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TECHNICAL REPORT NO. 318

December 1972

CHERNOFF EFFICIENCY OF LINEAR RANK STATISTICS

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Research was supported in part by the National Science Foundation under Grant G.P. 12093, the Air Force under Grant AF-AFOSR 69-1803 and The Office of Naval Research under Grant ONR N00014-67-A-0128-0017.

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0. Introduction and summary. Traditionally, the asymptotic relative efficiency (ARE) of one test of a statistical hypothesis relative to another can be computed by holding the significance level fixed, and, as the sample size increases, comparing the power of the tests along a sequence of alternatives which will tend to the hypothesis. This method, proposed by Pitman [21] , has been extremely fruitful, and is based on theory allied with the central limit theorem. There have been other methods suggested by varoius authors. For example, Hodges and Lehmann [16] considered a fixed alternative, and compared the rates at which the type II error tends to zero when the significance level is held fixed. Bahadur [2] developed a method of comparing tests, which is equivalent to holding the power, at a fixed alternative, bounded away from zero and one, and comparing the rates at which the significance levels tend to zero. How reliable is Pitman efficiency? Dixon [6] , [7] has emphasized that a comprehensive efficiency comparison of two tests cannot be made with a single number. Thus the Bahadur efficiency, which is a curve of values, is more informative. However, the question arises as to how comparable the Bahadur limit is to the finite sample values. For the Wilcoxon two sample rank test, the Bahadur limit (computed by Hoadley [15] , Woodworth [27] , or Stone [26]) does not compare closely with the finite sample values

(computed by Milton [20]) and so is somewhat unsatisfactory. In particular, The Bahadur limit decreases to zero too rapidly with large alternatives to be realistic when compared with the finite sample efficiencies. The Hodges-Lehmann efficiency may give better comparisons but suffers in its very definition from a lack of realism in that it keeps the more serious type I error fixed and permits the less serious type II error to go to zero with increasing sample size.

Thus it is hoped that the Chernoff efficiency 4 , which is defined at a fixed alternative by letting both the type I and type II errors go to zero at roughly the same (exponential) rate, will give more realistic efficiency comparisons.

The Bahadur, Hodges-Lehmann, and Chernoff efficiencies are based on the theory of large deviations which has been developed extensively only in the case of statistics which are sums of independently, identically distributed (i.i.d.) random variables. Lately, the theory of large deviations has been extended by Sanov [22], Klotz [18], Sethuraman [23], [24], Hoadly [14], Stone [26], Abrahamson [2], Siever [25], Woodworth [27], and others, to include statistics which are not sums of i.i.d. random variables. The Sanov-Hoadley-Woodworth approach is based on a theorem of Hoeffding [17], and uses the multinomial distribution with approximations. The Chernoff-Feller[9]-Klotz-Stone-Siever approach uses a moment generating function argument. In this thesis Hoadley's result is extended and applied to obtain the asymptotic relative efficiency of Chernoff for the Wilcoxon statistic. In addition, methods are outlined for extension to other linear rank statistics.

In section 1 of this paper, Hoadley's theory is applied to the Chernoff-Savage linear rank statistics [5], extended slightly with suitable restrictions to include unbounded score functions both under the null and alternative hypotheses. In section 2, theorems are given to estimate the probability of a large deviation(information number) of the Wilcoxon test and Fisher-Yates(normal score) test under the null and alternative hypotheses. In section 3, the Chernoff efficiency is discussed; and its relation to information numbers is established. The information indices, critical values, and Chernoff efficiencies of the Wilcoxon test relative to two-sample t-test for various normal shift alternatives are presented in Tables 3.1,4.3 and Figure 4.4 respectively. In section 4, a definition of efficiency for small sample theory that may be used for rough comparison with the asymptotic relative efficiency is discussed. The finite sample efficiency of the Wilcoxon, at various normal shift alternatives, relative to the two-sample t-test is computed for equal samples of size seven. The efficiencies are given in Tables 4.1,4.2 and Figure 4.5 respectively. Comparisons and discussions follow . In section 5, a detailed computation procedure is described.

1. The probability of large deviations for linear rank statistics. In order to state the main results of this section, it will be necessary to first give the basic notations, definitions, assumptions and preliminary lemmas. Let $X_1, X_2, ---, X_m$ and $Y_1, Y_2, Y_3, ---, Y_n$ be random samples of sizes m and n drawn from populations with cdf's F_0 and G_0 respectively. We assume there is a $\rho_0 > 0$ such that $0 < \rho_0 < \rho_N < 1 - \rho_0$ where $N = m + n$, $N\rho_N = m$ and $\rho_N \to \rho$ as $N \to \infty$. Those notations will be used throughout the paper. Further, we assume that F_0 and G_0 are continuous.

Define

$$W_{N_i} = \begin{cases} 1 & \text{if the i-th smallest observation in the combined sample is an X} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

We will be concerned with statistics of the following form, the so called Chernoff-Savage statistics:

$$mT_N = \sum_{i=1}^{N} E_{N_i} W_{N_i} \tag{1.2}$$

Where E_{N_i} are given constants. Many two-sample statistics occuring in nonparametric inference can be reduced to this form. Govindarajulu, LeCam and Raghavachari [10] use the following representation:

$$(1.3) \qquad T_N(F_{0m}, G_{0n}) = \int_{-\infty}^{\infty} J_N(NH_{0N}(x)/N+1)\,dF_{0m}(x)$$

where $F_{0m}(x)$=(number of $X_i \leq x)/m$, $G_{0n}(x)$=(number of $Y_j \leq x)/n$ and $H_{0N}(x) = \rho_N F_{0m}(x) + (1-\rho_N) G_{0m}(x)$. (1.2) and (1.3) are equivalent when $E_{N_i} = J_N(i/N+1)$. It will be assumed hereafter that the sequence $J_N(.)$ satisfies the following

Property B.

(i) For each N, J_N is constant over the intervals $[i/N+1, (i+1)/N+1)$, for $i=0,1,---,N$.

(ii) There exists a score function J over (0,1) such that

$$\frac{1}{N+1} \sum_{i=1}^{N} \left| J_N(i/N+1) - J(i/N+1) \right| \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty .$$

(iii) $J(.)$ is continuous, integrable over (0,1) and $|J(.)|$ is non-increasing on $(0,\frac{1}{2}]$ and non-decreasing on $(\frac{1}{2},1)$.

The following result holds for $J(.)$.

Lemma 1.1. For given $\varepsilon > 0$, there exist L, U; $0 < L < U < 1$, and $0 < M < \infty$ depending on ε such that

$$\int_0^L |J(u)|\,du + \int_U^1 |J(u)|\,du < \varepsilon$$

and $|J(u)| \leq M$ for $u \in [L,U]$.

Definition 1.2. A sequence of statistics $\{T_N\}$ defined in (1.3) satisfying Property B, will be called a type B sequence of Chernoff-Savage statistics.

