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OPTIMAL STOPPING OF THE SAMPLE DISTRIBUTION FUNCTION

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SUMMARY

Let $F_n(t)$ be the usual sample distribution function obtained by sampling from a uniform (0, 1) distribution. Let $B_n(t) = \sqrt{n}\,(F_n(t) - ct)$, where $c = 1 + \frac{\delta}{\sqrt{n}}$, $-\infty < \delta < \infty$. For $V_n(\delta) = \sup_\tau E(B_n(\tau))$, where the supremum is taken over all stopping times $\tau \leq 1$, we show that $\lim_{n \to \infty} V_n(\delta) = V(\delta)$, where $V(\delta)$ can be computed easily by using a normal distribution table. We find a stopping time σ ($\sigma(B_n)$ = least $t \geq 0$: $B_n(t) \geq \alpha\sqrt{1-t} - \delta$, α a known number) such that $E(B_n(\sigma)) \to V(\delta)$, as $n \to \infty$, $-\infty < \delta < \infty$.

In the course of proving the above, we show that $\sup_{0 \leq t \leq 1} |B_n(t)|$, $n \geq 1$, are uniformly integrable.

Optimal Stopping of the Sample Distribution Function *

1. Introduction:

Let $t_1 \leq t_2 \leq \ldots \leq t_n$ be the order statistics of a random sample from a uniform distribution on [0, 1]. Set $t_o = 0$ and $t_{n+1} = \infty$. For $0 \leq t \leq 1$, define

$$F_n(t) = j/n, \quad t_j \leq t < t_{j+1}, \quad j = 0, 1, 2, \ldots, n, \quad \text{and}$$

$$B_n(t) = B_n(t;\,\delta) = \sqrt{n}\,(F_n(t) - ct),$$

where

$$-\infty < \delta < \infty \quad \text{and} \quad c = (1 + \frac{\delta}{\sqrt{n}})$$

Define

$$V_n(\delta) = \sup_{\tau} E(B_n(\tau;\delta)),$$

where the supremum is taken over all stopping times $\tau \leq 1$.

For $0 \leq t \leq 1$, $-\infty < \delta < \infty$, let $B(t) = B(t;\delta)$ be standard Brownian motion conditioned (pinned) to pass through $-\delta$ at $t = 1$. Let

$$V(\delta) = \sup_{\tau} E(B(\tau;\,\delta)),$$

where the supremum is taken over all stopping times $\tau \leq 1$. Shepp [5] has shown that

$$V(\delta) = E(B(\sigma;\delta)) = \begin{cases} 0, & \delta \geq \alpha \\ -\delta + (1-\alpha^2)\int_0^\infty e^{s\delta - s^2/2}\,ds, & \delta < \alpha, \end{cases}$$

where α is the unique real solution of $\alpha = (1-\alpha^2)\int_0^\infty e^{s\alpha - s^2/2}\,ds$ ($\alpha = .83992\ldots$) and

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$$(1) \qquad \sigma = \sigma(B) = \begin{cases} \text{least } t \geq 0 & \text{such that } B(t) \geq \alpha\sqrt{1-t} - \delta . \\ 1 & \text{otherwise} \end{cases}$$

It is well known that

$$B_n(t;\delta) \overset{\mathcal{D}}{\Longrightarrow} B(t;\delta) , \qquad \text{as } n \to \infty ,$$

(where $\overset{\mathcal{D}}{\Longrightarrow}$ means "converges in distribution to"). This paper will show that the following two statements are true.

$$(2) \qquad \lim_{n\to\infty} V_n(\delta) = V(\delta) , \quad -\infty < \delta < \infty, \quad \text{and}$$

$$(3) \qquad \lim_{n\to\infty} E(B_n(\sigma;\delta)) = E(B(\sigma;\delta)) = V(\delta), \quad -\infty < \delta < \infty .$$

Formula (2) states that "the limit of the values (V_n) equals the value (V) of the limiting process". Since (2) is true, (3) states that "σ is almost optimal for the B_n process when n is large".

