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A FINITE DAM WITH POISSON INPUT AND POISSON RELEASE

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A Finite Dam with Poisson Input and Poisson Release

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1. INTRODUCTION. In a recent paper, Puri and Senturia [5] study the content $Z(t)$ of a reservoir with infinite depth. In that model instantaneous inputs and releases occur at random times such that their number, $N(t)$, in the interval $(0,t)$, for $t>0$ is a Poisson process with parameter $\lambda+\mu$, $\lambda,\mu>0$. The inputs form a sequence of independent nonnegative random variables $\{X_n\}$, which are independent of $N(t)$ and have a common distribution function B. In addition, independent of the $\{X_n\}$, the releases form a sequence of nonnegative independent random variables $\{Y_n\}$, which are independent of $N(t)$ and have a common distribution function D, negative exponential with parameter $\beta>0$. The reservoir has capacity h and is therefore full when $Z(t) = h$ for some $t\geq 0$. If an input at time τ_1, say, of random amount X exceeds $h-Z(t)$, the deficiency of the reservoir, an instantaneous overflow occurs, the level of the reservoir remaining at $Z(t) = h$ until the occurence of the next release, Y. Puri and Senturia obtain the Laplace transform of the content distribution for an initially full reservoir and for an arbitrary initial content. This was done for the case in which B is a general distribution function, while D is a negative exponential distribution function with parameter $\beta>0$. These content distributions, it is shown, tend as $t\to\infty$ to a limit distribution independent of the initial conditions.

One motivation for studying such storage models is an attempt to extend further a mathematical theory of quantal response assays. The classical theory developed by Finney [2], among others, depends upon a threshold level (tolerance limit) assumption. A new non-threshold approach to quantal response assays was presented by Puri and Senturia (see [5]). This approach utilizes a continuous time stochastic process which describes, for $t \geq 0$, the effective level of a drug in the subject's body. Such a stochastic process is developed along with, yet independent of, an appropriate response indicating stochastic process. The quantal response assay procedure considered in [5] is one in which at time $t = 0$ a single fixed dose, z, of some noxious substance is administered to the subject. There are, however, situations in which one would like to determine dose response relationships when repeated exposures to some noxious agent occur. In many industrial occupations, as for example coal mining, or asbestos product fabrication, such a condition obtains. In order to extend the applicability of the theory developed in [5] to these cases, it is appealing to turn attention to the models in storage theory. Moreover, because of the biological nature of the systems involved in quantal response assays it is important that we focus attention on finite rather than infinite capacity models.

One finds, however, that the vast literature of storage theory contains relatively few models that are applicable to the problem of quantal response assays. One reason for this is that in storage theory the assumption of infinite capacity has often been made. Unfortunately, this assumption is made not so much to satisfy the realities of the situation as to make subsequent analysis more tractable. Some infinite capacity storage models relevant to the present problem are discussed in [6]. Most of them are characterized by a deterministic release rule, which, as indicated in [6], is inappropriate for a quantal response model.

The situation is much the same when one turns to finite capacity models. One finds finite models, such as those of Roes ([7], [8]), Odoom, and Lloyd ([3]), and Ali Khan ([1]), all of which retain the same assumption of deterministic release, that is, unit release per unit time, as is made in most infinite capacity models. Thus the objection raised in [6] to the applicability of these models to this biological phenomenon remains. What one would like is a finite capacity storage model with the inputs and releases comprising sequences (either independent or dependent) of random variables.

In the present paper, to these ends, a finite version of the storage model studied by Puri and Senturia [4] is considered. Exact distributions of the content, $Z(t)$, are derived for an initially full reservoir, for an initially empty reservoir, and for a reservoir with an arbitrary initial content z, $0<z<h$. Limiting distributions, as $t\to\infty$, are also derived, and turn out, as expected, to be independent of the initial content z.

Let $\{Z(t), t>0\}$ be a stochastic process which represents the content of a reservoir with finite depth. That is, $Z(t)$

takes values in the interval $[0,h]$, $h<\infty$. The process $Z(t)$ is defined constructively as follows. Initially $Z(0) = z$, $0 \leq z \leq h$. The process $Z(t)$ remains constant at the level z for a random length of time, whose distribution function is

$$H(t) = \begin{cases} 1-\exp\{-(\lambda+\mu)t\}, & t \geq 0 \\ 0 & t<0 \end{cases}$$