In [22], Sanov shows that if F_{0N} is the empirical cdf of a sample of size N drawn from a population with cdf F_0, and Ω is a certain well-behaved set of cdf's not containing F_0, then

(1.4) $$\lim_{N\to\infty} N^{-1} \ln P\{F_{0N}\in\Omega\} = -\inf_{F\in\Omega} \int \ln(dF/dF_0)dF.$$

Hoadley [14] extended this theory to the c-sample case where the set of cdf's in question depends on N. More precisely, if D is the set of cdf's on $(-\infty,\infty)$ and with F_{0m} and G_{0n} the empirical cdf's for F_0 and G_0, then Hoadley's results (Theorem 1.1 of [14]) restated, for c=2, is the following

Theorem 1.3. *For the two-sample problem, if* $\Omega \subset D\times D$ *is* Q_0-*regular, then*

(1.5) $$\lim_{N\to\infty} -N^{-1} \ln P\{(F_{0m},G_{0n})\in\Omega\}$$

$$= \inf_{(F,G)\in\Omega}\left\{\rho\int \ln(dF/dF_0)dF+(1-\rho)\int \ln(dG/dG_0)dG\right\}$$

where $Q_0 = (F_0,G_0)$. For convenience, we give the definitions

(1.6) $$I(F,F_0) = \int \ln(dF/dF_0)dF$$

and

(1.7) $$I_{\rho}^{(2)}(\Omega,Q_0) = \inf_{(F,G)\in\Omega} \{\rho I(F,F_0) + (1-\rho) I(G,G_0)\}$$

$I^{(2)}(\Omega,Q_0)$ is called the two-sample information number when ρ, $1-\rho$ are the limiting relative sample size proportions. For the asymptotic theory, we let $m,n \to \infty$ in such a way that for $N=m+n \to \infty$, $m/N \to \rho$. For ease of reference, we restate the definition of Q_0-regularity as follows: A class $\Omega \subset D \times D$ is said to be Q_0-regular if for each ρ; $1 > \rho > 0$, the following conditions are satisfied:

(A) $I_{\rho}^{(2)}(\Omega,Q_0) < \infty$

(B) For every $\eta > 0$, there is a product-strip U, such that $U \subset \Omega$ and $I_{\rho}^{(2)}(U,Q_0) < I_{\rho}^{(2)}(\Omega,Q_0) + \eta$.

(C) For every $\eta > 0$, there is a finite number, $K=K(\eta)$, of product-strips, $U_1, U_2, ---, U_K$, such that $\Omega \subset \bigcup_{i=1}^{K} U_i$, $I_{\rho}^{(2)}(U_i,Q_0) < \infty$, and $I_{\rho}^{(2)}(\Omega,Q_0) < I_{\rho}^{(2)}(U_i,Q_0) + \eta$, for $i=1,2,---,K$.

Hoadley investigated conditions under which Theorem 1.3 can be applied to the class

(1.8) $$\Omega(c) = \{ Q \in D \times D : T(Q) \geq c \}$$

where $T: D \times D \to R=(-\infty,\infty)$. Since F and G are absolutely continuous with respect to $H=\rho F+(1-\rho)G$, by the Radon-Nikodym theorem, there exist $f,g(\geq 0)$ such that $f=dF/dH$ and $g=dG/dH$. Furthermore, assume there exist $f_{0,H}$, $g_{0,H}$ (≥ 0) such that $f_{0,H} = dF_0/dH$ and $g_{0,H} = dG_0/dH$. The function

$$(1.9) \qquad I(c;F_0,G_0)$$

$$= \inf_{(F,G)\in\Omega(c)} \left\{ \rho\int_{-\infty}^{\infty} f(x)\ln(f(x)/f_{0,H}(x))dH(x) + (1-\rho)\int_{-\infty}^{\infty} g(x)\ln(g(x)/g_{0,H}(x))dH(x) \right\}$$

is well-defined by letting

$$f\ \ln(f/f_{0,H}) = \begin{cases} 0 & \text{if } f=0 \\ +\infty & \text{if } f>0 \text{ and } f_{0,H}=0 \\ \text{itself} & \text{if } f>0 \text{ and } f_{0,H}>0 \end{cases}$$

and similarly for $g\ \ln(g/g_{0,H})$. When $\Omega(c)=\phi$ we define $I=\infty$. We further assume

$$(1.10) \qquad I(c;F_0,G_0) = \inf_{(F,G)\in\Omega(c)} \int_{-\infty}^{\infty} \varphi(f,g)dH(x)$$

where

$$(1.11) \qquad \varphi(f,g) = \rho f\ \ln(f/f_{0,H}) + (1-\rho)g\ \ln(g/g_{0,H}).$$

Let E denote the normed linear space of functions of bounded variation in $(-\infty,\infty)$, with $\|F\| = \sup_{-\infty<x<\infty} |F(x)|$, for $F \in E$. Thus for $Q=(F,G) \in E \times E$, define

$$(1.12) \qquad \|Q\|_2 = \max\left(\sup_{-\infty<x<\infty} |F(x)|,\ \sup_{-\infty<x<\infty} |G(x)|\right).$$

$\|\ \|_2$ is a norm on $E \times E$; hence we can consider $E \times E$ as a metric space with the metric induced by this norm. In [13], Hoadley proved the following

<u>Theorem 1.4.</u> <u>If</u> T <u>is uniformly continuous, then for every</u> $c > T(Q_0)$ <u>at which</u> $I(c;F_0,G_0)$ <u>is continuous</u>

$$(1.13) \qquad \lim_{N\to\infty} -N^{-1} \ln P\left\{T(F_{0m},G_{0n}) \geq c\right\} = I(c;F_0,G_0)$$

<u>where</u> $Q_0=(F_0,G_0)$ <u>and</u> $I(c;F_0,G_0)$ <u>is defined in</u> (1.9).

We shall prove further that Theorem 1.4 holds for all statistics $T_N(F_{0m},G_{0n})$, defined in (1.3), that satisfy Property B. We first show Theorem 1.4 holds for $T^{(M)}(F,G)$ defined as follows:

$$(1.14) \qquad T^{(M)}(F,G) = \int_{-\infty}^{\infty} J^{(M)}(H(x))\,dF(x)$$

with

$$(1.15) \qquad J^{(M)}(u) = \begin{cases} J(u) & \text{if } |J(u)| \leq M \\ M & \text{if } J(u) > M \\ -M & \text{if } J(u) < -M \end{cases}$$

for each positive $M > |J(\frac{1}{2})|$, and $I(c;F_0,G_0)$ is replaced by

$$I^{(M)}(c;F_0,G_0) = \inf_{(F,G)\in\Omega_M(c)} \int \varphi(f,g)dH(x) \tag{1.16}$$

where $\Omega_M(c) = \{ Q\in D\times D;\ T^{(M)}(Q) \geq c\}$ and $\varphi(f,g)$ defined in (1.11). In order to prove that Theorem 1.4 is also true for $T^{(M)}$, the following lemmas are used,

Lemma 1.5. *Let* $Q_1 = (F_1,G_1)$ *and* $Q_2 = (F_2,G_2)$. *Then* $T^{(M)}$ *is uniformly continuous in* $D\times D$ *with respect to the metric defined by (1.12), and* $|T^{(M)}(Q) - T(Q)| \longrightarrow 0$ *as* $M\to\infty$ *uniformly in* Q *when* $\Gamma(Q)= \int_{-\infty}^{\infty} J(H(x))dF(x) < \infty$.

