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A CANONICAL ANALYSIS OF MULTIPLE TIME SERIES

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1. Introduction

Data frequently occur in the form of k related time series simultaneously observed at some constant interval. In particular, economic, industrial and ecological data are often of this kind. Much work has been done on the problem of detecting, estimating and describing relationships of various kinds among such series - see e.g. Quenouille (1957), Hannan (1970), Box and Jenkins (1970), and Brillinger (1975). In this paper we shall consider a particular method for characterizing structure.

Consider a $k \times 1$ vector process $\{\underset{\sim}{\mathfrak{z}}_t\}$ and let $\underset{\sim}{z}_t = \underset{\sim}{\mathfrak{z}}_t - \underset{\sim}{\mu}$ where $\underset{\sim}{\mu}$ is a convenient $k \times 1$ vector of origin which is the mean if the process is stationary. Suppose

$$\underset{\sim}{z}_t = \hat{\underset{\sim}{z}}_{t-1}(1) + \underset{\sim}{a}_t \tag{1.1}$$

where

$$\hat{\underset{\sim}{z}}_{t-1}(1) = E(\underset{\sim}{z}_t | \underset{\sim}{z}_{t-1}, \underset{\sim}{z}_{t-2}, \ldots) = \sum_{\ell=1}^{p} \underset{\sim}{\pi}_\ell \underset{\sim}{z}_{t-\ell} \tag{1.2}$$

is the expectation of $\underset{\sim}{z}_t$ conditional on past history up to time t-1, the $\underset{\sim}{\pi}_\ell$ are $k \times k$ matrices, $\{\underset{\sim}{a}_t\}$ is a sequence of independently and normally distributed $k \times 1$ vector random variables with mean zero and covariance matrix $\underset{\sim}{\Sigma}$, and $\underset{\sim}{a}_t$ is independent of $\hat{\underset{\sim}{z}}_{t-1}(1)$. The model (1.1) can be written as

$$(\underset{\sim}{I} - \sum_{\ell=1}^{p} \underset{\sim}{\pi}_\ell B^{\ell}) \underset{\sim}{z}_t = \underset{\sim}{a}_t \tag{1.3}$$

where $\underset{\sim}{I}$ is the identity matrix and B is the backshift operator such that $B\underset{\sim}{z}_t = \underset{\sim}{z}_{t-1}$. The process $\{\underset{\sim}{z}_t\}$ is stationary if the determinantal polynomial in B, $\det(\underset{\sim}{I} - \sum_{\ell=1}^{p} \underset{\sim}{\pi}_\ell B)$, has all its zeroes lying outside the unit circle, and otherwise the process will be called nonstationary.

Now suppose $k = 1$. Then, if the process is stationary,

$$E[z_t^2] = E[\hat{z}_{t-1}(1)]^2 + E[a_t^2]$$

i.e. $$\sigma_z^2 = \sigma_{\hat{z}}^2 + \sigma_a^2 .$$

We can define a quantity λ measuring the predictability of a stationary series from its past as

$$\lambda = \frac{\sigma_{\hat{z}}^2}{\sigma_z^2} = 1 - \frac{\sigma_a^2}{\sigma_z^2} . \qquad (1.4)$$

Suppose we were considering k different stock market indicators such as the Dow Jones Average, the Standard and Poors index, etc. It is natural to conjecture that each of these might be represented as some aggregate of one or more common inputs which may be nearly nonstationary, together with other stationary or white noise components. This leads one to contemplate some form of canonical representation and in particular to consider a linear aggregate $\dot{z}_t = \underset{\sim}{m}' \underset{\sim}{z}_t$, where $\underset{\sim}{m}$ is a $k \times 1$ vector, which is most predictable in the sense of maximizing λ. The analysis in fact yields k canonical components from least to most predictable. The most predictable components will often approach nonstationarity and the least predictable will be stationary or even random. Thus it is sometimes convenient to think about the k-dimensional space of the observation $\underset{\sim}{z}_t$ as containing stationary and nonstationary subspaces.

Variables within the stationary space can reflect relationships which remain stable over time and may be associated with, or even point to, economic or physical laws.

2. Choice of the canonical variables

Suppose that $\underset{\sim}{z}_t$ is stationary. Denoting the covariance matrices of $\underset{\sim}{z}_t$ and $\hat{\underset{\sim}{z}}_{t-1}(1)$ respectively by $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})$ and $\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}})$, then, since in (1.1) $\underset{\sim}{a}_t$ and $\hat{\underset{\sim}{z}}_{t-1}(1)$ are independent, we have

$$\underset{\sim}{\Gamma}_0(\underset{\sim}{z}) = \underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}}) + \underset{\sim}{\Sigma} . \qquad (2.1)$$

Until further notice, we shall assume that $\underset{\sim}{\Sigma}$, and therefore $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})$, are positive definite.

Note that from (1.1)

$$\underset{\sim}{z}_t\underset{\sim}{z}'_{t-q} = \sum_{\ell=1}^{p} \underset{\sim}{\pi}_\ell \underset{\sim}{z}_{t-\ell}\underset{\sim}{z}'_{t-q} + \underset{\sim}{a}_t\underset{\sim}{z}'_{t-q} .$$

On taking expectations

$$\underset{\sim}{\Gamma}'_q(\underset{\sim}{z}) = \sum_{\ell=1}^{p} \underset{\sim}{\pi}_\ell \underset{\sim}{\Gamma}'_{q-\ell}(\underset{\sim}{z}) , \qquad q > 0 \qquad (2.2)$$

and

$$\underset{\sim}{\Gamma}_0(\underset{\sim}{z}) = \sum_{\ell=1}^{p} \underset{\sim}{\pi}_\ell \underset{\sim}{\Gamma}_\ell(\underset{\sim}{z}) + \underset{\sim}{\Sigma} , \qquad q = 0,$$

where $\underset{\sim}{\Gamma}_j(\underset{\sim}{z}) = E(\underset{\sim}{z}_{t-j}\underset{\sim}{z}'_t)$ is the lag j autocovariance matrix of $\underset{\sim}{z}_t$. Comparison with (2.1) shows that

$$\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}}) = \sum_{\ell=1}^{p} \underset{\sim}{\pi}_\ell \underset{\sim}{\Gamma}_\ell(\underset{\sim}{z}) . \qquad (2.3)$$

Now consider a scalar random variable $\dot{z}_t$ which is a linear combination

$$\dot{z}_t = \underset{\sim}{m}'\underset{\sim}{z}_t = m_1 z_{1t} + \dots + m_k z_{kt}$$

of the elements of $\underset{\sim}{z}_t$. Thus,

$$\dot{z}_t = \underset{\sim}{m}'\hat{\underset{\sim}{z}}_{t-1}(1) + \underset{\sim}{m}'\underset{\sim}{a}_t$$

or

$$\dot{z}_t = \hat{\dot{z}}_{t-1}(1) + \dot{a}_t \ ,$$

where $\hat{\dot{z}}_{t-1}(1) = E\,(\dot{z}_t | \dot{z}_{t-1}, \ \dot{z}_{t-2}, \dots)$ and $\hat{\dot{z}}_{t-1}(1)$ and $\dot{a}_t$ are independently distributed. It follows that

$$\sigma^2_{\dot{z}} = \sigma_{\hat{\dot{z}}}^{\ 2} + \sigma^{\ 2}_{\dot{a}}$$

i.e.

$$\underset{\sim}{m}'\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\,\underset{\sim}{m} = \underset{\sim}{m}'\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}})\,\underset{\sim}{m} + \underset{\sim}{m}'\underset{\sim}{\Sigma}\,\underset{\sim}{m} \ .$$

In the sense discussed above, the most predictable linear combination of the z_{jt} is obtained by maximizing, with respect to $\underset{\sim}{m}$,

$$\lambda = \frac{\sigma_{\hat{\dot{z}}}^{2}}{\sigma^2_{\dot{z}}} = 1 - \frac{\sigma^2_{\dot{a}}}{\sigma^2_{\dot{z}}} = \frac{\underset{\sim}{m}'\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}})\,\underset{\sim}{m}}{\underset{\sim}{m}'\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\,\underset{\sim}{m}} \ . \tag{2. 4}$$

Now, the maximum value of λ is the largest root of the determinantal equation

$$\det\{\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}}) - \lambda\,\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\} = 0 \tag{2. 5}$$

and the corresponding elements of $\underset{\sim}{m}$ are obtained, except for an arbitrary scale factor, as the solution of the homogeneous linear equations

$$\{\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}}) - \lambda \underset{\sim}{\Gamma}_0(\underset{\sim}{z})\}\,\underset{\sim}{m} = \underset{\sim}{0}\,. \qquad (2.6)$$