2. Proof of (3) .

In order to prove (2) and (3) we will need some of the theory of weak convergence of probability measures. See Billingsley [1] for details. Let

$$W(t) = W(t,\omega), \quad t \geq 0, \quad \omega \in \Omega,$$

be standard Brownian motion on $[0,\infty)$ defined on some probability space $(\Omega, \mathcal{F}, P_1)$. For $-\infty < \delta < \infty$, set

$$(4) \qquad \begin{aligned} B(s) = B(s,\omega;\delta) &= -\delta s + (1-s)\, W\left(\frac{s}{1-s}\right), & 0 \leq s \leq 1 \\ &= -\delta , & s = 1 , \end{aligned}$$

which is consistent with our earlier definition of B.

Let $C = C[0,1]$ be the metric space of continuous functions on $[0,1]$ with metric d given by

$$d(x,y) = \sup_{0 \le t \le 1} |x(t) - y(t)| .$$

$C[0,\infty)$ is the set of continuous functions on $[0,\infty)$. $D = D[0,1]$ is the metric space of functions on $[0,1]$ which are right continuous and have left hand limits. The metric d' on D is of no interest to us except for the following lemma.

Lemma 1. If $x \in C$, $x_n \in D$, $n \ge 1$, and $d'(x_n, x) \to 0$, as $n \to \infty$, then

$$\sup_{0 \le t \le 1} |x_n(t) - x(t)| \to 0, \text{ as } n \to \infty .$$

Proof: See [1], p. 112.

Let

$$g(s) = g_\delta(s) = \alpha\sqrt{1-s} - \delta , \quad 0 \le s \le 1, \text{ and}$$

$$g^*(t) = g^*_\delta(t) = \alpha\sqrt{t+1} - \delta, \quad t \ge 0 .$$

We note that for $x \in D$ and σ given by (1),

$$\sigma(x) = \sigma_\delta(x) = \inf\{t: x(t) \ge g_\delta(t)\} ,$$

where the infimum of the empty set is one. For $x \in C[0,\infty)$, set

(5) $$\pi(x) = \pi_\delta(x) = \inf\{t: x(t) \ge g^*_\delta(t)\},$$

where the infimum of the empty set is ∞. Shepp [5] has shown that

$$P_1\{\pi_\delta(W) < \infty\} = 1, \quad -\infty < \delta < \infty .$$

By (4) above,

(6) $\sigma_\delta(B) = \pi_\delta(W) / (\pi_\delta(W) + 1), \quad -\infty < \delta < \infty$,

hence,

(7) $P_1\{\sigma_\delta(B) < 1\} = 1, \quad -\infty < \delta < \infty$.

Let $\mu_n = \mu_{n,\delta}$ be the measure induced on D by B_n and let $\mu = \mu_\delta$ be the measure induced on D by B. Since $B_n \overset{\mathscr{D}}{\Rightarrow} B$ we say that μ_n converges weakly to μ (written $\mu_n \Rightarrow \mu$).

Let

$A = \{\omega \in \Omega$: $\sigma(B) < 1$ and for every $\epsilon > 0$, there exists a t in the interval $(\sigma(B), \sigma(B) + \epsilon)$ such that $B(t) > g(t)\}$, $-\infty < \delta < \infty$.

We refer to A as the event that "B is not tangent to g at $\sigma(B)$". By (4) and (6) A may be written as

$\{\omega \in \Omega$: $\pi(W) < \infty$ and for every $\epsilon > 0$, there exists, a $t \in (\pi(W), \pi(W) + \epsilon)$ such that $W(t) > g^*(t)\}$, $-\infty < \delta < \infty$.

Henceforth, we restrict ourselves to the case where $-\infty < \delta < \alpha$.

Lemma 2. $P_1(A) = 1, \; -\infty < \delta < \alpha$.

Proof: Since

$\delta < \alpha$, $\pi > 0$ and $W(t) < g^*(t)$, $0 \leq t < \pi$. For $m = 1, 2, \ldots$,

define

$B_m = \{W(t) > g^*(t)$, for some t, $\pi \leq t < \pi + \frac{1}{m}\}$.