where $\lambda,\mu>0$. At the end of this random length of time the process $Z(t)$ jumps to a new level. We shall say that such a jump is, with probability $\lambda/(\lambda+\mu)$, an instantaneous input, X, to the reservoir and is, with probability $\mu/(\lambda+\mu)$, an instantaneous release, Y, from the reservoir. The reservoir is full when the content attains the value h. If the input, X, exceeds $h-z$, or the deficiency of the reservoir, an instantaneous overflow occurs, so that $Z(t)$ takes the value h until the occurrence of a release. If the release, Y, exceeds the content at that instant, then only the available amount is released, and $Z(t)$ takes the value 0 until the occurrence of an input. The process continues in this manner, the waiting times between jumps of the process all following the same distribution H. We assume that the sequences $\{X_n\}$ and $\{Y_n\}$ are independent of each other and of the random waiting times. The X_n are assumed to be independent nonnegative random variables with common distribution function $B(x)$. The Y_n are assumed to be independent nonnegative random variables with common distribution function $D(y)$.

2. THE PROCESS Z(t).

2.1. AN INTEGRAL EQUATION FOR THE PROCESS Z(t).

The following notation will be used throughout.

$$W(t,z,x) = P(Z(t)\leq x | Z(0) = z),\ x<h$$

$$\Phi(\theta,z,x) = \int_0^\infty \exp(-\theta t)W(t,z,x)dt,\ Re(\theta)>0$$

$$I(x) = \begin{matrix} 1 & \text{for } x>0 \\ 0 & \text{otherwise} \end{matrix}$$

For $x<0$, $W(t,z,x) = 0$, while for $x\geq h$ $W(t,z,x) = 1$. Let $N(t)$ denote the number of jumps of the process $Z(t)$ in the interval $(0,t]$. Then $N(t)<\infty$ almost surely. The forward Kolmogorov integral equation for $W(t,z,x)$ is then valid, and we may concentrate on the nature of the last jump of the process $Z(t)$ before time t. The following forward Kolmogorov integral equation for $W(t,z,x)$ for the case $0<x<h$ can be readily established.

$$\begin{aligned}(1)\qquad W(t,z,x) &= I(x-z)\exp\{-(\lambda+\mu)t\} \\ &+ \mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^x W(t,z,x-y)dB(y) \\ &+ \mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^{h-x} W(t,z,x+y)dD(y) \\ &+ \mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)d\tau\,[1-D(h-x)].\end{aligned}$$

Converting equation (1) into its Laplace transform we have, for $0<x<h$ and $Re(\theta)>0$,

$$(2) \qquad \Phi(\theta,z,x)(\lambda+\mu+\theta) = I(x-z) + \lambda\int_0^x \Phi(\theta,z,x-y)\,dB(y) + \mu\left[\int_0^{h-x} \Phi(\theta,z,x+y)\,dD(y) + [1-D(h-x)]\theta^{-1}\right].$$

The existence and uniqueness can be shown by the principle of contraction mappings. The solution of equation (2) appears to be rather complicated in the present general form. A tractable solution is possible in the case in which B is a negative exponential distribution function with parameter $\alpha>0$ and D is a negative exponential distribution function with parameter $\beta>0$. We turn to this special case in the next sub-section.

2.2 SOLUTION FOR THE TRANSFORM $\Phi(\theta,z,x)$.

From now on we assume $B(y) = 1-\exp(-\alpha y)$, $y\geq 0$, 0 otherwise and $D(y) = 1-\exp(-\beta y)$, $y\geq 0$, 0 otherwise, and first exhibit the solution of (2) in the case $z = 0$.

THEOREM 1. *For $z = 0$ equation (2) has the unique solution, for $0<x<h$,*

$$(3) \qquad \Phi(\theta,0,x) = \theta^{-1} + (\alpha\theta)^{-1}(r_1+\alpha)\{\exp(r_1 h)(r_1-\beta)^{-1} \cdot [\exp(r_2 h)(r_2-\beta)^{-1} - (r_1+\alpha)\exp(r_1 h)(r_2+\alpha)^{-1}(r_1-\beta)^{-1}]^{-1} \cdot [\exp(r_2 x) - \exp(r_1 x)(r_1+\alpha)(r_2+\alpha)^{-1}] - \exp(r_1,x)\},$$

with

$$\int_0^\infty \exp(-\theta t)P(Z(t) = 0|Z(0) = 0)dt = \theta^{-1}[1-\alpha^{-1}(r_1+\alpha)\{\exp(r_2h)/(r_2-\beta) - \exp(r_1h)/(r_1-\beta)\}\{\exp(r_2h)/(r_2-\beta) - \exp(r_1h)(r_1+\alpha)(r_2+\alpha)^{-1}(r_1-\beta)^{-1}\}^{-1}], \tag{4}$$