2.1. The canonical transformation

In general there will be k real roots $\lambda_1, \ldots, \lambda_k$ which satisfy the determinantal equation (2.5) . They are the eigen values of the matrix $\underset{\sim}{\Gamma}_0^{-1}(\underset{\sim}{z})\,\underset{\sim}{\Gamma}(\hat{\underset{\sim}{z}})$. Suppose that the λ_j are ordered with λ_1 the smallest, and that the k corresponding linearly independent eigen vectors, $\underset{\sim}{m}'_1, \ldots, \underset{\sim}{m}'_k$, form the k rows of a matrix $\underset{\sim}{M}$. Then, we can construct a transformed process $\{\dot{\underset{\sim}{z}}_t\}$, where

$$\dot{\underset{\sim}{z}}_t = \hat{\dot{\underset{\sim}{z}}}_{t-1}(1) + \dot{\underset{\sim}{a}}_t \qquad (2.7)$$

with $\dot{\underset{\sim}{z}}_t = \underset{\sim}{M}\underset{\sim}{z}_t$, $\dot{\underset{\sim}{a}}_t = \underset{\sim}{M}\underset{\sim}{a}_t$, $\hat{\dot{\underset{\sim}{z}}}_{t-1}(1) = \sum_{\ell=1}^{p} \dot{\underset{\sim}{\pi}}_\ell \dot{\underset{\sim}{z}}_{t-\ell}$ and $\dot{\underset{\sim}{\pi}}_\ell = \underset{\sim}{M}\underset{\sim}{\pi}_\ell\underset{\sim}{M}^{-1}$.

Corresponding to (2.1), we now have

$$\underset{\sim}{\Gamma}_0(\dot{\underset{\sim}{z}}) = \underset{\sim}{\Gamma}_0(\hat{\dot{\underset{\sim}{z}}}) + \dot{\underset{\sim}{\Sigma}} \qquad (2.8)$$

where $\underset{\sim}{\Gamma}_0(\dot{\underset{\sim}{z}}) = \underset{\sim}{M}\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\underset{\sim}{M}'$, $\underset{\sim}{\Gamma}_0(\hat{\dot{\underset{\sim}{z}}}) = \underset{\sim}{M}\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}})\underset{\sim}{M}'$ and $\dot{\underset{\sim}{\Sigma}} = \underset{\sim}{M}\underset{\sim}{\Sigma}\underset{\sim}{M}'$.

Since from (2.1)

$$\underset{\sim}{I} = \underset{\sim}{\Gamma}_0^{-1}(\underset{\sim}{z})\,\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}}) + \underset{\sim}{\Gamma}_0^{-1}(\underset{\sim}{z})\,\underset{\sim}{\Sigma}\,, \qquad (2.9)$$

it readily follows that (i)

$$\underset{\sim}{M}'^{-1}\underset{\sim}{\Gamma}_0^{-1}(\underset{\sim}{z})\,\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}})\,\underset{\sim}{M}' = \underset{\sim}{\Lambda} \quad \text{and} \quad \underset{\sim}{M}'^{-1}\underset{\sim}{\Gamma}_0^{-1}(\underset{\sim}{z})\,\underset{\sim}{\Sigma}\,\underset{\sim}{M}' = \underset{\sim}{I} - \underset{\sim}{\Lambda} \qquad (2.10)$$

where $\underset{\sim}{\Lambda}$ is the k × k diagonal matrix with elements $(\lambda_1, \ldots, \lambda_k)$,

(ii) $\quad 0 \leq \lambda_j < 1, \quad j = 1, \ldots, k,$

and

(iii) for $i \neq j$

$$\underset{\sim}{m}_i' \underset{\sim}{\Sigma} \underset{\sim}{m}_j = \underset{\sim}{m}_i' \underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}}) \underset{\sim}{m}_j = 0 \quad .$$

In other words, $\underset{\sim}{M} \underset{\sim}{\Sigma} \underset{\sim}{M}'$, $\underset{\sim}{M} \underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}}) \underset{\sim}{M}'$ and, therefore, $\underset{\sim}{M} \underset{\sim}{\Gamma}_0(\underset{\sim}{z}) \underset{\sim}{M}'$ are all diagonal. Thus, the transformation (2.7) produces k component series $\{\dot{z}_{1t}, \ldots, \dot{z}_{kt}\}$ which

(i) are ordered from least predictable to most predictable,

(ii) are contemporaneously independent,

(iii) have predictable components $\{\hat{\dot{z}}_{1(t-1)}(1), \ldots, \hat{\dot{z}}_{k(t-1)}(1)\}$ which are contemporaneously independent, and

(iv) have unpredictable components $\{\dot{a}_{1t}, \ldots, \dot{a}_{kt}\}$ which are contemporaneously and temporally independent.

2.2 Scaling of the transformed series.

Now the above canonical analysis is clearly scale invariant in the sense that changes in scales of measurement of the original z_{jt} will not affect the λ_j or the relative weighting applied to the original variables in generating the canonical variables $\dot{z}_{jt}$. However, the canonical variables themselves have no natural scaling. If c is an arbitrary constant, then $\dot{z}_{jt}$ and $c\,\dot{z}_{jt}$ can equally be called the jth canonical variable. This corresponds to the fact that each of the k eigen vectors $\underset{\sim}{m}_1, \ldots, \underset{\sim}{m}_k$ contains an arbitrary scale factor. In particular, multiplication of the jth vector $\underset{\sim}{m}_j$ by c will magnify the corresponding variances of $\dot{z}_{jt}$, $\hat{\dot{z}}_{j(t-1)}(1)$ and $\dot{a}_{jt}$ by c^2. Also, for the predictable vector $\hat{\dot{z}}_{t-1}(1)$, the elements of the jth row of $\dot{\underset{\sim}{\pi}}_\ell$ will be multiplied by c and those of the jth column divided by c, leaving the remaining elements unaffected.

2. 3 Zero roots

It turns out that special interest attaches to situations where certain of the λ_j approach zero. When the k_1 roots, $\lambda_1, \ldots \lambda_{k_1}$, are zero, the matrix $\underset{\sim}{\Gamma}_0(\hat{z})$ in (2. 8) then can be written

$$\underset{\sim}{\Gamma}_0(\hat{z}) = \begin{array}{cc|c} \underset{\sim}{0} & \underset{\sim}{0} & k_1 \\ \underset{\sim}{0} & \underset{\sim}{D} & k_2 \\ k_1 & k_2 & \end{array} \tag{2. 11}$$

where $\underset{\sim}{D}$ is an diagonal matrix.

Writing

$$\dot{\underset{\sim}{z}}'_t = [\underset{k_1}{\dot{\underset{\sim}{z}}'_{1t}} \vdots \underset{k_2}{\dot{\underset{\sim}{z}}'_{2t}}] \quad \text{and} \quad \dot{\underset{\sim}{a}}'_t = [\underset{k_1}{\dot{\underset{\sim}{a}}'_{1t}} \vdots \underset{k_2}{\dot{\underset{\sim}{a}}'_{2t}}] \,, \tag{2. 12}$$

we now show that, with probability one,

$$\dot{\underset{\sim}{z}}_{1t} = \dot{\underset{\sim}{a}}_{1t} \,. \tag{2. 13}$$

Proof:

In (2. 7) write

$$\dot{\underset{\sim}{\pi}}_\ell = \begin{array}{cc|c} \dot{\underset{\sim}{\pi}}^{(\ell)}_{11} & \dot{\underset{\sim}{\pi}}^{(\ell)}_{12} & k_1 \\ \dot{\underset{\sim}{\pi}}^{(\ell)}_{21} & \dot{\underset{\sim}{\pi}}^{(\ell)}_{22} & k_2 \\ k_1 & k_2 & \end{array} \,, \quad \ell = 1, \ldots, p \tag{2. 14}$$

so that

$$\dot{\underset{\sim}{z}}_{1t} = \sum_{\ell=1}^{p} \left\{ \dot{\underset{\sim}{\pi}}^{(\ell)}_{11} \dot{\underset{\sim}{z}}_{1(t-\ell)} + \dot{\underset{\sim}{\pi}}^{(\ell)}_{12} \dot{\underset{\sim}{z}}_{2(t-\ell)} \right\} + \dot{\underset{\sim}{a}}_{1t} \,.$$

Then, (2.11) implies that, with probability one,

$$\boldsymbol{U}_1 \boldsymbol{M} \boldsymbol{z}_t = - \sum_{\ell=1}^{p-1} \boldsymbol{U}_{\ell+1} \boldsymbol{M} \boldsymbol{z}_{t-\ell}$$

where $\boldsymbol{U}_\ell = \left[\dot{\boldsymbol{\pi}}_{11}^{(\ell)} \vdots \dot{\boldsymbol{\pi}}_{12}^{(\ell)} \right]$. This means that the $k_1 \times 1$ transformed vector $- \boldsymbol{U}_1 \boldsymbol{M} \boldsymbol{z}_t$ can be expressed completely in terms of past values of $\boldsymbol{z}_t$. Now from (1.1) $\boldsymbol{U}_1 \boldsymbol{M} \boldsymbol{z}_t = \boldsymbol{U}_1 \boldsymbol{M} \hat{\boldsymbol{z}}_{t-1}(1) + \boldsymbol{U}_1 \boldsymbol{M} \boldsymbol{a}_t$, it follows that, with probability one, $\boldsymbol{U}_1 \boldsymbol{M} \boldsymbol{a}_t = \boldsymbol{0}$, which implies that $\boldsymbol{U}_1 \boldsymbol{M} \boldsymbol{\Sigma} \boldsymbol{M} \boldsymbol{U}_1' = \boldsymbol{0}$. Since $\boldsymbol{M} \boldsymbol{\Sigma} \boldsymbol{M}'$ is positive definite, we must have $\boldsymbol{U}_1 = \boldsymbol{0}$. In the same way, it is readily seen that $\boldsymbol{U}_2 = \ldots = \boldsymbol{U}_p = \boldsymbol{0}$ and hence (2.13) follows.