Proof: For proving the uniform continuity of $T^{(M)}$, it suffices to show that for given $\varepsilon>0$ there exists $\delta(\varepsilon) > 0$ independent of Q_1, Q_2 such that

$$\left|T^{(M)}(Q_1) - T^{(M)}(Q_2)\right| < \varepsilon \tag{1.17}$$

whenever $d= \| Q_1 -Q_2 \|_2 < \delta(\varepsilon)$. Now consider, for a given $\varepsilon>0$,

$$\left|T^{(M)}(Q_1) - T^{(M)}(Q_2)\right|$$

$$= \left|\int_{-\infty}^{\infty} J^{(M)}(H_1(x))dF_1(x) - \int_{-\infty}^{\infty} J^{(M)}(H_2(x))dF_2(x)\right|$$

$$= \int_{-\infty}^{\infty}\left|J^{(M)}(H_1(x)) - J^{(M)}(H_2(x))\right|dF_1(x) + \left|\int_{-\infty}^{\infty} J^{(M)}(H_2(x))d\{F_2(x)-F_1(x)\}\right| . \tag{1.18}$$

Since $J^{(M)}(.)$ is continuous and bounded by M over $[0,1]$, and hence is uniformly continuous over $[0,1]$ so we can make $|J^{(M)}(H_1(x))-J^{(M)}(H_2(x))| < \mathcal{E}/2$ if we select $\sup_{-\infty<x<\infty}|H_1(x)-H_2(x)| < \delta(\mathcal{E})$ for some $\delta(\mathcal{E})>0$ independently of H_1, H_2 over $(-\infty,\infty)$. This can be made to hold by making the metric d small enough, i.e. $d<\delta(\mathcal{E})$. Thus integrating, the first term is less than $\mathcal{E}/2$. Next, consider the second term. Since F_1 and F_2 are monotonically increasing on $(-\infty,\infty)$, then $F_2 - F_1$ is of bounded variation on $(-\infty,\infty)$. By a theorem in II.6.2(see page 38 of [13]), $J^{(M)}(H_2(x))$ can be expressed as the difference of two nondecreasing functions. Let $J^{(M)}(H_2(x))=J_1^{(M)}(H_2(x)) -J_2^{(M)}(H_2(x))$ where $J_1^{(M)}(H_2(x))$ and $J_2^{(M)}(H_2(x))$ are nondecreasing functions on $(-\infty,\infty)$. Then an integration by parts theorem(see page 93 of [13]) can be applied and we have

$$\left| \int_{-\infty}^{\infty} J^{(M)}(H_2(x))d\{F_2(x) - F_1(x)\}\right|$$

$$=\Big| [F_2(\infty)-F_1(\infty)]\, J^{(M)}(H_2(\infty))-[F_2(-\infty)-F_1(-\infty)]\, J^{(M)}(H_2(-\infty))$$

$$-\int_{-\infty}^{\infty}\{F_2(x)-F_1(x)\}\, d\{J_1^{(M)}(H_2(x))-J_2^{(M)}(H_2(x))\}$$

$$-\sum_x\Big\{\left[F_2(x+o)-F_1(x+o)-F_2(x)+F_1(x)\right]\left[J^{(M)}(H_2(x+o))-J^{(M)}(H_2(x))\right]$$

$$-\left[F_2(x)-F_1(x)-F_2(x-o)+F_1(x-o)\right]\left[J^{(M)}(H_2(x))-J^{(M)}(H_2(x-o))\right]\Big\}\Big|$$

$$= \left| \int_{-\infty}^{\infty} \{F_2(x-o) - F_1(x-o)\} d\{J_1^{(M)}(H_2(x)) - J_2^{(M)}(H_2(x))\} \right|$$

$$\leq \int_{-\infty}^{\infty} |F_2(x-o) - F_1(x-o)| \, dV_1^{(M)}(H_2(x))$$

$$+ \int_{-\infty}^{\infty} |F_2(x-o) - F_1(x-o)| \, dV_2^{(M)}(H_2(x))$$

$$< \delta \left\{ |J_1^{(M)}(1) - J_1^{(M)}(0)| + |J_2^{(M)}(1) - J_2^{(M)}(0)| \right\} .$$

The summation being taken over all common discontinuity points of $F_2(x) - F_1(x)$, $J^{(M)}(H_2(x))$ and

$V_i^{(M)}(H_2(x)) = \int_{-\infty}^{x} |dJ_i^{(M)}(H_2(y))|$, $i=1,2$. $V_i^{(M)}(H_2(\infty))$ is the total variation of $J_i^{(M)}(H_2(x))$. The first inequality holds using theorem II.14.2 of [13] (see page 67). The last inequality holds by using $|F_2(x)-F_1(x)| < \delta$ for all $x \in (-\infty,\infty)$ and the fact that $\int_{-\infty}^{\infty} |dJ_i^{(M)}(H_2(x))| = |J_i^{(M)}(1) - J_i^{(M)}(0)|$, for $i=1,2$. Thus we have

$$\left| \int_{-\infty}^{\infty} J^{(M)}(H_2(x)) d\{F_2(x) - F_1(x)\} \right| < 4\delta M.$$

Taking $\delta = \varepsilon/8M$, we have established (1.17). Finally we consider

$$\begin{aligned} &\left| T(Q) - T^{(M)}(Q) \right| \\ &= \left| \int_{-\infty}^{\infty} \left\{ J(H(x)) - J^{(M)}(H(x)) \right\} dF(x) \right| \\ &\leq 2 \int_{\{x:\, |J(H(x))| > M\}} |J(H(x))| \, dF(x) \\ &\leq \frac{2}{\rho} \int_{\{u:\, |J(u)| > M\}} |J(u)| \, du \quad . \end{aligned}$$

Thus the finiteness of $T(Q)$ gives $\left|T(Q) - T^{(M)}(Q)\right| \longrightarrow 0$ as $M \to \infty$ uniformly in Q. We have established the lemma.

<u>Lemma 1.6.</u> <u>$I(c;F_0,G_0)$ defined in</u> (1.9) <u>is nonnegative, nondecreasing and convex in c for all</u> $c \in (c_0, c^*)$ <u>where</u>

$c_0 = \int_{-\infty}^{\infty} J(H_0(x))dF_0(x)$ <u>and $c^* > c_0$</u> <u>is such that</u> $I(c;F_0,G_0) < \infty$. <u>Hence $I(c;F_0,G_0)$ is continuous in c for</u> $c_0 < c < c^*$. <u>Similarly we have $I^{(M)}(c;F_0,G_0)$ defined in</u> (1.16) <u>is continuous in c for all</u> $c \in (c_0, c^*)$.