where $Re(\theta)>0$ *and* r_1 *and* r_2 *are given with plus and minus sign respectively by*,

$$r_1(\theta),\ r_2(\theta) = \frac{-A(\theta) \pm [(A(\theta))^2 + 4\alpha\beta\theta(\lambda+\mu+\theta)]^{\frac{1}{2}}}{2(\lambda+\mu+\theta)} \tag{5}$$

where

$$A(\theta) = \alpha(\mu+\theta)-\beta(\lambda+\theta).$$

Proof: We show that one solution of (2) is of the form $\Phi(\theta,0,x) = \theta^{-1} + C_1(\theta)\exp(r_1h) + C_2(\theta)\exp(r_2h)$ with r_1 and r_2 as in (5). Substituting this form of the solution into (2) we obtain an identity in x. Comparing the coefficients of $\exp(r_1x)$, of $\exp(r_2x)$, of $\exp(-\alpha x)$ and of $\exp(\beta x)$ on both sides of this identity we obtain the three relations

$$C_1(\theta)(r_1+\alpha)^{-1} + C_2(\theta)(r_2+\alpha)^{-1} + (\alpha\theta)^{-1} = 0, \tag{6}$$

and

$$C_1(\theta)\exp(r_1h)(r_1-\beta)^{-1} + C_2(\theta)\exp(r_2h)(r_2-\beta)^{-1} = 0, \tag{7}$$

$$\lambda+\mu+\theta - \alpha\lambda(r+\alpha)^{-1} + \mu\beta(r-\beta)^{-1} = 0. \tag{8}$$

Equation (8) has the two roots r_1 and r_2 given in (5). Once these roots have been determined it is a straight forward matter to determine $C_1(\theta)$ and $C_2(\theta)$ from equations (6) and (7). These yield the result in (3). The uniqueness of solution of (2) guarantees that (3) is the only solution for the case $z = 0$.

Finally, (4) follows from

$$\int_0^{\infty} \exp(-\theta t) P(Z(t) = 0 | Z(0) = 0) dt = \lim_{x \to 0^+} -0.$$

We shall next state a lemma which is essential in the case $0<z<h$. Let H be an arbitrary function, defined on the nonnegative half of the real line, which is integrable in every finite subinterval of that half line and which can be expressed as the difference of two monotone nondecreasing functions. Let also

$$K(s) = \sum_{n=0}^{\infty} H^{(n)}(s),$$

$$H^{(0)}(s) \equiv 1, \quad H^{(n)}(s) = \int_0^s H^{(n-1)}(s-u)\, dH(u), \quad 0 \leq s \leq h-z, \quad n=1,2,\ldots \; . \tag{9}$$

<u>LEMMA 1</u>. (i) <u>The Volterra type equation</u>

$$F(\xi) = a + \int_0^{\xi} F(\xi - y)\, dH(y), \quad 0 \leq \xi \leq b < \infty \tag{10}$$

<u>in</u> F, <u>where</u> H <u>is given and has the properties listed above and</u> a <u>is a given constant, has solution given by</u>

$$F(\xi) = aK(\xi) \tag{11}$$

<u>and this solution is unique, provided</u> $K(\xi)$ <u>converges uniformly in</u> $0 \leq \xi \leq b < \infty$.

(ii) *Let B be such that* $B(s)/s\leq A<\infty$ *for* $0<s\leq\varepsilon$, *for some* $\varepsilon>0$ *and some constant* $A>0$. *Then for*

(12) $$H(s) = [\lambda B(s) + \mu(1-\exp(\beta s))](\lambda+\mu+\theta)^{-1}, \quad 0\leq s\leq h-z,$$

where z *is fixed but otherwise arbitrary*, K(s) *exists* and is finite for $0\leq s\leq h-z$.

Lemma 1 was proved in [4], where it was noted that when B has a density the condition $B(s)/s\leq A<\infty$ for $0<s\leq\varepsilon$, some $\varepsilon>0$, $A>0$, is satisfied and the lemma holds. In our model, B possesses a density and thus the uniform convergence of K is guaranteed. With that we turn to the case $0<z<h$ in the following

<u>THEOREM 2</u>. *For* $0<z<h$ *equation* (2) *has the unique solution, for* $z\leq x<h$,

