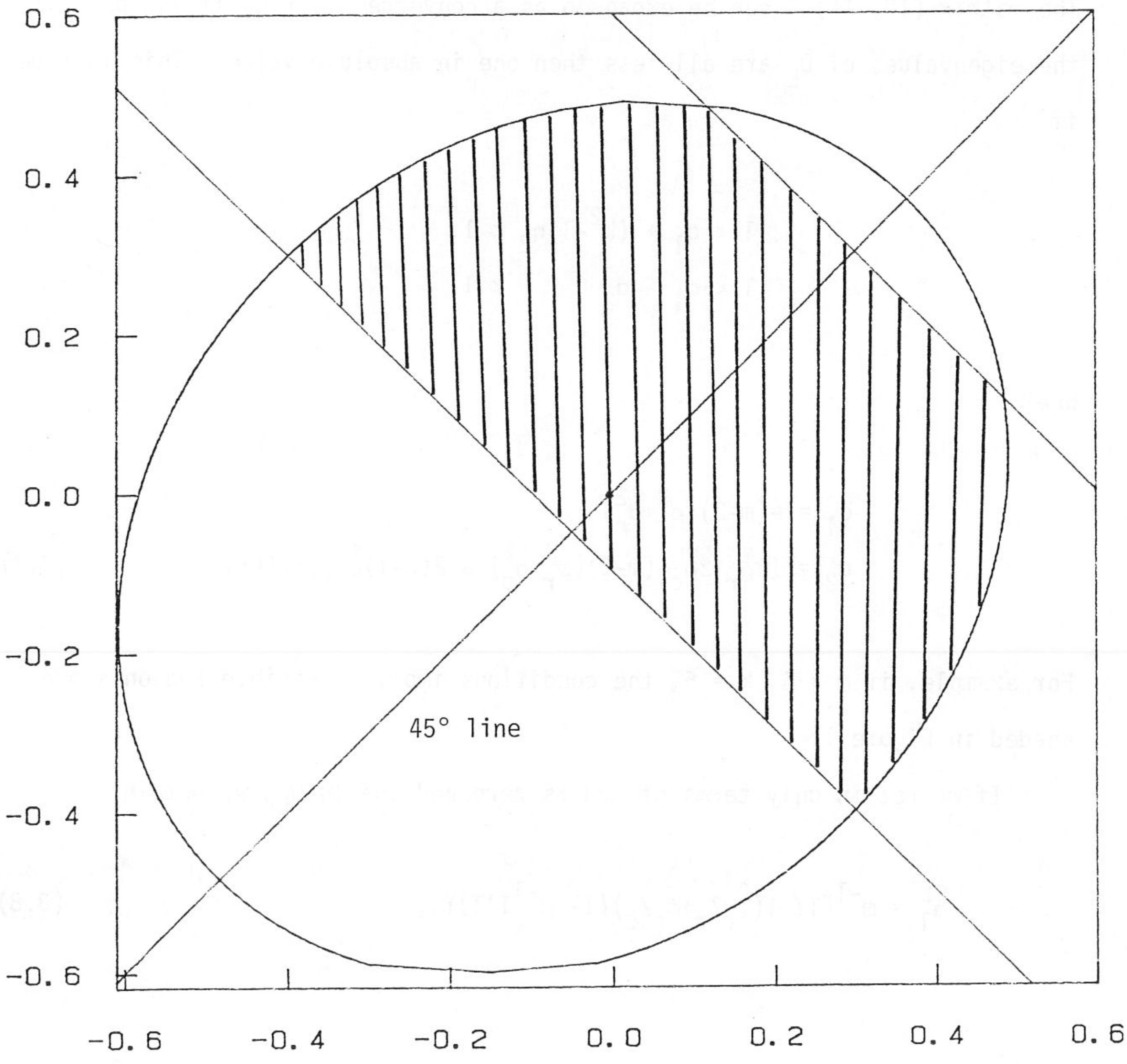
$$q_1 = -(m-2)(\hat{\rho}_r^2 + \hat{\rho}_c^2),$$
$$q_2 = \{2\hat{\rho}_r + 2\hat{\rho}_c - (m-3)(\hat{\rho}_r^2 + \hat{\rho}_c^2) - 2(k-1)\hat{\rho}_r \hat{\rho}_c\}/(k+1). \tag{3.7}$$

For example, if $m = 6$, $k = 5$, the conditions imply a feasible region shown shaded in Figure 1.