<u>Proof:</u> Since $I(c;F_0,G_0) < \infty$, for all c; $c_0 < c < c^*$, then F and G are respectively absolutely continuous with respect to F_0 and G_0 and hence the continuity of F and G follows directly from the assumption that F_0 and G_0 are continuous. Thus $I(c;F_0,G_0)$ and $\Omega(c)$ can be rewritten as follows if we denote $u = H(x)$:

$$I(c; f_0^*, g_0^*)$$

$$= \inf_{f^* \in \Omega^*(c)} \left\{ \int_0^1 \rho f^*(u) \ln(f^*(u)/f_0^*(u)) du \right.$$

$$\left. + (1-\rho) \int_0^1 g^*(u) \ln(g^*(u)/g_0^*(u)) du \right\}$$

where $\Omega^*(c) = \left\{ f^* : \int_0^1 J(u) f^*(u) du \geq c \right\}$ and $f^*(u) = f(H^{-1}(u))$ $f_0^*(u) = f_0(H^{-1}(u))$, $g^*(u) = g(H^{-1}(u))$, $g_0^* = g_0(H^{-1}(u))$. Since $\frac{f^*}{f_0^*} \ln(\frac{f^*}{f_0^*})$ and $\frac{g^*}{g_0^*} \ln(\frac{g^*}{g_0^*})$ are convex, Jensen's inequality applies and gives

$$I(c; f_0^*, g_0^*)$$

$$\geq \inf \left\{ \left[\rho \int_0^1 f^* du \right] \left[\ln \int_0^1 f^* du \right] + (1-\rho) \left[\int_0^1 g^* du \right] \left[\ln \int_0^1 g^* du \right] \right.$$

$$= 0$$

using $\int_0^1 f^* du = 1$ and $\int_0^1 g^* du = 1$. This shows $I(c; f_0^*, g_0^*)$ is nonnegative. For any c_1, c_2; $c_0 < c_1 \leq c_2 < c^*$, we have $\Omega^*(c_2) \subset \Omega^*(c_1)$ and $I(c_2; f_0^*, g_0^*) \geq I(c_1; f_0^*, g_0^*)$, since they are the respective infimums over $\Omega^*(c_i)$. Thus we have $I(c; f_0^*, g_0^*)$ is nondecreasing in c and $I(c; f_0^*, g_0^*) \leq I(c^*; f_0^*, g_0^*) < \infty$ for all c; $c_0 \in (c_0, c^*)$. Note that $\Omega(c)$ is nonempty by definition of I.

Let (f_i^*, g_i^*) be any element of $\Omega^*(c_i)$ such that for any $\varepsilon > 0$

$$I(c_i; f_0^*, g_0^*) + \varepsilon$$

$$\geq \rho \int_0^1 f_i^*(u) \ln(f_i^*(u)/f_0^*(u))du +$$

$$(1-\rho)\int_0^1 g_i^*(u)\ln(g_i^*(u)/g_0^*(u))du$$

for i=1,2. For any α satisfying $0 < \alpha < 1$, the convexity of $\frac{f^*}{f_0^*}\ln(\frac{f^*}{f_0^*})$ and $\frac{g^*}{g_0^*}\ln(\frac{g^*}{g_0^*})$ yields the following inequality.

$$\alpha\left\{I(c_1; f_0^*, g_0^*) + \varepsilon\right\} + (1-\alpha)\left\{I(c_2; f_0^*, g_0^*) + \varepsilon\right\}$$

$$\geq \rho\int_0^1\left\{\alpha\frac{f_1^*}{f_0^*}\ln(\frac{f_1^*}{f_0^*}) + (1-\alpha)\frac{f_2^*}{f_0^*}\ln(\frac{f_2^*}{f_0^*})\right\}f_0^* du +$$

$$(1-\rho)\int_0^1\left\{\alpha\frac{g_1^*}{g_0^*}\ln(\frac{g_1^*}{g_0^*}) + (1-\alpha)\frac{g_2^*}{g_0^*}\ln(\frac{g_2^*}{g_0^*})\right\}g_0^* du$$

$$\geq \rho\int_0^1 \frac{\alpha f_1^* + (1-\alpha) f_2^*}{f_0^*}\ln(\frac{\alpha f_1^* + (1-\alpha) f_2^*}{f_0^*}) f_0^* du +$$

$$(1-\rho)\int_0^1 \frac{\alpha g_1^* + (1-\alpha) g_2^*}{g_0^*}\ln(\frac{\alpha g_1^* + (1-\alpha) g_2^*}{g_0^*}) g_0^* du \text{ .}$$

Setting $f^{**} = \alpha f_1^* + (1-\alpha) f_2^*$ and $g^{**} = \alpha g_1^* + (1-\alpha) g_2^*$. Thus we have

$$\alpha I(c_1; f_0^*, g_0^*) + (1-\alpha) I(c_2; f_0^*, g_0^*) + \varepsilon$$

$$\geq \rho\int_0^1 f^{**}\ln(f^{**}/f_0^*)du + (1-\rho)\int_0^1 g^{**}\ln(g^{**}/g_0^*)du$$

$$\geq I(\alpha c_1+(1-\alpha)c_2; f_0^*, g_0^*).$$

Thus convexity follows letting $\varepsilon \to 0$. The last inequality holds because $(f^{**}, g^{**}) \in \Omega^*(\alpha c_1+(1-\alpha)c_2)$. Since

$$\int_0^1 J(u)f^{**}(u)du = \int_0^1 J(u)\left[\alpha f_1^*(u)+(1-\alpha)f_2^*(u)\right] du$$

$$\geq \alpha c_1 + (1-\alpha)c_2,$$

using $f_i^* \in \Omega^*(c_i)$. Thus we have established lemma 1.6. By Theorem 1.4, Lemma 1.5 and Lemma 1.6, we conclude the following

Theorem 1.7. *Let* $T^{(M)}$ *be defined as in* (1.14) *and* $I^{(M)}$ *as in* (1.15) *and let* $I^{(M)}(c^*; F_0, G_0) < \infty$ *for some* $c^* > c_0$, *where* c_0 *is defined as in Lemma 1.6. Then for every* $c \in (c_0, c^*)$, *we have*

$$\lim_{N\to\infty} -N^{-1} \ln P\left\{ T^{(M)}(F_{0m}, G_{0m}) \geq c \right\} = I^{(M)}(c; F_0, G_0).$$

Before proving the main theorem, we prove the following

Lemma 1.8. *Let* $I, I^{(M)}$ *be defined as in* (1.9), (1.15) *respectively. Then for given* $\varepsilon > 0$ *there exists a positive integer* $M_0(\varepsilon)$ *such that*

$$I(c-\varepsilon; F_0, G_0) \leq I^{(M)}(c; F_0, G_0) \leq I(c+\varepsilon; F_0, G_0)$$

for all $M \geq M_0(\varepsilon)$.

Proof: By lemma 1.5, $|T(Q) - T^{(M)}(Q)| \to 0$ as $M\to\infty$ uniformly in Q. Thus for given $\varepsilon>0$, there exists a positive integer $M_0(\varepsilon)$ such that for all $M \geq M_0(\varepsilon)$

$$(1.19) \qquad \int_{-\infty}^{\infty} J(H(x))dF(x) - \varepsilon < \int_{-\infty}^{\infty} J^{(M)}(H(x))dF < \int_{-\infty}^{\infty} J(H(x))dF + \varepsilon.$$

Since $\quad I^{(M)}(c;F_0,G_0)$

$$= \inf\left\{ \int_{-\infty}^{\infty} \varphi(f,g)dH(x) : \int_{-\infty}^{\infty} J^{(M)}(H(x))dF(x) \geq c \right\}$$

$$\geq \inf\left\{ \int_{-\infty}^{\infty} \varphi(f,g)dH(x) : \int_{-\infty}^{\infty} J(H(x))dF(x) + \varepsilon \geq c \right\}$$

$$= I(c-\varepsilon;F_0,G_0).$$

The inequality holds by using the right inequality in (1.19). A similar procedure yields $I^{(M)}(c;F_0,G_0) \leq I(c+\varepsilon;F_0,G_0)$. Combining the two results, we have the required inequality.