(13) $$\Phi(\theta,z,x) = \begin{cases} \theta^{-1}+C_1(\theta)\exp(r_1x)+C_2(\theta)\exp(r_2x)-(\lambda+\mu+\theta)^{-1}K(z-x), & 0\leq x<z \\ \theta^{-1}+C_1(\theta)\exp(r_1x)+C_2(\theta)\exp(r_2x) & , z\leq x<h \end{cases}$$

where

(14) $$C_1(\theta) = [(r_2+\alpha)^{-1}-(r_1-\beta)\exp\{(r_2-r_1)h\}(r_1+\alpha)^{-1}(r_2-\beta)^{-1}]^{-1}\cdot$$
$$\cdot[(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z\exp(\alpha v)K(z-v)dv]\cdot$$
$$\cdot(r_1-\beta)\exp\{(r_2-r_1)h\}(r_2-\beta)^{-1},$$

(15) $$C_2(\theta) = -[(r_2+\alpha)^{-1}-(r_1-\beta)\exp\{(r_2-r_1)h\}(r_1+\alpha)^{-1}(r_2-\beta)^{-1}]^{-1}\cdot$$
$$\cdot[(\alpha\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_0^z\exp(\alpha v)K(z-v)dv],$$

<u>and</u> r_1, r_2 <u>as</u> <u>in</u> (5), K <u>as</u> <u>in</u> (9) <u>with</u>,

(16) $$H(u) = [\mu(1-e^{-\beta u}) - \lambda(1-e^{\alpha u})](\lambda+\mu+\theta)^{-1}, \quad 0\le u\le h-z.$$

The proof of Theorem 2 employs the exact same technique as in Theorem 2 in [4]. That is, first Φ is designated as

$$\Phi(\theta,z,x) = \begin{cases} \Phi_1(\theta,z,x) & \text{for } 0<x<z \\ \Phi_2(\theta,z,x) & \text{for } z\le x<h \end{cases}.$$

Then noting that the solution Φ_2, to (2) in the case z=0 has the form

(17) $$\Phi_2 = \theta^{-1} + C_1(\theta)\exp(r_1x) + C_2(\theta)\exp(r_2x), \quad z\le x\le h,$$

we construct a solution to (2) by putting Φ_2 as in (17) and setting

$$\Phi_2(\theta,z,x) = \theta^{-1} + C_1(\theta)\exp(r_1x) + C_2(\theta)\exp(r_2x) + g(\theta,z,x), \quad 0\le x<z,$$

(18)

where $g(\theta,z,x)$ is a function to be determined. The same Volterra type equation,

$$\Gamma(\theta,s) = (\lambda+\mu+\theta)^{-1} + \int_0^s \Gamma(\theta,s-y)\,dH(y),$$

as obtained in [4] Theorem 2 results where now H is given in (16). The solution to this Volterra type equation is provided by Lemma 1. It is then a straightforward matter to determine the constants $C_1(\theta)$ and $C_2(\theta)$ following the same method as in [4] to which the interested reader is referred for details.

The case $z = h$ is now given in

THEOREM 3. *For* $z = h$ *equation* (2) *has the unique solution, for* $0<x<h$,

$$\Phi(\theta,h,x) = (\beta\theta)^{-1}[\exp(r_2h)/(\beta-r_2)-\exp(r_1h)(r_1+\alpha)(\beta-r_1)^{-1}(r_2+\alpha)^{-1}]^{-1}\cdot[\exp(r_2x)-\exp(r_1x)(r_1+\alpha)/(r_2+\alpha)], \tag{19}$$

with

$$\int_0^\infty \exp(-\theta t)P(Z(t) = h|Z(0) = h)dt = (\beta\theta)^{-1}[\exp(r_2h)/(\beta-r_2)-\exp(r_1h)(r_1+\alpha)(r_2+\alpha)^{-1}\cdot(\beta-r_1)^{-1}]^{-1}[\exp(r_2h)r_2/(\beta-r_2)-\exp(r_1h)r_1(r_1+\alpha)(r_2+\alpha)^{-1}(\beta-r_1)^{-1}], \tag{20}$$

where $\mathrm{Re}(\theta)>0$ *and* r_1 *and* r_2 *are given in* (5).