Thus, if k_1 of the roots, $\lambda_1, \ldots, \lambda_{k1}$, are zero. the transformed series $\{\dot{\boldsymbol{z}}_t\}$ can be expressed as

$$\dot{\boldsymbol{z}}_{1t} = \dot{\boldsymbol{a}}_{1t}$$

$$\dot{\boldsymbol{z}}_{2t} = \sum_{\ell=1}^{p} \dot{\boldsymbol{\pi}}_{21}^{(\ell)} \dot{\boldsymbol{z}}_{1(t-\ell)} + \sum_{\ell=1}^{p} \dot{\boldsymbol{\pi}}_{22}^{(\ell)} \dot{\boldsymbol{z}}_{2(t-\ell)} + \dot{\boldsymbol{a}}_{2t} \qquad (2.15)$$

so that the canonical transformation decomposes the original $k \times 1$ vector process $\{\boldsymbol{z}_t\}$ into two parts: (i) a part $\{\dot{\boldsymbol{z}}_{1t}\}$ which follows a k_1 dimensional white noise process and (ii) a part $\{\dot{\boldsymbol{z}}_{2t}\}$ which is stationary but the predictable part of $\dot{\boldsymbol{z}}_{2t}$ depends on both $\dot{\boldsymbol{z}}_{1(t-\ell)}$ and $\dot{\boldsymbol{z}}_{2(t-\ell)}$, $\ell = 1, \ldots, p$.

The practical importance of equation (2.15) is that it implies that there are k_1 relationships between the original variables of the "regression" type

$$m_{j1} a_{1t} + \ldots + m_{jk} z_{kt} = \mu_j + \dot{a}_{jt} \quad (j = 1, 2 \ldots k_1)$$

where the $\dot{a}_{jt}$ are contemporaneously and temporally independent. We shall later illustrate this situation with an example.

3. The AR(1) process

In this section, we discuss some properties of the canonical transformation when $\{\underset{\sim}{z}_t\}$ follows an k-dimensional autoregressive process of order 1. Thus with $p = 1$ and $\underset{\sim}{\pi}_1 = \underset{\sim}{\phi}$, expressions (1.1), (2.2) and (2.3) yield

$$\underset{\sim}{z}_t = \hat{\underset{\sim}{z}}_{t-1}(1) + \underset{\sim}{a}_t, \quad \text{where } \hat{\underset{\sim}{z}}_{t-1}(1) = \underset{\sim}{\phi}\,\underset{\sim}{z}_{t-1}\ ; \tag{3.1}$$

$$\underset{\sim}{\Gamma}_1'(\underset{\sim}{z}) = \underset{\sim}{\phi}\,\underset{\sim}{\Gamma}_0(\underset{\sim}{z}), \qquad \underset{\sim}{\Gamma}_0(\underset{\sim}{z}) = \underset{\sim}{\phi}\,\underset{\sim}{\Gamma}_1(\underset{\sim}{z}) + \underset{\sim}{\Sigma} \tag{3.2}$$

and

$$\underset{\sim}{\Gamma}_0(\hat{\underset{\sim}{z}}) = \underset{\sim}{\phi}\,\underset{\sim}{\Gamma}_1(\underset{\sim}{z})\ .$$

It follows that

$$\underset{\sim}{\Gamma}_0(\underset{\sim}{z}) = \underset{\sim}{\phi}\,\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\,\underset{\sim}{\phi}' + \underset{\sim}{\Sigma} \tag{3.3}$$

and the required roots λ_j and vectors $\underset{\sim}{m}_j$ are the k eigen values and eigen vectors of the matrix

$$\underset{\sim}{Q} = \underset{\sim}{\Gamma}_0^{-1}(\underset{\sim}{z})\,\underset{\sim}{\phi}\,\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\,\underset{\sim}{\phi}'\ . \tag{3.4}$$

The transformed process can now be written

$$\dot{\underset{\sim}{z}}_t = \dot{\underset{\sim}{\phi}}\,\dot{\underset{\sim}{z}}_{t-1} + \dot{\underset{\sim}{a}}_t \qquad \text{where } \dot{\underset{\sim}{\phi}} = \underset{\sim}{M}\,\underset{\sim}{\phi}\,\underset{\sim}{M}^{-1} \tag{3.5}$$

3.1. Nonstationary series and unit roots

In the above we have assumed that $\underset{\sim}{z}_t$ is stationary. In practice, many time series exhibit nonstationary behavior. A useful class of models to represent nonstationary series may be obtained by allowing the zeroes of the det $(\underset{\sim}{I} - \sum_{j=1}^{p} \underset{\sim}{\pi}_j B^j)$ of (1.3) to lie on the unit circle. For the AR(1) model in (3.1), let $\alpha_1, \ldots, \alpha_k$ be the eigen values of the matrix $\underset{\sim}{\phi}$. Then

$$\det(\underset{\sim}{I} - \underset{\sim}{\phi} B) = \prod_{j=1}^{k} (1-\alpha_j B)$$

so that the zeroes of $\det(\underset{\sim}{I} - \underset{\sim}{\phi} B)$ are simply $(\alpha_1^{-1}, \ldots, \alpha_j^{-1})$. If one or more of the α_j are on the unit circle, then $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})$ does not exist and the canonical transformation method will break down. However, it is of interest to study the limiting situation when k_2 of the α_j approach values on the unit circle. We now prove the following theorem.

Theorem 3.1. Suppose that $\underset{\sim}{z}_t$ follows the stationary AR(1) model in (3.1) where the covariance matrix $\underset{\sim}{\Sigma}$ of $\underset{\sim}{a}_t$ is positive definite. A sufficient and necessary condition for k_2 of the eigen values of $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})^{-1}\underset{\sim}{\phi}\,\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\underset{\sim}{\phi}'$ to tend to unity is that k_2 of the eigen values of $\underset{\sim}{\phi}$ approach values on the unit circle.

Proof:

Let $k = k_1 + k_2$ and the eigen values of $\underset{\sim}{\phi}$ be divided into two sets $\underset{\sim}{\alpha}_1 = (\alpha_1, \ldots, \alpha_{k_1})$ and $\underset{\sim}{\alpha}_2 = (\alpha_{k_1+1}, \ldots, \alpha_k)$ with no common element and such that α_j and its complex conjugate belong to the same set. The characteristic polynomial of $\underset{\sim}{\phi}$ can be written as the product

$$f(\alpha) = f_{k_1}(\alpha)\, f_{k_2}(\alpha) \tag{3.6}$$

where

$$f_{k_1}(\alpha) = \alpha^{k_1} - \gamma_1 \alpha^{k_1-1} - \ldots - \gamma_{k_1} \quad \text{and} \quad f_{k_2}(\alpha) = \alpha^{k_2} - s_1 \alpha^{k_2-1} - \ldots - s_{k_2}$$

are real polynomials of degrees k_1 and k_2 with roots $\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\alpha}_2$, respectively. Now there exists an $k \times k$ real nonsingular matrix $\underset{\sim}{C}$ such that $\underset{\sim}{C}\,\underset{\sim}{\phi}\,\underset{\sim}{C}^{-1}$ is of the block diagonal form

$$\underset{\sim}{C}\underset{\sim}{\phi}\underset{\sim}{C}^{-1} = \begin{array}{cc|c} \underset{\sim}{R} & \underset{\sim}{0} & k_1 \\ \underset{\sim}{0} & \underset{\sim}{S} & k_2 \\ \hline k_1 & k_2 & \end{array} \tag{3.7}$$

where

$$\underset{\sim}{R} = \begin{array}{c|c|c} \underset{\sim}{0} & \underset{\sim}{I} & k_1-1 \\ \hline \gamma_{k_1} & \gamma_{k_1-1}, \ldots, \gamma_1 & 1 \\ \hline 1 & k_1-1 & \end{array} \quad \text{and} \quad \underset{\sim}{S} = \begin{array}{c|c|c} \underset{\sim}{0} & \underset{\sim}{I} & k_2-1 \\ \hline s_{k_2} & s_{k_2-1}, \ldots, s_1 & 1 \\ \hline 1 & k_2-1 & \end{array} .$$