Lemma 1.9. For given $\varepsilon>0$, there exist two positive integers $N_0(\varepsilon), M_0(\varepsilon)$ such that for all $N \geq N_0(\varepsilon)$ and $M \geq M_0(\varepsilon)$

$$\left| T^{(M)}(F_{0m},G_{0n}) - \int_{-\infty}^{\infty} J(NH_{0N}(x)/N+1)dF_{0m}(x) \right| < \varepsilon \quad .$$

Proof: Consider

$$\left| \int_{-\infty}^{\infty} \left\{ J(NH_{0N}(x)/N+1) - J^{(M)}(NH_{0N}(x)/N+1) \right\} dF_{0m}(x) \right|$$

$$\leq 2 \int_A \left| J(NH_{0N}(x)/N+1) \right| dF_{0m}(x)$$

$$\leq \frac{2(N+1)}{m} \int_A \left|J(NH_{0N}(x)/N+1)\right| d\{NH_{0N}(x)/N+1\}$$

$$= \frac{2(N+1)}{m} \sum_{i:|J|>M} \left|J(i/N+1)\right| \frac{1}{N+1}$$

$$\leq \frac{2(N+1)}{m} \int_{|J|>M} |J(u)|\, du$$

where $A = \{x: |J(NH_{0N}(x)/N+1)| > M\}$. The last inequality holds by part (iii) of property B. Since $\int_0^1 |J(u)|\, du < \infty$ we have for given $\varepsilon > 0$, there exists $M_0(\varepsilon)$ such that

$$\int_{|J|>M} |J(u)|\, du < \varepsilon$$

for all $M \geq M_0(\varepsilon)$. By $m/(N+1) \to \rho$ as $N \to \infty$, and for this particular $\varepsilon > 0$, there exists an integer $N_0(\varepsilon)$ such that

$$\rho - \varepsilon < m/(N+1) < \rho + \varepsilon$$

for all $N \geq N_0(\varepsilon)$. Therefore we have for all $N \geq N_0(\varepsilon)$ and all $M \geq M_0(\varepsilon)$

$$(1.20) \qquad \left| \int_{-\infty}^{\infty} \{J(NH_{0N}(x)/N+1) - J^{(M)}(NH_{0N}(x)/N+1)\} dF_{0m}(x) \right|$$

$$< 2\varepsilon/(\rho - \varepsilon).$$

Again using $m/(N+1) \to \rho$ as $N \to \infty$ then for some $\varepsilon > 0$, there exists an integer $N_0(\varepsilon)$ such that for all $N \geq N_0(\varepsilon)$ and for all x; $-\infty < x < \infty$,

$$\left|\rho F_{0m}(x) + (1-\rho)G_{0n}(x) - mF_{0n}(x)/(N+1) - nG_{0n}(x)/(N+1)\right| < \varepsilon \ .$$

Thus the uniform continuity of $J^{(M)}$ over $[0,1]$ gives

$$(1.21) \qquad \left|\int_{-\infty}^{\infty} J^{(M)}(NH_{0N}(x)/N+1)\,dF_{0m}(x) - T^{(M)}(F_{0m},G_{0n})\right| < \varepsilon .$$

Combining (1.20) and (1.21) we have the required result.

<u>Theorem 1.10. If T_N is a type B sequence of Chernoff-Savage statistics and $I(c^*;F_0,G_0) < \infty$ for some $c^* > c_0 = \int_{-\infty}^{\infty} J(H_0(x))\,dF_0(x)$. Then for every c; $c^* > c > c_0$, we have</u>

$$\lim_{N\to\infty} -N^{-1} \ln P\left\{T_N(F_{0m},G_{0n}) \geq c\right\} = I(c;F_0,G_0)$$

<u>Proof:</u> Consider

$$\left|\int_{-\infty}^{\infty}\left\{J_N(NH_{0N}(x)/N+1) - J(NH_{0N}(x)/N+1)\right\}dF_{0m}(x)\right|$$

$$\leq \frac{N+1}{m}\int_{-\infty}^{\infty}\left|J_N(NH_{0N}(x)/N+1) - J(NH_{0N}(x)/N+1)\right|\frac{N}{N+1}\,dH_{0N}(x)$$

$$= \frac{N+1}{m}\sum_{i=1}^{N}\left|J_N(i/N+1) - J(i/N+1)\right|\frac{1}{N+1} \ .$$

The sum approaches zero as $N \to \infty$ by part(ii) of Property B. Therefore for given $\varepsilon > 0$ there exists an integer $N_0(\varepsilon)$ such that for all $N \geq N_0(\varepsilon)$

(1.22) $$\int_{-\infty}^{\infty} J(NH_{0N}(x)/N+1)dF_{0m}(x) - \varepsilon/2$$

$$< T_N(F_{0m}, G_{0n}) < \int_{-\infty}^{\infty} J(NH_{0N}(x)/N+1)dF_{0m}(x) + \varepsilon/2.$$

By Lemma 1.9, for given $\varepsilon > 0$ there exist $N_0(\varepsilon), M_0(\varepsilon)$ such that for all $N \geq N_0(\varepsilon)$ and $M \geq M_0(\varepsilon)$

(1.23) $$T^{(M)}(F_{0m}, G_{0n}) - \varepsilon/2$$

$$< \int_{-\infty}^{\infty} J(NH_{0N}(x)/N+1)dF_{0m}(x) < T^{(M)}(F_{0m}, G_{0n}) + \varepsilon/2.$$

Combining (1.22) and (1.23) we have for all $N \geq N_0(\varepsilon)$, $M \geq M_0(\varepsilon)$

(1.24) $$T^{(M)}(F_{0m}, G_{0n}) - \varepsilon < T_N(F_{0m}, G_{0n}) < T^{(M)}(F_{0m}, G_{0n}) + \varepsilon .$$

Thus (1.24) gives for all $N \geq N_0(\varepsilon)$ and all $M \geq M_0(\varepsilon)$

$$P\left\{T^{(M)}(F_{0m}, G_{0n}) - \varepsilon \geq c\right\}$$

$$\leq P\left\{T_N(F_{0m}, G_{0n}) \geq c\right\} \leq P\left\{T^{(M)}(F_{0m}, G_{0n}) + \varepsilon \geq c\right\}$$

and

$$-N^{-1} \ln P\left\{T^{(M)}(F_{0m}, G_{0n}) \geq c - \varepsilon\right\}$$

$$\leq -N^{-1} \ln P\left\{T_N(F_{0m}, G_{0n}) \geq c\right\}$$

$$\leq -N^{-1} \ln P\left\{T^{(M)}(F_{0m}, G_{0n}) \geq c + \varepsilon\right\} .$$