The proof is omitted. The solution follows the same technique as that in Theorem 1 except that we exhibit a solution in the form $\Phi = C_i\exp(r_1x) + C_2\exp(r_2x)$ with r_1 and r_2 as in (5). In addition, (20) follows from

$$\int_0^\infty \exp(-\theta t)P(Z(t) = h|Z(0) = h)dt = \theta^{-1}-\lim_{x\uparrow h}\Phi(\theta,h,x).$$

From Theorems 1,2, and 3 one can derive the moments of $Z(t)$. Let, for $\mathrm{Re}(s)>0$,

$$c(t,s) = \int_{-\infty}^\infty \exp(sx)d_xW(t,z,x) = \exp(hs)-s\int_0^h \exp(sx)W(t,z,x)dx .$$

Then

$$\int_0^\infty \exp(-\theta t)c(t,s)dt = \theta^{-1}\exp(hs)-s\int_0^h \exp(sx)\int_0^\infty \exp(-\theta t)W(t,z,x)dtdx$$

by virtue of the integrability of W(t,z,x) and Fubini's theorem. From (13) it then follows that

$$\begin{aligned}\int_0^\infty \exp(-\theta t)c(t,s)\,dt = \theta^{-1}-s\{&C_1(\theta)(r_1+s)^{-1}(\exp\{(r_1+s)h\}-1)\\ &+C_2(\theta)(r_2+s)^{-1}(\exp\{(r_2+s)h\}-1)\\ &-(\lambda+\mu+\theta)^{-1}\int_0^z \exp(sx)K(z-x)dx\},\end{aligned} \tag{21}$$

where $C_1(\theta)$ and $C_2(\theta)$ are the same as in (14) and (15) respectively. Thus

$$\begin{aligned}\int_0^\infty \exp(-\theta t)E[Z(t)|Z(0)=z]dt &= \frac{\partial}{\partial s}[\int_0^\infty \exp(-\theta t)c(t,s)dt\}\Big|_{s=0}\\ &= (\lambda+\mu+\theta)^{-1}\int_0^z K(z-x)dx-C_1(\theta)r_1^{-1}(\exp(r_1h)-1)-C_2(\theta)r_2^{-1}(\exp(r_2h)-1),\end{aligned} \tag{22}$$

and similarly

$$\begin{aligned}&\int_0^\infty \exp(-\theta t)E[Z(t)|Z(0)=z]dt\\ &= 2[(\lambda+\mu+\theta)^{-1}\int_0^z xK(z-x)dx+C_1(\theta)r_1^{-1}\{\exp(r_1h)(r_1^{-1}-h)-1\}\\ &\quad +C_2(\theta)r_2^{-1}\{\exp(r_2h)(r_2{}^{-1}-h)-1\}]\ .\end{aligned} \tag{23}$$

The limit behavior of Z(t) at $t\to\infty$ is given in the following theorem.

THEOREM 4. *Under the conditions of Theorem 2*

$$\lim_{t\to\infty} P(Z(t) \le x) = \psi(x), \quad 0 \le x \le h,$$

independent of the value of z, *where the distribution* ψ *is given for* $0 \le x \le h$ *by*

$$\psi(x) = \begin{cases} (\alpha+\beta)[\beta(\lambda+\mu)\{\mu^{-1}\exp\{-ph\}-\alpha(\beta\lambda)^{-1}\}]^{-1} \cdot [\exp\{-px\}-\alpha(\lambda+\mu)\{\lambda(\alpha+\beta)\}^{-1}], & \alpha\mu>\beta\lambda \\ \{\beta[\beta^{-1}-\lambda(\alpha\mu)^{-1}\exp(-ph)]\}^{-1} \cdot \{1-\lambda(\alpha+\beta)\exp(-px)[\alpha(\lambda+\mu)]^{-1}, & \alpha\mu<\beta\lambda, \end{cases}$$

(24)

where $p = (\alpha\mu-\beta\lambda)/(\lambda+\mu)$.

Proof: By a standard Tauberian argument (Widder[9], p. 192)

$$\psi(x) = \lim_{\theta\downarrow 0} \theta\Phi(\theta,z,x) \text{ for } 0 \le x \le h.$$

Applying this argument to Φ in (3), (13), and (19) we arrive at (24). Thus the limit is independent of the initial condition $Z(0) = z$.

The interpretation of this limit is that if average inputs per unit time exceed average releases per unit time, then $Z(t)$ has a nondegenerate limiting distribution with positive mass at h. Moreover, if average inputs per unit time are less than average releases per unit time, then $Z(t)$ also has a nondegenerate limiting distribution with positive mass at 0.

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13. ABSTRACT

This paper focuses on a model of a reservoir with finite capacity and with instantaneous inputs and releases occurring at random times such that their number in a given time interval is a Poisson process. The inputs form a sequence of independent identically distributed non-negative random variables as do the releases. Exact distributions of the content of the reservoir are derived for an initially full reservoir, for an initially empty reservoir, and for a reservoir with an arbitrary initial content. Limiting distributions, as time tends to infinity, are also derived and turn out to be independent of the initial content.

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