$B_m \supset B_{m+1}$ and $A = \bigcap_{m=1}^{\infty} B_m$. Hence

$$P_1(A) = \lim P_1(B_m), \text{ as } m \to \infty .$$

Furthermore,

$$P_1(B_m) \geq P_1 \{W(t) > g^*(\pi + \frac{1}{m}), \text{ for some } t, \pi \leq t < \pi + \frac{1}{m}\} .$$

This implies, by the strong Markov property, that

(8) $$P_1(B_m) \geq P_1 \{ \sup_{0 \leq t \leq \frac{1}{m}} W(t) > \alpha (\sqrt{1 + \frac{1}{m}} - 1)\} ,$$

since $g^*(x + \frac{1}{m}) - g^*(x)$ is a decreasing function of x. The distribution of $\sup_{0 \leq t \leq a} W(t)$ is well known (see, for example [1], p. 72). Hence (8) yields

$$P_1(B_m) \geq 2 P_1 \{W(\frac{1}{m}) > \alpha \sqrt{m} (\sqrt{1 + \frac{1}{m}} - 1) \}$$

$$= 2[1 - \Phi (\alpha/(m(\sqrt{1 + \frac{1}{m}} + 1)))] \to 1 ,$$

$$\text{as } m \to \infty ,$$

where Φ is the distribution function of a N(0, 1) random variable.

Q. E. D.

For $x \in D$, let

$$x(t-) = \lim x(s) , \text{ as } s \uparrow t,$$

$$J^*(x) = \sup_{0 \leq t \leq 1} | x(t) - x(t-)| , \text{ and}$$

$$h(x) = h_\delta(x) = \begin{cases} 0 \text{ if } x(0) \neq 0 \text{ or } J^*(x) > 1 \\ x(\sigma_\delta(x)) , \text{ otherwise.} \end{cases}$$

For any fixed $\delta < \alpha$, h_δ is a bounded function. Furthermore,

(9) $$E(B_n(\sigma)) = \int h(x)\, d\mu_n(x), \quad n \geq 1, \text{ and}$$

(10) $$E(B(\sigma)) = \int h(x)\, d\mu(x) .$$

Let $D_h = D_h(\delta)$ be the set of discontinuities of h_δ. D_h is a measurable subset of D (see [1], p. 30).

Lemma 3. If $\mu_n \Rightarrow \mu$ and h is a real bounded measurable function with

$\mu(D_h) = 0$, then $\int h(x)\, d\mu_n(x) \to \int h(x)\, d\mu$, as $n \to \infty$.

Proof: See [1], p. 31.

In order to apply Lemma 3 to the our problem, we need to show that $\mu(D_h) = 0$. Let

$$C^* = C^*(\delta) = \{x \in C: \sigma(x) < 1,\ x(0) = 0,\ x(1) = -\delta,$$
$$\text{and } x \text{ is not tangent to } g \text{ at } \sigma(x)\} .$$

By Lemma 2 and (7), $\mu(C^*) = 1$. Therefore, it suffices to prove the following.

Lemma 4. If $x \in C^*$, then h is continuous at x.

Proof: Let $x_n \in D$ with $x_n \to x$, as $n \to \infty$. For $x \in C^*$, $0 < \sigma(x) < 1$. Let $\epsilon > 0$ satisfy $\epsilon < \min(\sigma(x), 1 - \sigma(x))$. Since $g - x$ is continuous, there exists a $\delta_1 > 0$ such that

$$g(s) - x(s) > 2\delta_1, \quad 0 \leq s \leq \sigma(x) - \epsilon .$$

Since $x \in C^*$, there exists an r, $\sigma(x) < r < \sigma(x) + \epsilon$, such that $x(r) - g(r) = 2\delta_2 > 0$. Let

$$\delta_3 = \min\left(\frac{1}{4}, \delta_1, \delta_2\right) .$$

By Lemma 1 there exists an N such that for all $n \geq N$,

$$\sup_{0 \le t \le 1} |x_n(t) - x(t)| < \delta_3 ,$$

and $J^*(x_n) < 1$, $n \geq N$, since $\delta_3 \leq \frac{1}{4}$. Thus

$$\sigma(x) - \epsilon < \sigma(x_n) < \sigma(x) + \epsilon , \qquad n \geq N.$$

Since g is strictly decreasing,

$$\delta_3 + g(\sigma(x) - \epsilon) > x_n(\sigma(x_n)) > g(\sigma(x) + \epsilon)), \quad n \geq N.$$