Theorem 1.7 applies and gives upon taking the limit

$$(1.25) \qquad I^{(M)}(c-\varepsilon; F_0, G_0)$$

$$\leq \lim_{N\to\infty} -N^{-1} \ln P\left\{ T_N(F_{0m}, G_{0n}) \geq c \right\}$$

$$\leq I^{(M)}(c+\varepsilon; F_0, G_0)$$

for all $M \geq M_0(\varepsilon)$. Combining (1.25) and Lemma 1.8, we have

$$I(c-2\varepsilon; F_0, G_0)$$

$$\leq \lim_{N\to\infty} -N^{-1} \ln P\left\{ T_N(F_{0m}, G_{0n}) \geq c \right\}$$

$$\leq I(c+2\varepsilon; F_0, G_0).$$

Since ε is arbitrary and $I(c; F_0, G_0)$ is continuous at c, then

$$\lim_{N\to\infty} -N^{-1} \ln P\left\{ T_N(F_{0m}, G_{0n}) \geq c \right\} = I(c; F_0, G_0).$$

Thus we have established the theorem.

<u>Remark:</u> If $F_0=G_0$ then $I(c; F_0, G_0)$ will be called the null large deviation probability of the linear rank statistics with score function J. For notational convenience, we set $I_H(c; J) = I(c; F_0, G_0)$. Let

$$(1.26) \qquad \Omega^{**}(c) = \left\{ (F,G) \in D \times D : T(F,G) \leq c \right\}$$

and

$$(1.27) \qquad \Omega^{**}_M(c) = \left\{ (F,G) \in D \times D : T^{(M)}(F,G) \leq c \right\}$$

where $T(F,G)$ and $T^{(M)}(F,G)$ are respectively defined as in

Lemma 1.5 and (1.14). Then define

$$(1.28) \qquad I^{**}(c;F_0,G_0) = \inf_{(F,G)\in\Omega^{**}(c)} \int \varphi(f,g)dH(x)$$

and

$$(1.29) \qquad I_M^{**}(c;F_0,G_0) = \inf_{(F,G)\in\Omega_M^{**}(c)} \int \varphi(f,g)dH(x)$$

where $\varphi(f,g)$ is defined in (1.11). The approach applied to prove Lemma 1.6 can be used to show that $I^{**}(c;F_0,G_0)$, $I_M^{**}(c;F_0,G_0)$ are nonnegative, nonincreasing and convex in c for all $c\in(c^{**},c_0)$ where $c_0=\int_{-\infty}^{\infty} J(H_0(x))dF_0(x)$ and c^{**} is such that $I^{**}(c;F_0,G_0)<\infty$ and hence they are continuous in c ; $c^{**}<c<c_0$. Furthermore we have

$$(1.30) \qquad I^{**}(c+\varepsilon;F_0,G_0) \le I_M^{**}(c;F_0,G_0) \le I^{**}(c-\varepsilon;F_0,G_0).$$

This fact can be proved by using same procedure which is applied to show Lemma 1.8. Thus

$$\lim_{N\to\infty} -N^{-1} \ln P\left\{T^{(M)}(F_{0m},G_{0n}) \le c\right\}$$

$$= \lim_{N\to\infty} -N^{-1} \ln P\left\{-T^{(M)}(F_{0m},G_{0n}) \ge -c\right\}$$

$$= \lim_{N\to\infty} -N^{-1} \ln P\left\{T_1^{(M)}(F_{0m},G_{0n}) \ge c_1\right\}$$

where $T_1^{(M)} = -T^{(M)}$ and $c_1 = -c$. By Lemma 1.5, $T_1^{(M)}$ is uniformly continuous. Define

$$I_1^{(M)}(c_1;F_0,G_0) = \inf_{(F,G)\in\Omega_M(c_1)} \int \varphi(f,g)dH(x)$$

where $\Omega_M(c_1) = \left\{(F,G)\in D\times D \;:\; T_1^{(M)} \geq c_1\right\}$. Since $I_1^{(M)}(c_1;F_0,G_0) = I_M^{**}(c;F_0,G_0)$, by Theorem1.7 we have

$$(1.31) \qquad \lim_{N\to\infty} -N^{-1} \ln P\left\{T^{(M)}(F_{0m},G_{0n}) \leq c\right\} = I^{**}(c;F_0,G_0).$$

Combining (1.30) and (1.31), the procedure used to show Theorem 1.10 can be applied to prove the following

<u>Theorem 1.11.</u> <u>If</u> T_N <u>is a type B sequence of Chernoff-Savage statistics and</u> $I^{**}(c^*;F_0,G_0) < \infty$ <u>for some</u> $c^* < c_0$ <u>where</u> $c_0 = \int_{-\infty}^{\infty} J(H_0(x))dF_0(x)$. <u>Then for every</u> c; $c^* < c < c_0$, <u>we have</u>

$$\lim_{N\to\infty} -N^{-1} \ln P\left\{T_N(F_{0m},G_{0n}) \leq c\right\} = I^{**}(c;F_0,G_0).$$

<u>Remark:</u> $I^{**}(c;F_0,G_0)$ will be called the information number of linear rank statistics with score function J under the alternative $F_0 \neq G_0$. For notational convenience, we set $I_A(c;J) = I^{**}(c;F_0,G_0)$.

2. Evaluation of $I_H(c;J)$ and $I_A(c;J)$. In this section we give expressions for the probability of a large deviation for linear rank statistics under the null and alternative hypotheses, and then derive useful formulae of $I_H(c;J)$ and $I_A(c;J)$ for the Wilcoxon and Fisher-Yates(normal scores) statistics. A more explicit expression for $I_H(c;J)$ is given in the following(see [15] p.376 and [27] p.259)

Theorem 2.1. *If J satisfies Property B then for every* $c > \int_{-\infty}^{\infty} J(H_0(x))f_0(x)dH(x)$ *there exist* h *and* v *such that*

$$(2.1) \qquad I_H(c;J)$$

$$= 2hc - 2hv - \int_0^1 \ln\left\{(1-\rho)+\rho\exp\left[2h(J(x)-v)/\rho\right]\right\}dx$$

where (h,v) *is the unique solution of*

$$(2.2) \qquad 1 = \int_0^1 \frac{\exp\left[2h(J(x)-v)/\rho\right]}{(1-\rho)+\rho\exp\left[2h(J(x)-v)/\rho\right]}dx$$

and

$$(2.3) \qquad c = \int_0^1 \frac{J(x)\exp\left[2h(J(x)-v)/\rho\right]}{(1-\rho)+\rho\exp\left[2h(J(x)-v)/\rho\right]}dx .$$

Proof: The linear rank statistics are distribution free under the null hypothesis so that we can, without loss of generality, assume $F_0(x)=G_0(x)=U(x)=x$ where U is the cdf of the uniform distribution on $[0,1]$. Thus $f_0(x)=g_0(x)=1$ for all $x\in[0,1]$. Define

$$(2.4) \qquad FI(c;J)=\rho\int_0^1 f(x)\ln f(x)dx + (1-\rho)\int_0^1 g(x)\ln g(x)dx.$$

