
DEPARTMENT OF STATISTICS

University of Wisconsin
Madison, Wisconsin
53706

TECHNICAL REPORT NO. 623

Ocotober 1980

DISTRIBUTION OF THE LIKELIHOOD RATIO STATISTIC FOR TESTING SPHERICITY STRUCTURE FOR A COMPLEX NORMAL COVARIANCE MATRIX

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Abstract

The null distribution of the likelihod ratio test criterion for testing sphericity structure in a complex normal multivariate Hermitian covariance matrix $\underset{\sim}{\Sigma}$ (i.e., for testing the hypothesis that all off diagonal elements of $\underset{\sim}{\Sigma}$ are zero while the diagonal elements are equal in sets) has been obtained in series form. A short table of percentage points is also given.

Some key words: Exact null complex multivariate distribution;

Likelihood ratio criterion; Test of sphericity structure,

Tables of percentage points.

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1. <u>Introduction</u>. Let the joint density function of complex random matrices $\underset{\sim}{X}$: $p \times N$ and $\underset{\sim}{S}$: $p \times p$ be (see Goodman (1963))

$$(1.1) \qquad CN(\underset{\sim}{X};\underset{\sim}{\mu},\underset{\sim}{\Sigma})\cdot CW(\underset{\sim}{S};p,n,\underset{\sim}{\Sigma})$$

where $CN(\underset{\sim}{X};\underset{\sim}{\mu},\underset{\sim}{\Sigma})$ is the complex multivariate normal distribution defined by

$$(1.2) \qquad CN(\underset{\sim}{X};\underset{\sim}{\mu},\underset{\sim}{\Sigma}) = (\pi)^{-pN}|\underset{\sim}{\Sigma}|^{-N}\exp[-\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{X}-\underset{\sim}{\mu})\overline{(\underset{\sim}{X}-\underset{\sim}{\mu})'}],$$

$CW(\underset{\sim}{S};p,n,\underset{\sim}{\Sigma})$ is the complex Wishart distribution defined by

$$(1.3) \qquad CW(\underset{\sim}{S};p,n,\underset{\sim}{\Sigma}) = \{\Gamma_p(n)\}^{-1}|\underset{\sim}{\Sigma}|^{-n}|\underset{\sim}{S}|^{n-p}\exp(-\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1}\underset{\sim}{S}),$$

where

$$(1.4) \qquad n = N-1 \text{ and } \tilde{\Gamma}_p(n) = (\pi)^{p(p-1)/2}\{\prod_{j=1}^{p}\Gamma(n-i+1)\}.$$

$\underset{\sim}{\Sigma}$ and $\underset{\sim}{S}$ are Hermitian positive definite matrices of order P and $\underset{\sim}{\mu}$: $p \times N$ is a complex matrix. In this paper we wish to consider the testing of a sphericity [redacted] structure for the Hermitian covariance matrix Σ, i.e. we wish to test the hypothesis that all off diagonal elements of $\underset{\sim}{\Sigma}$ are zero and the diagonal elements are equal in sets. The sphericity test and the test of hypothesis of intraclass correlation model are special cases of such a hypothesis. Since the problems of testing the hypotheses on complex multivariate populations play an important role in drawing inference on the multiple stationary Gaussian time series (see Goodman (1963), Hannan (1970), Ligget (1972), Priestley et al (1973), Wahba (1971)), suitable methods are developed to obtain the distribution of the likelihood ratio criterion in computational forms from which exact percentage points of the criterion may be computed for any sample size.

2. The likelihood ratio criterion and its distribution. Consider the null hypothesis

$$(2.1)\qquad H_0: \underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_1^2 \underset{\sim}{I}_{p_1} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \sigma_2^2 \underset{\sim}{I}_{p_2} & \cdots & \underset{\sim}{0} \\ \vdots & & & \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \sigma_q^2 \underset{\sim}{I}_{p_q} \end{bmatrix}, \quad \sigma_i^2 > 0 \text{ unknown},$$

against the general alternatives, where $\underset{\sim}{I}_{p_i}$ is the identity matrix of order p_i and $p_1 + p_2 + \ldots + p_q = p$. Let

$$\underset{\sim}{S}: \begin{bmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} & \cdots & \underset{\sim}{S}_{1q} \\ \underset{\sim}{S}_{21} & \underset{\sim}{S}_{22} & \cdots & \underset{\sim}{S}_{2q} \\ \vdots & & & \\ \underset{\sim}{S}_{q1} & \cdots & \cdots & \underset{\sim}{S}_{qq} \end{bmatrix}$$

be partitioned similar to (2.1). Then it can be easily seen that the likelihood ratio criterion for testing H_0 against the general alternatives $A_0 \neq H_0$ is based on the statistic

$$(2.2)\qquad \lambda = |\underset{\sim}{S}|^{N/2} / \{ \prod_{i=1}^{q} [(\mathrm{tr}\, \underset{\sim}{S}_{ii}/p_i)^{p_i}]^{N/2} \}.$$