Therefore,

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} x_n(\sigma(x_n)) = g(\sigma(x)) = h(x),$$

Q. E. D.

3. Proof of (2).

Recall that random variables $X_1, X_2, \ldots,$ are uniformly integrable if, and only if,

$$\lim_{\eta\to\infty} \sup_{n\geq 1} \int_{\{|X_n| > \eta\}} |X_n| \, dP = 0.$$

Lemma 5. $X_1, X_2, \ldots,$ are uniformly integrable if, and only if ,

(i) $\sup E|X_n| < \infty$, and

(ii) $\lim_{\eta\to 0} \sup \int_A |X_n| \, dP = 0,$

where the supremum is taken over all sets A with $P(A) < \eta$.

Proof: See [4], p. 62 .

Lemma 6. Let $X_1, X_2, \ldots,$ be random variables. If there is a random variable Y and an $\eta_o > 0$ such that $E|Y| < \infty$ and

$$P(|X_n| \geq \eta) \leq P(|Y| \geq \eta), \quad n \geq 1, \quad \eta \geq \eta_o,$$

then X_n, $n \geq 1$, are uniformly integrable.

Proof: See [1], p. 32.

For $n \geq 1$, $-\infty < \delta < \alpha$, let

$$Y_n = Y_n(\delta) = \sup_{0 \leq t \leq 1} \left| B_n(t; \delta) \right| .$$

Theorem 1. The random variables $Y_n(\delta)$, $n \geq 1$, defined above are uniformly integrable for all δ, $-\infty < \delta < \alpha$.

Proof: First consider the case $\delta = 0$. Let $W_1, W_2, \ldots,$ be independent random variables, each having distribution function

$$G(x) = 1 - e^{-x}, \quad x \geq 0 .$$

If $\xi_n = W_1 + W_2 + \ldots + W_n$, then (see [2], p. 285)

$$Y_n(0) \overset{\mathcal{D}}{=} \sqrt{n} \max_{1 \leq k \leq n} \left| \frac{\xi_k}{\xi_{n+1}} - \frac{k}{n} \right|, \quad n \geq 1.$$

Note that

$$Y_n(0) \overset{\mathcal{D}}{=} \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| (\xi_k - k) + n\left(\frac{\xi_k}{\xi_{n+1}} - \frac{\xi_k}{n}\right) \right|$$

$$\leq \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| \xi_k - k \right| + \sqrt{n} \left(\xi_{n+1} \left| \frac{1}{\xi_{n+1}} - \frac{1}{n} \right| \right)$$

$$= \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| \xi_k - k \right| + \sqrt{n} \left| 1 - \frac{\xi_{n+1}}{n} \right|$$

$$= \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} \left| \xi_k - k \right| + \left| \frac{\xi_n - n}{\sqrt{n}} \right| + \frac{W_{n+1}}{\sqrt{n}} .$$

$\xi_k - k$ is a martingale with $E[(\xi_k - k)^2] = k$. Thus, by the Kolmogorov extension of Chebyshev's inequality ([2], p. 65)

$$P\left(\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} |\xi_k - k| > \eta\right) \leq \frac{1}{\eta^2}, \quad \eta > 0, \ n \geq 1.$$

By Chebyshev's inequality,

$$P\left(\frac{1}{\sqrt{n}} |\xi_n - n| \geq \eta\right) \leq \frac{1}{\eta^2}, \quad \eta > 0, \quad n \geq 1.$$

Finally,

$$P\left(\frac{1}{\sqrt{n}} W_{n+1} > \eta\right) \leq \frac{1}{\eta^2}, \quad n \geq 1,$$

for η sufficiently large. It now follows easily from Lemma 6 that $Y_n(0)$, $n \geq 1$, are uniformly integrable.