Letting $\underset{\sim}{V} = \underset{\sim}{C}\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\underset{\sim}{C}'$ and $\underset{\sim}{W} = \underset{\sim}{C}\,\underset{\sim}{\Sigma}\,\underset{\sim}{C}'$ and partitioning

$$\underset{\sim}{V} = \begin{array}{cc|c} \underset{\sim}{V}_{11} & \underset{\sim}{V}_{12} & k_1 \\ \underset{\sim}{V}_{12}' & \underset{\sim}{V}_{22} & k_2 \\ \hline k_1 & k_2 & \end{array} \quad \text{and} \quad \underset{\sim}{W} = \begin{array}{cc|c} \underset{\sim}{W}_{11} & \underset{\sim}{W}_{12} & k_1 \\ \underset{\sim}{W}_{12}' & \underset{\sim}{W}_{22} & k_2 \\ \hline k_1 & k_2 & \end{array}$$

we obtain

$$\underset{\sim}{V}_{11} = \underset{\sim}{R}\underset{\sim}{V}_{11}\underset{\sim}{R}' + \underset{\sim}{W}_{11}, \quad \underset{\sim}{V}_{12} = \underset{\sim}{R}\underset{\sim}{V}_{12}\underset{\sim}{S}' + \underset{\sim}{W}_{12} ,$$

and
$$\tag{3.8}$$

$$\underset{\sim}{V}_{22} = \underset{\sim}{S}\underset{\sim}{V}_{22}\underset{\sim}{S}' + \underset{\sim}{W}_{22} .$$

By writing $\underset{\sim}{V}_{22} = \underset{\sim}{A}\,\underset{\sim}{A}'$ and

$$\underset{\sim}{V} = \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A} \end{bmatrix} \begin{bmatrix} \underset{\sim}{V}_{11} & \underset{\sim}{V}_{12}\underset{\sim}{A}'^{-1} \\ \underset{\sim}{A}^{-1}\underset{\sim}{V}_{12}' & \underset{\sim}{I} \end{bmatrix} \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}' \end{bmatrix} \tag{3.9}$$

it is readily seen from (3.3) and (3.4) that the λ_j are the roots of the determinantal polynomial

$$\det \left\{ (1-\lambda) \begin{bmatrix} \underset{\sim}{V}_{11} & \underset{\sim}{V}_{12}\underset{\sim}{A}'^{-1} \\ \underset{\sim}{A}^{-1}\underset{\sim}{V}'_{12} & \underset{\sim}{I} \end{bmatrix} - \begin{bmatrix} \underset{\sim}{W}_{11} & \underset{\sim}{W}_{12}\underset{\sim}{A}'^{-1} \\ \underset{\sim}{A}^{-1}\underset{\sim}{W}'_{12} & \underset{\sim}{A}^{-1}\underset{\sim}{W}_{22}\underset{\sim}{A}'^{-1} \end{bmatrix} \right\} = 0. \quad (3.10)$$

To prove sufficiency, we need to show that if the k_2 eigen values

$$\alpha_j \to e^{i\omega_j}, \qquad j = k_1 + 1, \ldots, k \quad (3.11)$$

then (3.10) will tend to

$$(1-\lambda)^{k_2} \det \{(1-\lambda) \underset{\sim}{V}_{11} - \underset{\sim}{W}_{11}\} = 0 . \quad (3.12)$$

It suffices to prove that (3.11) implies that

$$\underset{\sim}{A}^{-1} \to \underset{\sim}{0} \quad \text{and} \quad \underset{\sim}{A}^{-1}\underset{\sim}{V}'_{12} \to \underset{\sim}{0} . \quad (3.13)$$

From (3.8)

$$\underset{\sim}{V}_{22}^{-1} \underset{\sim}{S} \underset{\sim}{V}_{22} \underset{\sim}{S}' = [\underset{\sim}{I} + \underset{\sim}{S}'^{-1} \underset{\sim}{V}_{22}^{-1} \underset{\sim}{S}^{-1} \underset{\sim}{W}_{22}]^{-1} \quad (3.14)$$

so that

$$\det \underset{\sim}{S}^2 = \det [\underset{\sim}{I} + \underset{\sim}{S}'^{-1} \underset{\sim}{V}_{22}^{-1} \underset{\sim}{S}^{-1} \underset{\sim}{W}_{22}]^{-1} .$$

When (3.11) holds, $\det \underset{\sim}{S}^2 = d_{k_2}^2 \to 1$. Since $\underset{\sim}{S}$ is nonsingular and $\underset{\sim}{W}_{22}$ is positive definite, it follows that $\underset{\sim}{V}_{22}^{-1} \to \underset{\sim}{0}$ and hence $\underset{\sim}{A} \to \underset{\sim}{0}$.

To show $\underset{\sim}{A}^{-1}\underset{\sim}{V}'_{12} \to \underset{\sim}{0}$, we have, from (3.14) and (3.8),

$$\underset{\sim}{I} = [\underset{\sim}{P}\underset{\sim}{P}' + \underset{\sim}{A}^{-1}\underset{\sim}{W}_{22}^{-1}\underset{\sim}{A}'^{-1}]^{-1} \qquad (3.15)$$

and

$$\underset{\sim}{A}^{-1}\underset{\sim}{V}'_{12} = [\underset{\sim}{P}' + \underset{\sim}{A}^{-1}\underset{\sim}{W}_{22}^{-1}\underset{\sim}{A}'^{-1}]^{-1}\underset{\sim}{A}^{-1}\underset{\sim}{V}'_{12}\underset{\sim}{R}' + \underset{\sim}{A}^{-1}\underset{\sim}{W}'_{12} \qquad (3.16)$$

where $\underset{\sim}{P} = \underset{\sim}{A}^{-1}\underset{\sim}{S}\underset{\sim}{A}$. Thus, when (3.11) holds, in (3.15) $\underset{\sim}{P} \rightarrow \underset{\sim}{P}_o$ where $\underset{\sim}{P}_o$ is an orthogonal matrix, and hence (3.16) becomes

$$\underset{\sim}{P}'_o\underset{\sim}{A}^{-1}\underset{\sim}{V}'_{12} = \underset{\sim}{A}^{-1}\underset{\sim}{V}'_{12}\underset{\sim}{R} . \qquad (3.17)$$

Since by supposition, $\underset{\sim}{S}$ and $\underset{\sim}{R}$ have no common eigen values, it follows from a well known result in matrix equation (see e.g. Gantmacher 1959, Vol. I, p. 220) that $\underset{\sim}{A}^{-1}\underset{\sim}{V}'_{12} \rightarrow \underset{\sim}{0}$. This completes the proof of the sufficiency part of the theorem.

Next to show necessity, recall that $|\alpha_j| < 1$ and $0 \leq \lambda_j < 1$, for $j=1, \ldots, k$. Thus, if k_2 of the λ_j tends to one, then exactly k_2 of the α_j must approach values on the unit circle. For, if otherwise, and suppose $k' \neq k_2$ of the α_j approach values on the unit circle, then from the sufficiency part of the theorem which we have just proved, k' of the λ_j must approach one, which contradicts the supposition. The theorem thus follows.

The eigen vectors

It is easy to see that the systems of equations $[\underset{\sim}{Q} - \lambda\underset{\sim}{I}]\underset{\sim}{m} = \underset{\sim}{0}$ is equivalent to

$$\{(1-\lambda)\underset{\sim}{I} - \underset{\sim}{V}^{-1}\underset{\sim}{W}\}\ \underset{\sim}{h} = \underset{\sim}{0} \qquad (3.18)$$

where $\underset{\sim}{C}'\underset{\sim}{h} = \underset{\sim}{m}$. When (3.11) holds, from (3.8), (3.9) and (3.13)

$$\underset{\sim}{V}^{-1}\underset{\sim}{W} \to \begin{array}{cc} \left[\begin{array}{cc} \underset{\sim}{V}_{11}^{-1}\underset{\sim}{W}_{11} & \underset{\sim}{V}_{11}^{-1}\underset{\sim}{W}_{12} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{array}\right] & \begin{array}{c} k_1 \\ k_2 \end{array} \\ \begin{array}{cc} k_1 & \qquad k_2 \end{array} & \end{array}$$

Thus, the matrix of eigen vectors $\underset{\sim}{M}'$ must be of the form

$$\underset{\sim}{C}'^{-1}\underset{\sim}{M}' = \underset{\sim}{H}' = \begin{array}{cc} \left[\begin{array}{cc} \underset{\sim}{H}_{11}' & -\underset{\sim}{W}_{11}^{-1}\underset{\sim}{W}_{12} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{array}\right] & \begin{array}{c} k_1 \\ k_2 \end{array} \\ \begin{array}{cc} k_1 & \qquad k_2 \end{array} & \end{array} \tag{3.19}$$

where the columns of $\underset{\sim}{H}_{11}'$ are the eigen vectors of $\underset{\sim}{V}_{11}^{-1}\underset{\sim}{W}_{11}$.