If we retain only terms of orders zero and one in Q_L, we obtain

$$\hat{\alpha}_1^L = m^{-1}[TY - T(\hat{\phi}_r Z_r + \hat{\phi}_c Z_c)(I - m^{-1}T'T)Y], \tag{3.8}$$

Figure 1. The shaded region shows the values of (ρ_r, ρ_c) for which the expansion of the maximum likelihood estimator involving Q_L is valid, for the case m = 6, k = 5 described in the text. For the case $\rho_r = \rho_c = \rho$ this region reduces to the intersection of the shaded region and the 45° line shown, with a multiplicative scaling factor of $2^{-1/2}$ applied.



where $\hat{\phi}_u = 2\hat{\rho}_u - (m-3)\hat{\rho}_u^2 + 2\hat{\rho}_r\hat{\rho}_c$ and u = r, c in turn. In deriving $\hat{\alpha}_1^L$ we have used the following facts (u=r or c):

$$\begin{aligned} Z_u^2 &= (m-2)I + (m-3)Z_u, \\ TZ_rZ_c &= 1_t1_t'T-T-TZ_r-TZ_c \\ T(I-m^{-1}T'T) &= 0, \\ 1_n'(I-m^{-1}T'T) &= 0. \end{aligned} \tag{3.9}$$

Comparing (3.8) with (3.4), we see that the estimators are identical except that $m\lambda_u$ in (3.4) replaces $\hat{\phi}_u$ in (3.8). Because of the validity of the expansion, $\hat{\alpha}_1^L$ will converge to the maximum likelihood estimator $\hat{\alpha}^L$, for fixed $\hat{\rho}_r$ and $\hat{\rho}_c$, if $\hat{\alpha}_1^L$ is applied iteratively in the following sense

$$\hat{\alpha}_{j+1}^L = m^{-1}\{TY-T(\hat{\phi}_rZ_r+\hat{\phi}_cZ_c)(Y-T'\hat{\alpha}_j^L)\}, \tag{3.10}$$

where $\hat{\alpha}_0^L = m^{-1}TY$. In fact, iteration is not needed here for $\delta \geq 1$, because the lattice design produces a $TV^{-1}T'$ of patterned form which can be explicitly inverted (Rao, 1973, p. 67) to give $\hat{\alpha}^L$ in terms of $\hat{\rho}_r$ and $\hat{\rho}_c$ as

$$\{(a-b)I+b11'\}T(I-Q)Y, \tag{3.11}$$

where

$$a = \{(m-1)q_1-(k^2-2)mq_2\}/[\{(m-1)q_1-(k^2-1)mq_2\}\{(m-1)q_1+mq_2\}],$$

$$b = mq_2/[\{(m-1)q_1-(k^2-1)mq_2\}\{(m-1)q_1+mq_2\}]. \qquad (3.12)$$

The maximum likelihood $\hat{\rho}_r$ and $\hat{\rho}_c$ are obtained as follow. Setting the first derivatives of the log likelihoods with respect to ρ_1 and ρ_2 equal to zero provides equations:

$$\frac{n\hat{X}'Z_u\hat{B}\hat{X}}{\hat{X}'\hat{B}^2\hat{X}} + \frac{\left\{\frac{\partial}{\partial\rho_u}|B|\right\}_{\rho_r=\hat{\rho}_r,\rho_c=\hat{\rho}_c}}{|\hat{B}|} = 0, \qquad u = r,c \qquad (3.13)$$

where $\hat{X}$ and $\hat{B}$ are the values of X and B with $\hat{\alpha}^L$, $\hat{\rho}_r$ and $\hat{\rho}_c$ replacing α, ρ_r and ρ_c. These two simultaneous equations involve, in their second terms, the (i,i+1) and the (i,i+k) elements, the designations being reduced modulo n where they exceed n, respectively of $\hat{B}^{-1}$ and must be solved, a tedious calculation. It is simpler in practice to substitute a selected grid of $(\hat{\rho}_r,\hat{\rho}_c)$ values into $\hat{\alpha}^L$ and hence evaluate the likelihood over that grid, so picking out the maximum value of the likelihood.

For least squares estimation, the term on the right of (3.13) is replaced by zero so that

$$\begin{bmatrix} \hat{\rho}_{r,LS} \\ \hat{\rho}_{c,LS} \end{bmatrix} = \begin{bmatrix} b_{rr} & b_{rc} \\ b_{cr} & b_{cc} \end{bmatrix}^{-1} \begin{bmatrix} X'Z_rX \\ X'Z_cX \end{bmatrix} \tag{3.14}$$

where $b_{uv} = X'Z_uZ_vX$. The least squares estimate of α, $\hat{\alpha}^L_{LS}$ is of the same form as $\hat{\alpha}^L$ but has ρ_u replaced by $\hat{\rho}_{u,LS}$ everywhere.

Special Case $\rho_r = \rho_c = \rho$.

In many practical situations, it is reasonable to assume that $\rho_r = \rho_c = \rho$. For comments on this choice, see Kempton and Howes (1981, p. 63). All formulas reduce to their appropriate forms via simple substitution of $\rho_r = \rho_c = \rho$. All that is needed is to write $Z = Z_r + Z_c$ whenever it occurs and to drop subscripts from λ, ϕ, and ρ. The equation parallel to Eqs. (3.13) is the result of adding these two equations together.