For $\delta \neq 0$,

$$B_n(t;\delta) = B_n(t;0) - \delta t, \qquad 0 \leq t \leq 1.$$

Hence, for $n \geq 1$

$$Y_n(\delta) = \sup_{0 \leq t \leq 1} |B_n(t;\delta)| \leq Y_n(0) + |\delta|.$$

Therefore $Y_n(\delta)$, $n \geq 1$, are uniformly integrable.

Q. E. D.

If X_n, $n \geq 1$, are random variables and a_n, $n \geq 1$, are positive constants, then $X_n = O_p(a_n)$ if, and only if,

$$\lim_{\eta \to o} \sup_{n \geq 1} P\left(\left|\frac{1}{a_n} X_n\right| > \eta\right) = O.$$

<u>Lemma 7.</u> There exists a probability space $(\Omega', \mathcal{F}', P')$ with $B_n(t; 0)$ and $B(t; 0)$ defined on it such that

$$X_n(0) = \sup_{0 \leq t \leq 1} | B_n(t; 0) - B(t; 0) | = O_{P'}(n^{-1/4} \sqrt{\log n}) .$$

<u>Proof:</u> See [3].

Since $B_n(t; \delta) = B_n(t; 0) - \delta t$ and $B(t; \delta) = B(t; 0) - \delta t$, the conclusion of Lemma 7 may be replaced by

$$X_n(\delta) = \sup_{0 \leq t \leq 1} | B_n(t; \delta) - B(t; \delta)|$$

$$= O_{P'}(n^{-1/4} \sqrt{\log n})$$

It is well known (see, for example [1], p. 72) that for $X' = \sup_{0 \leq t \leq 1} B(t; 0)$, $E|X'| < \infty$. Furthermore, $X' \overset{\mathcal{D}}{=} - X' \overset{\mathcal{D}}{=} \inf_{0 \leq t \leq 1} B(t; 0)$.

Now let

$$X(\delta) = \inf_{0 \leq t \leq 1} B(t; \delta) .$$

$$0 \geq X(\delta) \geq X(0) - \delta^{+} ,$$

so that

$$E | X(\delta) | < \infty , \quad -\infty < \delta < \alpha .$$

<u>Theorem 2.</u>

$$\lim_{n \to \infty} V_n(\delta) = V(\delta) , \quad -\infty < \delta < \alpha .$$

<u>Proof:</u> We adopt the above notation. In particular, $(\Omega', \mathcal{F}', P')$, B_n, and B are as given by Lemma 7. Fix δ. Given $\eta > 0$, there is an $\epsilon_1 > 0$ such that for every $\epsilon \leq \epsilon_1$,

$$(11) \qquad \sup_{\{A:P'(A) < \epsilon\}} \int_A \sup_{0 \leq t \leq 1} B_n(t;\delta)\, dP' < \eta/3$$

by Theorem 1. Furthermore, there is an $\epsilon_2 > 0$ such that for every $\epsilon \leq \epsilon_2$,

$$(12) \qquad \sup_{\{A:P'(A) < \epsilon\}} \left| \int_A \inf B(t;\delta) dP' \right| < \eta/3 ,$$

since the integrand is integrable. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. By Lemma 7 there is a Θ such that for every $\theta \geq \Theta$

$$(13) \qquad P'\{ \sup_{0 \leq t \leq 1} |B_n(t;\delta) - B(t;\delta)| \geq \theta n^{-1/4} \sqrt{\log n} \} \leq \epsilon ,$$

for every $n \geq 1$.