<u>The transformed matrix $\dot{\underset{\sim}{\Phi}}$.</u>

It follows from (3.19) that

$$\dot{\underset{\sim}{\Phi}} = \underset{\sim}{M}\underset{\sim}{\Phi}\underset{\sim}{M}^{-1} = \begin{array}{cc} \left[\begin{array}{cc} \underset{\sim}{\Phi}_{11} & \underset{\sim}{0} \\ \dot{\underset{\sim}{\Phi}}_{21} & \dot{\underset{\sim}{\Phi}}_{22} \end{array}\right] & \begin{array}{c} k_1 \\ k_2 \end{array} \\ \begin{array}{cc} k_1 & \quad k_2 \end{array} & \end{array} \tag{3.20}$$

where $\dot{\underset{\sim}{\Phi}} = \underset{\sim}{H}_{11}\underset{\sim}{R}\,\underset{\sim}{H}_{11}^{-1}$, $\dot{\underset{\sim}{\Phi}}_{22} = \underset{\sim}{S}_o$, $\dot{\underset{\sim}{\Phi}}_{21} = (\underset{\sim}{S}_o\underset{\sim}{W}_{12}'\underset{\sim}{W}_{11}^{-1} - \underset{\sim}{W}_{12}\underset{\sim}{W}_{11}^{-1}\underset{\sim}{R})\underset{\sim}{H}_{11}^{-1}$ and $\underset{\sim}{S}_o$ is the limiting matrix of $\underset{\sim}{S}$ when all its roots approach values on the unit circle.

To summarize, for the AR(1) model in (3.1),

(i) if, and only if, k_2 of the eigen values α_j of $\underset{\sim}{\Phi}$ approach values on the unit circle, then k_2 of the eigen values λ_j of $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})^{-1}\underset{\sim}{\phi}\underset{\sim}{\Gamma}_0(\underset{\sim}{z})\underset{\sim}{\phi}'$ will approach unity,

(ii) the transformed model for $\underset{\sim}{\dot{z}}_t$ is, in the limit,

$$\begin{aligned} \underset{\sim}{\dot{z}}_{1t} &= \underset{\sim}{\dot{\phi}}_{11}\underset{\sim}{\dot{z}}_{1(t-1)} + \underset{\sim}{\dot{a}}_{1t} \\ \underset{\sim}{\dot{z}}_{2t} &= \underset{\sim}{\dot{\phi}}_{21}\underset{\sim}{\dot{z}}_{1(t-1)} + \underset{\sim}{\dot{\phi}}_{22}\underset{\sim}{\dot{z}}_{2(t-1)} + \underset{\sim}{\dot{a}}_{2t} \end{aligned} \qquad (3.22)$$

where $\underset{\sim}{\dot{z}}'_t = \{\underset{\sim}{\dot{z}}'_{1t} \vdots \underset{\sim}{\dot{z}}'_{2t}\}$ and $\underset{\sim}{\dot{a}}_t = \{\underset{\sim}{\dot{a}}'_{1t} \vdots \underset{\sim}{\dot{a}}'_{2t}\}$. The canonical transformation therefore decomposes $\underset{\sim}{z}_t$ into two parts:

(i) a part $\underset{\sim}{\dot{z}}_{1t}$ which follows a stationary AR(1) process and

(ii) a part $\underset{\sim}{\dot{z}}_{2t}$ which is approaching nonstationarity and also depends on $\underset{\sim}{\dot{z}}_{1(t-1)}$.

3.2 Zero and unit roots.

For the AR(1) model, suppose that k_1 of the λ_j are zero, k_3 of them approach unity and the remaining $k_2 = k - k_1 - k_3$ are intermediate in size. Then, from the results in (2.15) and (3.21) and upon partitioning $\underset{\sim}{\dot{z}}_t$, $\underset{\sim}{\dot{a}}_t$ and $\underset{\sim}{\dot{\phi}}$ into

$$\underset{\sim}{\dot{z}}'_t = \{\underbrace{\underset{\sim}{\dot{z}}'_{1t}}_{k_1} \vdots \underbrace{\underset{\sim}{\dot{z}}'_{2t}}_{k_2} \vdots \underbrace{\underset{\sim}{\dot{z}}'_{3t}}_{k_3}\}, \qquad \underset{\sim}{\dot{a}}'_t = \{\underbrace{\underset{\sim}{\dot{a}}'_{1t}}_{k_1} \vdots \underbrace{\underset{\sim}{\dot{a}}'_{2t}}_{k_2} \vdots \underbrace{\underset{\sim}{\dot{a}}'_{3t}}_{k_3}\}$$

and

$$\underset{\sim}{\dot{\phi}} = \begin{array}{cc} \left[\begin{array}{ccc} \underset{\sim}{\dot{\phi}}_{11} & \underset{\sim}{\dot{\phi}}_{12} & \underset{\sim}{\dot{\phi}}_{13} \\ \underset{\sim}{\dot{\phi}}_{21} & \underset{\sim}{\dot{\phi}}_{22} & \underset{\sim}{\dot{\phi}}_{23} \\ \underset{\sim}{\dot{\phi}}_{31} & \underset{\sim}{\dot{\phi}}_{32} & \underset{\sim}{\dot{\phi}}_{33} \end{array}\right] & \begin{array}{c} k_1 \\ k_2 \\ k_3 \end{array} \\ \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} & \end{array}, \qquad (3.22)$$

the transformed process $\{\dot{\underset{\sim}{z}}_t\}$ takes the form

$$\begin{aligned}
\dot{\underset{\sim}{z}}_{1t} &= && && && + \dot{\underset{\sim}{a}}_{1t}\\
\dot{\underset{\sim}{z}}_{2t} &= \dot{\underset{\sim}{\phi}}_{21}\dot{\underset{\sim}{z}}_{1(t-1)} && + \dot{\underset{\sim}{\phi}}_{22}\dot{\underset{\sim}{z}}_{2(t-1)} && && + \dot{\underset{\sim}{a}}_{2t}\\
\dot{\underset{\sim}{z}}_{3t} &= \dot{\underset{\sim}{\phi}}_{31}\dot{\underset{\sim}{z}}_{1(t-1)} && + \dot{\underset{\sim}{\phi}}_{32}\dot{\underset{\sim}{z}}_{2(t-1)} && + \dot{\underset{\sim}{\phi}}_{33}\dot{\underset{\sim}{z}}_{3(t-1)} && + \dot{\underset{\sim}{a}}_{3t}\;.
\end{aligned} \tag{3.23}$$

Thus, there are (i) a k_1-dimensional white noise process $\{\dot{\underset{\sim}{z}}_{1t}\}$, (ii) a k_2-dimensional stationary process $\{\dot{\underset{\sim}{z}}_{2t}\}$ such that the predictable part of $\dot{\underset{\sim}{z}}_{2t}$ depend only on $\dot{\underset{\sim}{z}}_{1(t-1)}$ and $\dot{\underset{\sim}{z}}_{2(t-1)}$ and (iii) a k_3 dimensional near nonstationary process $\{\dot{\underset{\sim}{z}}_{3t}\}$ such that the predictable part of $\dot{\underset{\sim}{z}}_{3t}$ depend on $\dot{\underset{\sim}{z}}_{1(t-1)}$, $\dot{\underset{\sim}{z}}_{2(t-1)}$ and $\dot{\underset{\sim}{z}}_{3(t-1)}$.

3.3 Variance components for the AR(1) process.

Whatever the scaling of the transformed process $\{\dot{\underset{\sim}{z}}_t\}$ in (3.5), since the jth element $\dot{z}_{jt}$ is $\dot{z}_{jt} = \sum_{i=1}^{k} \dot{\phi}_{ji}\,\dot{z}_{i(t-1)} + \dot{a}_{jt}$, where $(\phi_{j1}, \dots, \phi_{jk})$ is the jth row of $\dot{\underset{\sim}{\phi}}$, and since $\dot{z}_{1(t-1)}, \dots, \dot{z}_{k(t-1)}$ and $\dot{a}_{jt}$ are independent, it follows that

$$\sigma^2_{\dot{z}_j} = \sum_{i=1}^{k} \phi^2_{ji}\,\sigma^2_{\dot{z}_i} + \sigma^2_{\dot{a}_j}\;. \tag{3.24}$$

The contribtuions of $\dot{z}_{1(t-1)}, \dots, \dot{z}_{k(t-1)}$ and $\dot{a}_{jt}$ to the variance of $\dot{z}_{jt}$ are, therefore, $\dot{\phi}^2_{j1}\sigma^2_{\dot{z}_1}$, $\dots$, $\dot{\phi}^2_{jk}\sigma^2_{\dot{z}_k}$ and $\sigma^2_{\dot{a}_j}$, respectively. It is convenient to consider these variance components in term of their proportional contribution to $\sigma^2_{\dot{z}_j}$, i.e. to consider $(\dot{\phi}^2_{ji}\,\sigma^2_{\dot{z}_i})/\sigma^2_{\dot{z}_j}$ and $\sigma^2_{\dot{a}_j}/\sigma_{\dot{z}_j} = 1 - \lambda_j$. This can be done conveniently by arranging the canonical variables with scaling such that the variances of $\dot{z}_{jt}$ are all unity.