4. LATTICE SQUARE DESIGN EXAMPLE

We shall re-examine data given by Cochran and Cox (1957, pp. 490-493) comprising a lattice square design with $k = 4$, $\delta = 1$, $m = 5$, $n = 80$. The appropriate formulas for Yates' estimators λ_r and λ_c are given in Appendix A and here result in the numerical values $\lambda_r = 0.04787$ and $\lambda_c = 0.03037$ (p. 493). The maximum likelihood estimates for ρ_r and ρ_c are $\hat{\rho}_r = 0.108$ and $\hat{\rho}_c = -0.030$ and these lead to estimates for the 16 treatment parameters given in the second column of Table 1. These may be compared with the "adjusted means" of Cochran and Cox (1957, p. 491) which are the corresponding Yates' estimates, shown as the third column of Table 1. We see that, although two entirely different models are used, individual estimates are broadly speaking, compatible. In view of the fact that only one at most of these models can be correct for the data set, we do not compare them via a model-dependent criterion such as weighted or unweighted sum of squares of residuals.

Table 1. Comparison of treatment effects for a 4 by 4 lattice square design with five replications. Second column: Maximum likelihood estimates under the assumption of a simultaneous autoregressive model. Third column: Yates' estimates from Cochran and Cox (1957, p. 491).

Treatment	Mle	Yates'
1	4.98	6.45
2	12.71	13.68
3	8.90	8.73
4	11.41	11.36
5	9.89	9.44
6	5.96	7.58
7	7.52	7.37
8	9.79	9.32
9	10.91	10.01
10	15.77	14.91
11	18.12	17.59
12	12.91	12.70
13	12.07	10.69
14	13.40	14.27
15	10.07	9.28
16	10.05	11.09
Sum	174.46*	174.47*

*These figures are subject to small rounding errors; both should be 174.48.

5. PAPADAKIS AND MAXIMUM LIKELIHOOD

We reconsider the assumption made in connection with Eq. (3.1). Suppose that now β_r is the combined neighbouring effect from the "plot to the left in a row", and the "plot to the right in a row", these left and right adjacencies being determined in torus, wrap-around manner (Martin, 1982). A similar "up and down in a column" definition is applied now to γ_c. Let N_r and N_c be the n by n neighbour-specification matrices for rows and columns whose ith row contains 1 in positions j for which plot j is row- and column-adjacent respectively, to plot i, and zero otherwise. Then the maximum likelihood estimator of α^* is

$$\hat{\alpha}_p^* = m^{-1}\{TY-mT(\psi_r N_r+\psi_c N_c)(I-m^{-1}T'T)Y\} \tag{5.1}$$

where ψ_r and ψ_c are coefficients for the adjustments needed for the row- and column-adjacency effects. Parallel to the work in Section 3, there will be two forms for ψ_r and ψ_c according to whether an intra-neighbour analysis (β_r,γ_c fixed effects) or an inter-neighbour analysis (β_r,γ_c random effects) is specified.

If we compare (5.1) with Eq. (3) of Draper and Faraggi (1984), we see that they would be equivalent if $m\psi_r = m\psi_c = \hat{\phi}$ and if $N = N_r+N_c$. In fact, (5.1) is the Papadakis estimator for the case $\psi_r \neq \psi_c$, up to definition of ψ_r and ψ_c. The Papadakis formulation would involve regression estimators of ψ_r and ψ_c instead. We thus see that the Papadakis estimator can be regarded, up to definition of ψ_r and ψ_c, as a maximum likelihood estimator in the particular model formulation described above.

6. REMARKS

A previous paper (Draper and Faraggi, 1984) discussed field trials in one and two dimensions for designs of Type II(a), Type III, and Type III cyclically row permuted (Williams, 1952) for one and two dimensional layouts. In the two dimensional layout, no directional differences were assumed and a single ρ value was used for both rows and columns. The extension to ρ_r and ρ_c for that case is easily effected using formulas of the type exhibited in this paper. The maximum likelihood estimator has the form of (2.10), with

$$V^{-1} = (I-\rho_r N_r-\rho_c N_c)^2 \tag{6.1}$$

where N_r is an n by n row-neighbour-specification matrix defined by

$$N_r = I_{\delta(k+1)} \otimes I_k \otimes N, \tag{6.2}$$

where N_c is an n by n column-neighbour-specification matrix defined by

$$N_c = I_{\delta(k+1)} \otimes N \otimes I_k, \tag{6.3}$$

and where N is the cyclic k by k matrix

$$N = \begin{bmatrix} 0\,1\,0\,0 \ldots 0\,1 \\ 1\,0\,1\,0 \ldots 0\,0 \\ \ldots. \\ 1\,0\,0\,0 \quad\;\; 1\,0 \end{bmatrix}. \tag{6.4}$$