Then the Lagrange multiplier method can be applied to seek the infimum of $FI(c;J)$ subject to $\int_0^1 J(x)f(x)dx \geq c$. Now for $h>0$, we have

$$- FI(c;J)$$

$$\leq \rho\int_0^1 f(x)\ln(1/f(x))dx + (1-\rho)\int_0^1 g(x)\ln(1/g(x))dx$$

$$+ h\left\{\int_0^1 J(x)f(x)dx - c\right\}$$

$$\leq \rho\int_0^1 \ln\{(1/f(x))\exp[hJ(x)/\rho]\}f(x)dx - hc$$

$$+ (1-\rho)\int_0^1 g(x)\ln(1/g(x))dx$$

$$\leq \rho\ln\left\{\int_0^1 \exp[hJ(x)/\rho]dx\right\}-hc +(1-\rho)\int_0^1 g(x)\ln(1/g(x))dx.$$

The last inequality follows by using Jensen's inequality. Notice that the last expression does not depend on f; hence for a fixed g, FI(c;J) is minimized by any f satisfying

$$(2.5) \qquad \text{(i)} \quad \int_0^1 J(x)f(x)dx = c$$

$$\text{(ii)} \quad \frac{1}{f(x)}\exp[hJ(x)/\rho] = 1/s_1 \quad \text{constant.}$$

Using the identity $\rho f(x) + (1-\rho)g(x) \equiv 1$ for all x; $0 \leq x \leq 1$, we can equivalently seek the infimum of FI(c;J) subject to $(1-\rho)\int_0^1 J(x)g(x)dx \leq \int_0^1 J(x)dx - \rho c$. A similar argument can be applied interchanging the role of f and g, to obtain the solution

$$(2.6) \qquad \text{(i)} \quad \int_0^1 J(x)f(x)dx = c$$

$$\text{(ii)} \quad \frac{1}{g(x)}\exp[-hJ(x)/\rho] = 1/s_2 \quad \text{constant.}$$

Combining (2.5) and (2.6), we have

$$(2.7) \qquad c = \int_0^1 J(x)f(x)dx$$

$$f(x) = s_1\exp[hJ(x)/\rho]$$

$$g(x) = s_2\exp[-hJ(x)/\rho] \quad .$$

Thus the infimum of FI(c;J) is attained by these f(x) and g(x) satisfying (2.7). Substitution gives

$$(2.8) \qquad I_H(c;J) = 2ch + \rho\ln(s_1) + (1-\rho)\ln(s_2) - \frac{h}{\rho}\int_0^1 J(x)dx$$

where (h,s_1,s_2) is the solution of

(2.9) $$c = \int_0^1 J(x)f(x)dx$$

$$1/s_1 = \int_0^1 \exp[hJ(x)/\rho]\,dx$$

$$1/s_2 = \int_0^1 \exp[-hJ(x)/\rho]\,dx.$$

Since $\rho f(x)+(1-\rho)g(x) \equiv 1$ for all $x \in [0,1]$, then (2.9) can be rewritten as

(2.10) $$1 = \int_0^1 \frac{s_1\exp[hJ(x)/\rho]}{\rho s_1\exp[hJ(x)/\rho] + (1-\rho)s_2\exp[-hJ(x)/\rho]}\,dx$$

$$c = \int_0^1 \frac{s_1 J(x)\exp[hJ(x)/\rho]}{\rho s_1\exp[hJ(x)/\rho] + (1-\rho)s_2\exp[-hJ(x)/\rho]}\,dx \quad .$$

Further simplifications reduce (2.10) to

(2.11) $$1 = \int_0^1 \frac{\exp\{2h[J(x)-v]/\rho\}}{(1-\rho)+\rho\exp\{2h[J(x)-v]/\rho\}}\,dx$$

and

(2.12) $$c = \int_0^1 \frac{J(x)\exp\{2h[J(x)-v]/\rho\}}{(1-\rho)+\rho\exp\{2h[J(x)-v]/\rho\}}\,dx$$

where $2hv = -\rho\ln(s_1/s_2)$. Finally we simplify $I_H(c;J)$

as follows:

$$I_H(c;J)$$

$$= 2ch + \rho \ln(s_1/s_2) + \ln(s_2) - \frac{h}{\rho}\int_0^1 J(x)dx$$

$$= 2ch - 2hv + \int_0^1 \ln\{s_2 \exp[-hJ(x)/\rho]\}dx$$

$$= 2ch - 2hv + \int_0^1 \ln\left\{\frac{s_2\exp[-hJ(x)/\rho]}{\rho s_1 \exp[hJ(x)/\rho] + (1-\rho)s_2\exp[-hJ(x)/\rho]}\right\}dx$$

$$= 2hc - 2hv - \int_0^1 \ln\{(1-\rho)+\rho\exp[2h(J(x)-v)/\rho]\}\,dx$$

where (h,v) is the unique solution of (2.11) and (2.12). The uniqueness of (h,v) has been proved in the Theorem 4 of [27]. Thus we complete the proof of Theorem 2.1.

Remark:

(i) This result was first reported by Woodworth [27]. The correspondence between the present notation and Woodworth's is as follows: $h = r\rho/2$, $c = r/\rho$, $v = s$. (see (3.13),(3.14) and (3.15) of [27]).

(ii) In [27], Woodworth proved(see Lemmas 4,5 of the appendix) that c(h) is a strictly increasing and continuous function of $h \geq 0$. This property was employed in the computation.

A similar approach used to prove Theorem 2.1 can be applied to derive an explicit expression for $I_A(c;J)$. We have the following expression provided $J(.)$ satisfies Property B and $c < \int_{-\infty}^{\infty} J(H_0(x))f_0(x)dH(x)$.

$$(2.13)\quad I_A(c;J) = \rho\ln(s_3) + (1-\rho)\ln(s_4) + \frac{a}{\rho}\int_0^1 J(x)dx - 2ac$$

where (a,s_3,s_4) is the unique solution of

$$(2.14)\qquad c = \int_{-\infty}^{\infty} s_3 f_0(x)J(H(x))\exp[-aJ(H(x))/\rho]\,dH(x)$$

and

$$(2.15)\qquad 1 = \int_{-\infty}^{\infty} s_3 f_0(x)\exp[-aJ(H(x))/\rho]\,dH(x)$$

$$(2.16)\qquad 1 = \int_{-\infty}^{\infty} s_4 g_0(x)\exp[aJ(H(x))/\rho]\,dH(x).$$

Remark:

(i) $I_A(c;J)$ is attained by

$$(2.17)\qquad f(x) = F'(x) = s_3 f_0(x)\exp[-aJ(H(x))/\rho]$$

and

$$(2.18)\qquad g(x) = G'(x) = s_4 g_0(x)\exp[aJ(H(x))/\rho]\ .$$

(ii) $c(a)$ can be proved to be a strictly decreasing and continuous function of $a \geq 0$ by using the methods of Woodworth [27], (see Lemmas 4,5 of his appendix).

The following theorem makes Property B more specific.