For the general process (1.1), to arrange for this scaling the matrix $\boldsymbol{M}$ must be chosen such that $\boldsymbol{M}\boldsymbol{\Gamma}_0(z)\,\boldsymbol{M}' = \boldsymbol{I}$. Corresponding to (2.7), let the transformed series in this scaling be written as

$$\overset{*}{\boldsymbol{z}}_t = \hat{\overset{*}{\boldsymbol{z}}}_{t-1}(1) + \overset{*}{\boldsymbol{a}}_t \quad . \qquad (3.25)$$

Then,

$$\boldsymbol{\Gamma}_0(\overset{*}{\boldsymbol{z}}) = \boldsymbol{\Gamma}_0(\hat{\overset{*}{\boldsymbol{z}}}) + \overset{*}{\boldsymbol{\Sigma}} \qquad (3.26)$$

where $\boldsymbol{\Gamma}_0(\overset{*}{\boldsymbol{z}}) = \boldsymbol{I}$, $\boldsymbol{\Gamma}_0(\hat{\overset{*}{\boldsymbol{z}}}) = \boldsymbol{\Lambda}$, $\overset{*}{\boldsymbol{\Sigma}} = \boldsymbol{I} - \boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}$ is the diagonal matrix in (2.10). For the AR(1) process, $\hat{\overset{*}{\boldsymbol{z}}}_{t-1}(1) = \overset{*}{\boldsymbol{\phi}}\overset{*}{\boldsymbol{z}}_{t-1}$, and hence

$$\overset{*}{\phi}{}^2_{ji} = (\phi^2_{ji}\,\sigma^2_{z_i})\,/\,\sigma^2_{z_j} \qquad \text{and} \qquad \overset{*}{\boldsymbol{\phi}}\,\overset{*}{\boldsymbol{\phi}}{}' = \boldsymbol{\Lambda} \quad . \qquad (3.27)$$

In this scaling, then, the rows of $\overset{*}{\boldsymbol{\phi}}$ are orthogonal and the sum of squares of the jth row is λ_j .

The preceding canonical analysis will now be illustrated by an example.

4. An Example: US Hog Corn and Wage Series.

Quenouille (1957) studied a 5-variate time series containing 82 yearly observations from 1867-1948. He made adjustments where necessary, logarithmically transformed each variate and then linearly coded the logs, so as to produce numbers of comparable magnitude in the different series. His resulting five series which he denoted by $x_{1t}, x_{2t}, \ldots, x_{5t}$ are plotted in Figure 4.1 and are identified in Table 4.1

Figure 4.1.U. S. Hog Data

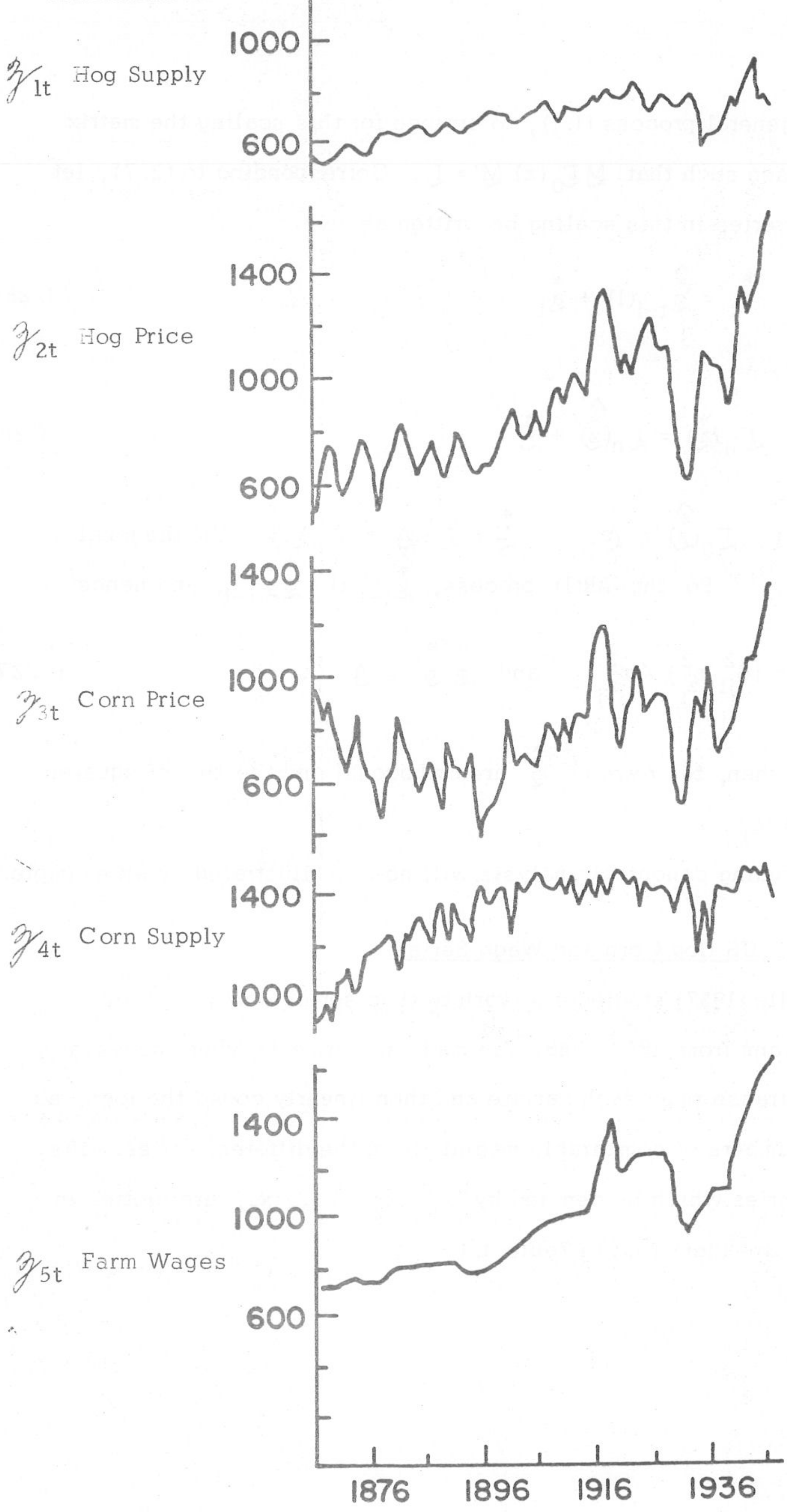

Table 4.1

Variate	Symbol	As logged and linearly coded by Quenouille	Used in our analysis
Hog Supply	H_s	x_{1t}	$\mathfrak{z}_{1t} = x_{1t}$
Hog Price	H_p	x_{2t}	$\mathfrak{z}_{2t} = x_{2(t+1)}$
Corn Price	R_p	x_{3t}	$\mathfrak{z}_{3t} = x_{3t}$
Corn Supply	R_s	x_{4t}	$\mathfrak{z}_{4t} = x_{4t}$
Farm Wages	W	x_{5t}	$\mathfrak{z}_{5t} = x_{5(t+1)}$

4.1 The AR(1) model.

Quenouille fitted the data to a first order autoregressive process but was doubtful as to the adequacy of the model. We found, however, that the fit can be improved by appropriately shifting series 2 and 5 backward by one period as indicated above.