Using eq. (4.2) of Pillai and Nagarsenker (1971) for the null moments of the sphericity criterion in the normal complex case, the moments of $W = \lambda^{2/N}$ is given by

$$(2.3)\qquad E(W^h) = \prod_{i=1}^{q} \left\{ p_i^{p_i h} \frac{\Gamma(np_i)}{\Gamma[(n+h)p_i]} \right\} \prod_{\alpha=1}^{p} \frac{\Gamma\{(n+1-\alpha)+h\}}{\Gamma(n+1-\alpha)},$$

where $n = N-1$.

In part (a) of this section we obtain the distribution of W as a Beta series while in part (b) we obtain its distribution as a chisquare series.

2(a) Distribution of W as a Beta series. From (2.3) on taking inverse Mellin transform, the density of W is given by

$$(2.4)\qquad p(\omega) = \frac{\prod_{i=1}^{q} \Gamma(np_i)}{\prod_{\alpha=1}^{p} \Gamma\{\frac{1}{2}(N-\alpha)\}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \omega^{-h-1} \prod_{i=1}^{q} [p_i^{hp_i}/\Gamma(np_i+hp_i)] \prod_{\alpha=1}^{p} \Gamma\{(N-\alpha)+h\}dh.$$

Putting $(N-\lambda)+h = s$ in (2.4), where λ is an adjustable constant which can be chosen to govern the rate of convergence, we have

$$(2.5)\qquad p(\omega) = K\cdot\omega^{(N-\lambda)-1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} W^{-s} [\prod_{i=1}^{q} p_i^{sp_i}/\Gamma\{p_i(s+\lambda-1)\}] \prod_{\alpha=1}^{p} \Gamma(s+\lambda-\alpha)ds$$

where

$$K = \{\prod_{i=1}^{q} \Gamma(np_i)/\prod_{i=1}^{p} \Gamma[(N-\alpha)]\}(\prod_{i=1}^{q} p_i^{-(N-\lambda)p_i}) \tag{2.6}$$

and $c = (N-\lambda)$.

Proceeding as in Nagarsinker and Pillai (1972), the distribution of W is obtained in the form

$$P[W \leq x] = K\cdot \sum_{i=1}^{\infty} (R_i)\, I_x(N-\lambda, v_i)/\ (v+i) \tag{2.7}$$

where $v = (p^2-q)/2$, $I_x(p,q) = \int_0^x x^{p-1}(1-x)^{q-1}\,dx$ and the coefficients R_i are computed as in Nagarsenker and Pillai (1972).

2(b) Distribution of W as a chisquare series. Inverting the characteristic function of $L = -(2N-U)\log W$ where U is an adjustable constant which can be chosen arbitrarily to govern the rate of convergence of the resulting chi-square series, the distribution function of L is obtained as

$$F(\ell) = p(L \geq \ell) = K_1 \sum_{v=0}^{\infty} B_j(2/2N-U)^{j+v}\, G_{2j+2v}(\chi^2) \tag{2.8}$$

where

$$K_1 = (2\pi)^{(p=q)/2}\,[\prod_{j=1}^{q} p_j^{-(N-1)p_j+\frac{1}{2}}\cdot \Gamma(np_j)]/\prod_{\alpha=1}^{p}(N-\alpha)]$$

and $G_{2j+2v} = \int_{\ell}^{\infty} g_{2j+2v}(\chi^2)\,d\chi^2$, $g_{2j+2v}(\chi^2)$ being the density of a chisquare variable on $2j+2v$ degrees of freedom. The coefficients B_j are computed as in Nagarsenker and Pillai (1972).

3. Approximation using Box's series. Let $\lambda = W^n$. Then following Anderson (1958), p.207) the asymptotic expansion for the cdf of $-2n\rho\log W$ is given by

$$Pr(-2n\rho\log W \leq z) = Pr(\chi_f^2 \leq z) + w_2(Pr\{\chi_{f+4}^2 \leq z\} - Pr\{\chi_f^2 \leq z\}) + O(n^{-3}) \tag{3.1}$$

where $f = (p^2-q)$, $(1-\rho) = \dfrac{2p^3 - p - \sum_{i=1}^{q}(1/p_i)}{6n(p^2-q)}$ and

$$w_2 = \frac{1}{2\rho n}\left[\sum_{k=1}^{p} B_2(\tfrac{n}{2}(1-\rho) + (1-k)) - \sum_{j=1}^{q}\frac{B_2(np_j(1-\rho)/2)}{p_j}\right].$$