We can re-write (2.10) as

$$m^{-1}(I-m^{-1}TQT')^{-1}T(I-Q)Y \tag{6.5}$$

where

$$Q = 2\hat{\rho}_r N_r + 2\hat{\rho}_c N_c - \hat{\rho}_r^2 N_r^2 - \hat{\rho}_c^2 N_c^2 - 2\hat{\rho}_r\hat{\rho}_c N_r N_c . \tag{6.6}$$

Here,

$$N_r^2 = 2I + N_r^{(2)},\ N_c^2 = 2I + N_c^{(2)},\ N_r N_c = N_{rc}^{(11)},$$

where $N_r^{(2)}$, $N_c^{(2)}$, and $N_{rc}^{(11)}$ denote n by n neighbour-specification matrices which contain 1's and 0's such that

$$N_r^{(2)} = I_{\delta(k+1)} \otimes I_k \otimes N^{(2)} \tag{6.7}$$

$$N_c^{(2)} = I_{\delta(k+1)} \otimes N^{(2)} \otimes I_k \tag{6.8}$$

$$N_{rc}^{(11)} = I_{\delta(k+1)} \otimes N \otimes N \tag{6.9}$$

where $N^{(2)}$ is the cyclic $k \times k$ matrix

$$N^{(2)} = \begin{bmatrix} 0\,0\,1\,0 & \ldots\,0\,1\,0 \\ 0\,0\,0\,1 & \ldots\,0\,0\,1 \\ 1\,0\,0\,0\,1 & \ldots\,0\,0\,0 \\ \ldots. & \\ 0\,1\,0\,0 & \ldots\,1\,0\,0 \end{bmatrix} \qquad (6.10)$$

and N is defined in (6.4). For example, the ith row of $N_r^{(2)}$ has 1's in those columns j for which cell j is located next but one to cell i in a row of the design, and has zeros otherwise. Table 2 illustrates the general situation and has those (row) cells marked with symbol $N_r^{(2)}$. Similar remarks apply for $N_c^{(2)}$ and $N_{rc}^{(11)}$ for columns and diagonals respectively, as shown in Table 2. When cell i is near or on an edge, the appropriate torus-generated cells must be used.

		$N_c^{(2)}$		
	$N_{rc}^{(11)}$	N_c	$N_{rc}^{(11)}$	
$N_r^{(2)}$	N_r	Cell i	N_r	$N_r^{(2)}$
	$N_{rc}^{(11)}$	N_c	$N_{rc}^{(11)}$	
		$N_c^{(2)}$		

Table 2. Cell patterns which lead to the n by n neighbour-specification matrices N_r, N_c, $N_r^{(2)}$, $N_c^{(2)}$, $N_{rc}^{(11)}$.

The expansion of (6.5), when valid, as far as terms of zero and first order in Q, can be written

$$m^{-1}\{TY-TQ(I-m^{-1}T'T)Y\}$$

$$= m^{-1}\{TY-T(2\hat{\rho}_r N_r+2\hat{\rho}_c N_c-\hat{\rho}_r^2 N_r^{(2)}-\hat{\rho}_c^2 N_c^{(2)}-2\hat{\rho}_r\hat{\rho}_c N_{rc}^{(11)})(I-m^{-1}T'T)Y\}, \quad (6.11)$$

where we have used the fact that $T(I-m^{-1}T'T) = 0$.

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Appendix A. Values of λ_r, λ_c for $\delta = \frac{1}{2}$, 1, 2 for Lattice Square Designs.

δ	With recovery of inter-block information		Without recovery of inter-block information
	λ_r	λ_c	$\lambda_r = \lambda_c$
$\frac{1}{2}$	$\frac{2(E_r-E_e)}{k(k+1)E_r}$	$\frac{2(E_c-E_e)}{k(k+1)E_c}$	$\frac{2}{k(k+1)}$
1	$\frac{(E_r-E_e)(kE_c-E_e)}{(k-1)(k^2E_rE_c-E_e^2)}$	$\frac{(E_c-E_e)(kE_r-E_e)}{(k-1)(k^2E_rE_c-E_e^2)}$	$\frac{1}{(k-1)k}$
2	$\frac{W_e-W_r}{k\{W_r+W_c+(k-1)W_e\}}$	$\frac{W_e-W_c}{k\{W_r+W_c+(k-1)W_e\}}$	$\frac{1}{(k-1)k}$

In the above table,

$W_e = E_e^{-1}$, $W_r = (2k-1)/(2kE_r-E_e)$, $W_c = (2k-1)/(2kE_c-E_e)$,

E_r = mean square due to rows adjusted for treatments,

E_c = mean square due to columns adjusted for treatments and rows,

E_e = residual mean square.