Employing the model $\underset{\sim}{z}_t = \underset{\sim}{\phi}\underset{\sim}{z}_{t-1} + \underset{\sim}{a}_t$ in (3.1), where $\underset{\sim}{z}_t = \underset{\sim}{\mathfrak{z}}_t - \underset{\sim}{\mu}$, and denoting estimates of the covariance matrices $\underset{\sim}{\Gamma}_j(\underset{\sim}{z})$ by $\underset{\sim}{C}_j$. The quantities needed for analysis are as follows:

$$10^{-4}\underset{\sim}{C}_o = \begin{bmatrix} 0.6831 & 1.2523 & 0.6535 & 0.9533 & 1.5224 \\ & 6.1939 & 3.7845 & 2.0209 & 5.5708 \\ & & 3.6877 & 0.2633 & 3.4746 \\ & & & 2.1407 & 2.1925 \\ & & & & 5.7206 \end{bmatrix}$$

$$10^{-4}\underset{\sim}{C}_1 = \begin{bmatrix} 0.5864 & 1.3670 & 0.7513 & 0.8632 & 1.5151 \\ 1.2038 & 5.2334 & 3.1639 & 1.8849 & 5.0392 \\ 0.4616 & 3.5820 & 2.7173 & 0.5605 & 3.0633 \\ 1.0108 & 1.8972 & 0.8338 & 1.6260 & 2.2508 \\ 1.3993 & 5.1586 & 3.2153 & 1.9817 & 5.3246 \end{bmatrix}$$

$$\hat{\underset{\sim}{\Phi}} = \underset{\sim}{C}_1'\underset{\sim}{C}_o^{-1} = \begin{bmatrix} 0.3922 & -0.0555 & 0.0082 & 0.2552 & 0.0915 \\ 0.1088 & 0.0926 & 0.5019 & 0.3280 & 0.3520 \\ -0.7797 & -0.5758 & 0.9139 & 0.6155 & 0.5393 \\ 0.8715 & 0.2979 & -0.0732 & 0.3845 & -0.2785 \\ -0.0536 & -0.2239 & 0.1033 & 0.2420 & 1.0076 \end{bmatrix}$$

The canonical analysis yields the following estimated eigen values and eigen vectors of $\underset{\sim}{\Gamma}_o^{-1}(\underset{\sim}{z})\underset{\sim}{\Gamma}_o(\hat{\underset{\sim}{z}})$. The latter are scaled according to (3.25) so that all the components of the transformed process $\{\overset{*}{\underset{\sim}{z}}\}$ have unit estimated variances.

				$\hat{\underset{\sim}{m}}_j'$			
j	$\hat{\lambda}_j$	H_s	H_p	R_p	R_s	W	
1	0.0232	(1.0000	0.3876	-0.2524	-0.5896	-0.2665)	× .0284
2	0.1421	(0.2080	1.0000	-0.8614	-0.3382	-0.3655)	× .0111
3	0.5061	(0.8925	-0.6433	-0.8277	-0.4784	1.0000)	× .0074
4	0.6901	(-0.9358	-0.2410	-0.4391	-0.5614	1.0000)	× .0129
5	0.8868	(0.6687	-0.1206	-0.0134	0.0396	1.0000)	× .0039

The transformed process is $\overset{*}{\underset{\sim}{z}}_t = \overset{*}{\underset{\sim}{\phi}}\,\overset{*}{\underset{\sim}{z}}_{t-1} + \overset{*}{\underset{\sim}{a}}_t$ with

$$\hat{\overset{*}{\underset{\sim}{\phi}}} = \begin{bmatrix} .1213 & -.0778 & .0465 & -.0110 & .0113 \\ .2215 & .2766 & -.1241 & -.0309 & .0119 \\ -.0321 & .3167 & .6334 & .0444 & -.0404 \\ .0885 & -.0025 & -.0492 & .8235 & .0416 \\ -.0801 & .0378 & .0396 & -.0363 & .9360 \end{bmatrix}$$

and the resulting series $\overset{*}{\mathcal{Z}} = \hat{\underset{\sim}{M}}\,\mathcal{Z}_t$ are shown in Figure 4.2.

The estimated (proportional) contributions, $\hat{\overset{*}{\phi}}{}^2_{ji}$ and $1 - \hat{\lambda}_j$, of $\overset{*}{z}_{1(t-1)}$, ..., $\overset{*}{z}_{5(t-1)}$ and $\overset{*}{a}_{jt}$ to the variance of $\overset{*}{z}_{jt}$ are given below in Table 4.2.

Figure 4. 2. The Transformed Series

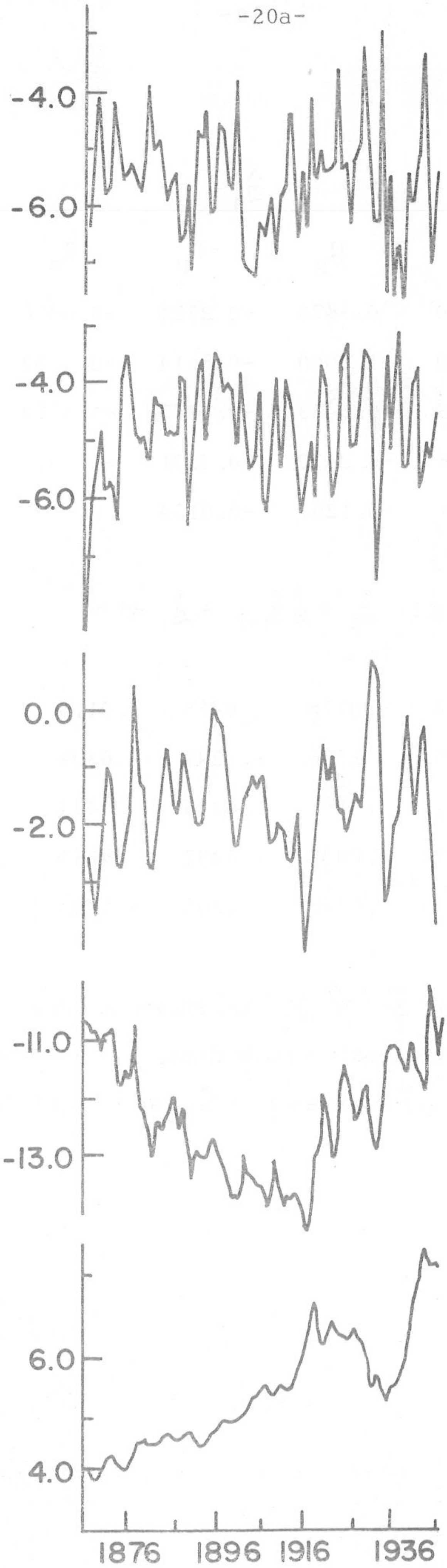

Table 4.2

	$\overset{*}{z}_{1(t-1)}$	$\overset{*}{z}_{2(t-1)}$	$\overset{*}{z}_{3(t-1)}$	$\overset{*}{z}_{4(t-1)}$	$\overset{*}{z}_{5(t-1)}$	$\overset{*}{a}_{jt}$
$\overset{*}{z}_{1t}$	.015	.006	.002	.000	.000	.977
$\overset{*}{z}_{2t}$	.049	.077	.015	.001	.000	.858
$\overset{*}{z}_{3t}$	.001	.100	.401	.002	.002	.494
$\overset{*}{z}_{4t}$	.008	.000	.002	.678	.002	.310
$\overset{*}{z}_{5t}$	.006	.001	.002	.001	.876	.113

We see from the above calculations that there is very little contribution to $\overset{*}{z}_{1t}$ and $\overset{*}{z}_{2t}$ from past history. These two transformed series are essentially white noise. The remarkable feature of $\overset{*}{z}_{3t}$, $\overset{*}{z}_{4t}$ and $\overset{*}{z}_{5t}$ is their heavy dependence on their own past, and this is especially so for the latter two components. It is almost true that $\overset{*}{z}_{4t}$ and $\overset{*}{z}_{5t}$ can be expressed as two independent univariate first order autoregress processes

$$\overset{*}{z}_{4t} = .82\ \overset{*}{z}_{4(t-1)} + \overset{*}{a}_{4t}$$

and

$$\overset{*}{z}_{5t} = .94\ \overset{*}{z}_{5(t-1)} + \overset{*}{a}_{5t} \qquad (4.1)$$

4.2 Interpretation

We see that for the most predictable component $\overset{*}{z}_{5t}$ corresponding to $\hat{\lambda}_5 = .8868$, the autoregressive parameter is close to unity indicating that the series is nearly nonstationary. That is, approximately, $(1-B)\ \overset{*}{z}_{5t} = \overset{*}{a}_{5t}$. Now $\overset{*}{z}_{5t} = \underset{\sim}{m}'_5 \underset{\sim}{z}_t$ and from the estimated eigen vector $\hat{\underset{\sim}{m}}_5$, $\overset{*}{z}_{5t}$ is essentially a linear combination of farm wages (W) and hog supply (H_s),

$$\overset{*}{z}_{5t} \doteq .0039 \, (z_{5t} + .67 \, z_{1t}) \qquad (4.2)$$

It follows that

$$(1 - B) \, z_{5t} + .67 \, (1-B) \, z_{1t} \qquad (4.3)$$

are very nearly randomly distributed about a fixed zero mean, pointing to a stable relationship between the first differences of H_s and W. Since the variables are logged, (4.3) seems to imply that <u>incrementally</u> a percentage increase in hog supply is associated with a percentage decrease in farm wages.