5. Numerical results and conclusions. The 0.05 and 0.01 significance points of W were computed using double precision arithmetic for various N in the following cases: $p=q=3,5,6$ and each $p_j=1$. The values are given correct to five significant figures in Table 1. The accuracy of the tabled values calculated using the series (2.7) and (2.8) is tested against its exact beta distribution for p=2 and the results agreed completely. It is to be noted that the relative merit of the two series lies in the fact that when N is small the former is more efficient than the latter in the sense that it needs considerably less number of terms of the series to attain the desired degree of accuracy for computing the percentage points while the latter is more efficient than the former when N is moderately large. Thus for N less than 20, series (2.7) was used and the number of terms required for five figure accuracy varied from 30 to 50, increasing with p while the series (2.8) needed many more terms. But for higher values of N series (2.8) was used since the number of terms in this case varied from 20 to 30, depending upon p and the accuracy was checked by using the beta series (2.7) which needed considerably more terms. Also the exact significance points of W for the special case in which all $p_j=1$ were checked using Box's asymptotic expansion for $-2n\rho\log W$ given in (3.1) for large values of n.

TABLE 1. Percentage points of W when

each $p_i = 1$

N \ P	3		5		6	
	.05	.01	.05	.01	.05	.01
6	$.0^{3}88068$	$.0^{4}79188$				
7	.010439	$.0^{2}22186$				
8	.034555	.011080				
9	.069778	.028436	$.0^{4}59328$			
10	.11068	.052752	$.0^{3}58786$	$.0^{3}13220$		
11	.15328	.081576	$.0^{2}24376$	$.0^{3}73464$	$.0^{4}58129$	$.0^{4}11343$
12	.19526	.11277	$.0^{2}64106$	$.0^{2}23522$	$.0^{3}34471$	$.0^{4}91569$
13	.23538	.14479	.012905	$.0^{2}54533$	$.0^{2}11811$	$.0^{3}38720$
14	.27309	.17661	.021914	.010299	$.0^{2}29115$	$.0^{2}11118$
15	.30821	.20761	.033175	.016940	$.0^{2}58001$	$.0^{2}24867$
16	.34076	.23742	.046306	.025270	$.0^{2}99893$	$.0^{2}46898$
17	.37065	.26585	.06090	.035097	.015507	$.0^{2}78327$
18	.39866	.29282	.076587	.046189	.022294	.011960
19	.42438	.31833	.093026	.058306	.030238	.017060
20	.44817	.34240	.10994	.071225	.039198	.023083
25	.54402	.44357	.19528	.14170	.094127	.064019
30	.61243	.51960	.27358	.21214	.15563	.11493
35	.66336	.57809	.34151	.27662	.21567	.16800
40	.70262	.62422	.39954	.33376	.27114	.21924
45	.73377	.66144	.44911	.38387	.32119	.26698
50	.75906	.69206	.49164	.42775	.36595	.31073
60	.79758	.73935	.56041	.50027	.44156	.38656
70	.82553	.77414	.61328	.55723	.50218	.44895
80	.84671	.80078	.65503	.60290	.55149	.50061
90	.86331	.82182	.68877	.64023	.59221	.54384
100	.87668	.83885	.71657	.67125	.62633	.58043

References

[1] Anderson, T. W. (1958). An Introduction to Multivariate Statistical Analysis. Wiley, New York.

[2] Goddman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution. Ann. Math. Statist. 34, 152-176.

[3] Hannan, E. J. (1970). Multiple Time Series. Wiley, New York.

[4] Liggett, Jr., W. S. (1972). Passive sonar: fitting models to multiple time series in Proceedings of the NATO Advanced Study. Institute on Signal Processing. Academic Press, New York.

[5] Nagarsenker, B. N. and Pillai, K. C. S. (1972). The distribution of the sphericity test criterion. Mimeograph Series No. 284, Department of Statistics, Purdue University.

[6] Pillai, K. C. S. and Nagarsenker, B. N. (1971). On the distribution of the sphericity test criterion in classical and complex normal populations having unknown covariance matrices. Ann. Math. Statist. 42, 764-767.

[7] Priestley, M. B., Subba Rao, T. and Tong, M. (1973). Identification of the structure of multivariate stochastic systems in Multivariate Analysis III (P. R. Krishnaiah, Ed.) Academic Press, New York.

[8] Wahba, G. (1971). Some tests of independence for stationary multivariate time series. J. Roy. Statist. Soc., Ser. B, 33, 153-166.

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