Fix $\theta \geq \Theta$. (13) implies that

$$P'\{B_n(t) \leq B(t) + \theta n^{-1/4} \sqrt{\log n} ,\ 0 \leq t \leq 1\} \geq 1 - \epsilon, \text{ for every } n \geq 1.$$

Set

$$A = A_{n,\epsilon} = \{B_n(t) \leq B(t) + \theta n^{-1/4} \sqrt{\log n} ,\ 0 \leq t \leq 1 \} .$$

The optimal stopping time for the process $B_n(t)$ is

$$\begin{aligned} \tau_n &= \text{least } t \geq 0 \text{ such that } B_n(t) \geq g_n(t), \\ &= 1 \quad , \text{ otherwise} \end{aligned}$$

(see, e.g. [6]), where all that is known about g_n is that it can be taken to be continuous, nonincreasing, and $g_n(1) = -\delta$. Consider the stopping time τ_n^* for the B process given by

$$\tau_n^* = \text{least } t \geq 0 \quad \text{such that } B(t) \geq g_n(t) - \theta n^{-1/4} \sqrt{\log n} \ .$$

On the set A, $\tau_n^* \leq \tau_n$ and

$$B_n(\tau_n) \leq B(\tau_n^*) + \theta n^{-1/4} \sqrt{\log n} + n^{-1/2}$$

(Note: $n^{-1/2}$ is the maximum "excess over the boundary" g_n of B_n at time τ_n). Therefore,

$$(14) \qquad V_n(\delta) = E(B_n(\tau_n)) = \int_A B_n(\tau_n) + \int_{A^c} B_n(\tau_n)$$

$$\leq \int_{\Omega'} B(\tau_n^*) - \int_{A^c} B(\tau_n^*) + \theta n^{-1/4} \sqrt{\log n} + n^{-1/2}$$

$$+ \int_{A^c} \sup_{0 \leq t \leq 1} B_n(t).$$

Pick N sufficiently large that

$$\theta n^{-1/4} \sqrt{\log n} + n^{-1/2} \leq \eta/3 \ , \quad n \geq N \ .$$

By (11), (12), and (14),

$$V_n(\delta) \leq E(B(\tau_n^*)) + \eta \leq V(\delta) + \eta \ , \quad n \geq N.$$

or, since η is arbitrary

$$\limsup_{n \to \infty} V_n(\delta) \leq V(\delta) \ ,$$

On the other hand, by (3),

$$V_n(\delta) \geq E(B_n(\sigma)) \to V(\delta) \ , \quad n \to \infty \quad .$$

Thus $\lim_{n \to \infty} V_n(\delta) = V^*(\delta)$, $-\infty < \delta < \alpha$. Q. E. D.

It is easy to see that for $n, \ m \geq 1$,

$$0 \leq V_n(\alpha) \leq V_n(\alpha - \frac{1}{m}) \quad .$$

Therefore

$$0 \leq \limsup_{n \to \infty} V_n(\alpha) \leq V(\alpha - \frac{1}{m}) \to 0 \ , \quad \text{as} \quad m \to \infty \ .$$

This gives us $V_n(\delta) \to 0 = V(\delta)$, as $n \to \infty$, for every $\delta \geq \alpha$. (It is easy to see that $V_n(\delta)$ is non-increasing in δ).

4. Remarks . In [6] we show how to compute $V_n(\delta)$ for small n and all δ, $-\infty < \delta < \infty$. We compare our exact (numerical) results for $V_n(\delta)$ with $V(\delta)$ for some choices of n and δ in Table 1. Applications of Theorem 2 are also given in [6] and will appear in a later report.

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Table 1

Comparison of $V_n(\delta)$ and $V(\delta)$.

	δ							
		-1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1
	1	1.0000	.7500	.6667	.5000	.3333	.2500	0
	4	1.2000	.8074	-	.4493	-	.1606	0
	9	1.2144	-	.6690	.4267	.2192	-	0
n	16	1.2148	.7924	-	.4142	-	.1205	0
	25	1.2132	-	-	.4062	-	-	0
	36	1.2115	.7835	.6494	.4006	.1904	.1068	0
	$V(\delta)$	1.1930	.7569	.6219	.3688	.1589	.0774	0

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