The nearly random components $\overset{*}{z}_1$ and $\overset{*}{z}_2$ (omitting the subscript t) associated with small values of λ are also of considerable interest. Their existence implies that any two linear combinations of the component series in the hyperplane

$$Z = \alpha \overset{*}{z}_1 + \beta \overset{*}{z}_2 = c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4 + c_5 z_5 \qquad (4.4)$$

<u>vary nearly randomly about fixed means</u>. In choosing the component it is natural to seek combinations which are scientifically meaningful.

Now the dollar value of the hogs sold is proportional to $H_p H_s$ and the dollar value of the corn needed to feed them is $R_p R_s$. If then a Z exists involving these dollar values it will be such that approximately $c_1 = c_2$ and $c_3 = c_4$. By least squares or otherwise it is easy to find the linear combination for which this is nearly true. Specifically, by setting $\alpha = 30.01$ and $\beta = 59.51$ we obtain the relationship

$$Z = z_1 + z_2 - 0.78 \, z_3 - 0.73 \, z_4 - 0.48 \, z_5 \quad . \qquad (4.5)$$

That is to say Z in (4.5) is approximately randomly distributed about a fixed mean.

Taking antilogs this implies that

$$\frac{H_p H_s}{(R_p R_s)^{0.75} W^{0.50}} \tag{4.6}$$

is approximately constant. The numerator is obviously a measure of return to the farmer and the denominator a measure of his expenditure. The analysis points to the near constancy of this relation reminding us of the "economic law" that a viable business must operate so as to balance expenditure and income.

Another relationship of considerable interest is that between the hog supply and the ratio of hog to corn prices. It is well known, Wallace and Bressman (1949), that farmers decision on hog production is heavily influenced by this ratio. Now if we choose

$$\alpha = 34.81 \quad \text{and} \quad \beta = -93.01$$

we then obtain

$$Z = .78 z_1 - .64 z_2 + .64 z_3 - .23 z_4 + .11 z_5 \tag{4.7}$$

Upon taking antilogs, this implies that, very approximately,

$$H_s^{.78} \doteq k \left(\frac{H_p}{R_p} \right)^{.64} \tag{4.8}$$

where k is some constant, indicating that a stable relationship existed between hog supply and the price ratio.

4.3 The order of the process

As we mentioned earlier, doubts were raised by Quenouille as to the adequacy of the autoregressive process in representing the Hog data. Following Bartlett (1947) a multivariate goodness of fit test is provided at any given stage of fitting as follows.

Let $\underset{\sim}{S}^{(p)}$ represent the $k \times k$ matrix of sums of squares and cross products of residuals after fitting an autoregressive process of order p in (1.1) to a k-variate series of length n. Then, on the assumption that a process of order not higher than p produces an adequate fit, the necessity for the term of order p may be judged by computing

$$\Lambda = \det \underset{\sim}{S}^{(p)} / \det \underset{\sim}{S}^{(p-1)} . \tag{4.9}$$

An approximate significance test is provided by referring the criterion

$$\mathcal{M} = \{n - 1\tfrac{1}{2} - pk\} \log_e \Lambda \tag{4.10}$$

to a table of χ^2 with k^2 degrees of freedom.

The analysis is given below for the realigned Hog Series. To avoid end effects the fitting of the autoregressive process is begun throughout from the sixth observation. Thus the series length is effectively n = 76 for each series.

Order p of fitted autoregressive process	Criterion $\mathcal{M} = \{74.5-5p\}\log_e \Lambda$
1	398.5
2	94.2
3	29.7
4	44.8
5	44.6

The individual items are distributed approximately as χ^2 with $k^2 = 25$ degrees of freedom. Evidently there is some evidence of inadequacy even of a process of order five. However, it is also clear that a great deal of the variation is being accounted for primarily by the first and to some additional extent by the second order process. In fact, a canonical analysis using the second order autoregressive process have been carried out and the results are very similar to those discussed earlier in Sections 4.1 and 4.2.

5. Further Considerations

5.1 Singularity of the matrix $\underset{\sim}{\Sigma}$

So far it has been assumed that the covariance matrix of $\underset{\sim}{a}_t$, $\underset{\sim}{\Sigma}$, in (2.1) is positive definite. Situation occurs when $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})$ is positive definite but $\underset{\sim}{\Sigma}$ is singular. Specifically, supposing that the rank of $\underset{\sim}{\Sigma}$ is k_1, then it readily follows from (2.5) and (2.9) that the k_2 roots $\lambda_{k_1+1}, \ldots, \lambda_k$ are exactly equal to one, and the transformed covariance $\dot{\underset{\sim}{\Sigma}}$ matrix of $\dot{\underset{\sim}{a}}_t$ in (2.8) takes the form

$$\dot{\underset{\sim}{\Sigma}} = \begin{array}{cc} \left[\begin{array}{c|c} \underset{\sim}{D}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{array}\right] & \begin{array}{c} k_1 \\ k_2 \end{array} \\ \begin{array}{cc} k_1 & \quad k_2 \end{array} & \end{array} \qquad (5.1)$$

where $\underset{\sim}{D}_1$ is an diagonal matrix with positive elements. Partitioning $\dot{\underset{\sim}{z}}_t$, $\dot{\underset{\sim}{a}}_t$ and $\dot{\underset{\sim}{\pi}}_\ell$ as given in (2.12) and (2.14), we see that $\dot{\underset{\sim}{a}}_{2t} = \underset{\sim}{0}$ with probability one. Thus the transformed model is

$$\begin{aligned} \dot{\underset{\sim}{z}}_{1t} &= \sum_{\ell=1}^{p} \dot{\underset{\sim}{\pi}}_{11}^{(\ell)} \dot{\underset{\sim}{z}}_{1(t-\ell)} + \dot{\underset{\sim}{\pi}}_{12}^{(\ell)} \dot{\underset{\sim}{z}}_{2(t-\ell)} + \dot{\underset{\sim}{a}}_{1t} \\ \dot{\underset{\sim}{z}}_{2t} &= \sum_{\ell=1}^{p} \dot{\underset{\sim}{\pi}}_{21}^{(\ell)} \dot{\underset{\sim}{z}}_{1(t-\ell)} + \dot{\underset{\sim}{\pi}}_{22}^{(\ell)} \dot{\underset{\sim}{z}}_{2(t-\ell)} \end{aligned} \tag{5.2}$$

In other words, the k_2-dimensional vector $\dot{\underset{\sim}{z}}_{2t}$ is completely predictable from the past values $\dot{\underset{\sim}{z}}_{1(t-\ell)}$ and $\dot{\underset{\sim}{z}}_{2(t-\ell)}$, $\ell = 1, \ldots, p$.

In practice, situations may occur that $\underset{\sim}{\Sigma}$ is nearly singular. From the results here and those discussed earlier in Section 3.1, we see that for the AR(1) process, certain of the roots λ_j will be nearly equal to one either when some of the eigen values of $\underset{\sim}{\phi}$ approach values on the unit circle or when $\underset{\sim}{\Sigma}$ is nearly singular. The problem of how to distinguish between these two cases is currently being investigated.

5.2 Singularity of the matrix $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})$

Examples can also occur when $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})$ is singular. Consider, for instance, a trivariate series $\{z_{1t}, z_{2t}, z_{3t}\}$ and suppose z_{1t} is labelled "persons employed in the state of Wisconsin in month t", z_{2t} as "persons unemployed", and z_{3t} as the "work force". Then it could happen that z_{3t} is in fact obtained as the simple sum $z_{1t} + z_{2t}$ in which case $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})$ would be necessarily singular. Other examples from the physical sciences are discussed in Box, Erjavec, Hunter and MacGregor (1973). It is not unusual to find exact and quite complex linear relationships imposed by the method in which the data is put together. Two situations can occur depending on whether or not the nature of any exact linear relationships existing in the data is already known.

If known, then the problem may be avoided by eliminating, in advance of the analysis, any dependencies and applying the analysis to a linearly independent subset of r of the k series. This can often be done very easily by removing from immediate consideration any derived series. When the nature of exact relationships in the data which might exist are not known, a principle component analysis of the estimate $\underset{\sim}{C}_0(\underset{\sim}{z})$ of $\underset{\sim}{\Gamma}_0(\underset{\sim}{z})$ should be first conducted. The existence of k-r roots which are nearly zero indicates the existence of k-r linearly independent exact relationships which define a hyperplane in the k space given by the k-r corresponding eigen vectors.

The canonical analysis of this paper may now be carried out on any subset of r linearly independent series which lie in the nonsingular space. Even when it is believed that the nature of any linear dependencies is already known, it is always a wise precaution to carry out the preliminary principle component analysis of $\underset{\sim}{C}_0(z)$ anyway. Occasionally the relationships found are unexpected and can be informative.

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This paper proposes a canonical transformation of a k dimensional stationary autoregressive process. The components of the transformed process are ordered from least predictable to most predictable. The least predictable components are often nearly white noise and the most predictable can be nearly nonstationary. Transformed variables which are white noise can reflect relationships which may be associated with or point to economic or physical laws. A 5-variate example are given.

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