<u>Theorem 2.2. If J_N satisfies part(i) of Property B and converges in the first mean to J which satisfies part(iii) of Property B then part(ii) of Property B is satisfied.</u>

The detailed proof will be presented in the appendix. Following are two examples of rank statistics for testing shift alternatives. We consider normal shift alternatives $F_0 = \Phi(x-\theta)$ and $G_0 = \Phi(x)$ where Φ is the standard normal cumulative distribution function.

<u>Example 2.3.</u> The Wilcoxon test is based on the two-sample scores statistic $J(u) = u - \frac{1}{2}$, and $J_N(u) = i/(N+1) - \frac{1}{2}$ $u \in [i/(N+1), (i+1)/(N+1))$, $i=0,1,---,N$. It is easy to verify that J_N and J satisfy the part (i) and (iii) of Property B respectively and that J_N converges to J in the first mean. Thus Theorem 2.2 yields Property B. From (2.1), (2.2) and (2.3) we obtain after a little manipulation

(2.19) $I_H(c;\text{Wilcoxon})$

$$= 4ch+\rho\ln(K(\rho)) -$$

$$\ln\left\{\rho^2+(1-\rho)^2K^2(\rho)+\rho(1-\rho)K(\rho)\left[\exp(h/\rho)+\exp(-h/\rho)\right]\right\}/2$$

where h is the unique solution of

(2.20) $$c(h) = \int_0^1 \frac{(u-\frac{1}{2})K(\rho)\exp[2h(u-\frac{1}{2})/\rho]}{(1-\rho)+\rho K(\rho)\exp[2h(u-\frac{1}{2})/\rho]}du$$

with

(2.21) $$K(\rho) = \frac{(1-\rho)[1-\exp(2h)]}{\rho[\exp(2h-h/\rho)-\exp(h/\rho)]} .$$

This result was reported by Stone [26], Hoadley [15], and Woodworth [27] in a slightly different form. Again from (2.13) through (2.16) we have after a routine calculation

(2.22) $$I_A(c;\text{Wilcoxon}) = \rho\ln(s_3) + (1-\rho)\ln(s_4) - 2ac$$

where (a,s_3,s_4) is the unique solution of

(2.23) $$c = \int_{-\infty}^{\infty} \frac{s_3}{\sqrt{2\pi}}(H(x)-\tfrac{1}{2})\exp\{-(x-\theta)^2/2-a(H(x)-\tfrac{1}{2})/\rho\}dx$$

(2.24) $$1 = \int_{-\infty}^{\infty} \frac{s_3}{\sqrt{2\pi}} \exp\{-(x-\theta)^2/2-a(H(x)-\tfrac{1}{2})/\rho\}dx$$

(2.25) $$1 = \int_{-\infty}^{\infty} \frac{s_4}{\sqrt{2\pi}} \exp\{-x^2/2+a(H(x)-\tfrac{1}{2})/\rho\}dx$$

with $H(x) = \rho F(x) + (1-\rho)G(x)$ and

(2.26) $$F'(x) = \frac{s_3}{\sqrt{2\pi}} \exp\{-(x-\theta)^2/2-a(H(x)-\tfrac{1}{2})/\rho\}$$

(2.27) $$G'(x) = \frac{s_4}{\sqrt{2\pi}} \exp\{-x^2/2+a(H(x)-\tfrac{1}{2})/\rho\} .$$

<u>Remark:</u> If we take $T_N(F_{0m},G_{0n}) = \int_{-\infty}^{\infty} G_{0n}(x)dF_{0m}(x)$ as

Hoadley does in page 374 of [15], then the equivalent forms of (2.22) through (2.27) can be expressed as follows:

$$(2.22)^* \quad I_A(c;\text{Wilcoxon}) = \rho\ln(s_3) + (1-\rho)\ln(s_4) - 2ac + a$$

where (a, s_3, s_4) is the unique solution of

$$(2.23)^* \quad c = \int_{-\infty}^{\infty} \frac{s_3}{\sqrt{2\pi}} G(x)\ \exp\{-(x-\theta)^2/2 - aG(x)/\rho\}dx$$

$$(2.24)^* \quad 1 = \int_{-\infty}^{\infty} \frac{s_3}{\sqrt{2\pi}} \exp\{-(x-\theta)^2/2 - aG(x)/\rho\}dx$$

$$(2.25)^* \quad 1 = \int_{-\infty}^{\infty} \frac{s_4}{\sqrt{2\pi}} \exp\{-x^2/2 + aF(x)/(1-\rho)\}dx$$

with $H(x) = \rho F(x) + (1-\rho)G(x)$ and

$$(2.26)^* \quad F'(x) = \frac{s_3}{\sqrt{2\pi}} \exp\{-(x-\theta)^2/2 - aG(x)/\rho\}$$

$$(2.27)^* \quad G'(x) = \frac{s_4}{\sqrt{2\pi}} \exp\{-x^2/2 + aF(x)/(1-\rho)\}\ .$$

<u>Example 2.4</u>. The Fisher-Yates(normal scores) test is based on the two-sample scores statistic with $J(u) = \Phi^{-1}(u)$, the inverse of the standard normal distribution function, and $J_N(u)$ defined as the expected value of the i-th smallest order statistic from a standard normal sample of size N. Theorem a of [11] V.1.4 shows that J_N converges to J in quadratic mean. Thus Theorem 2.2 gives Property B. From (2.1) through (2.3) we obtain

$$(2.28) \quad I_H(c;ns) = 2hc - 2hv - \int_{-\infty}^{\infty} \ln\{(1-\rho) + \rho\exp[2h(x-v)/\rho]\}\varphi(x)dx$$

where $\varphi(x) = d\Phi(x)/dx$, and (h,v) is the unique solution of

$$(2.29) \qquad 1 = \int_{-\infty}^{\infty} \frac{\exp[2h(x-v)/\rho]}{(1-\rho)+\rho\exp[2h(x-v)/\rho]}\varphi(x)dx$$

and

$$(2.30) \qquad c = \int_{-\infty}^{\infty} \frac{x\exp[2h(x-v)/\rho]}{(1-\ )+\ \exp[2h(x-v)/\rho]}\varphi(x)dx \ .$$

This result is given by Woodworth [27]. Again from (2.13) through (2.16) we obtain

$$(2.31) \qquad I_A(c;ns) = \rho\ln(s_3) + (1-\rho)\ln(s_4) - 2ac$$

where (a,s_3,s_4) is the unique solution of

$$(2.32) \qquad c = \int_{-\infty}^{\infty} \frac{s_3}{\sqrt{2\pi}}\Phi^{-1}(H(x))\exp\{-(x-\theta)^2/2 - a\,\Phi^{-1}(H(x))/\rho\}dx$$

and

$$(2.33) \qquad 1 = \int_{-\infty}^{\infty} \frac{s_3}{\sqrt{2\pi}}\exp\{-(x-\theta)^2/2 - a\,\Phi^{-1}(H(x))/\rho\}dx$$

$$1 = \int_{-\infty}^{\infty} \frac{s_4}{\sqrt{2\pi}}\exp\{-x^2/2 + a\,\Phi^{-1}(H(x))/\rho\}dx.$$

As before $H(x) = \rho F(x) + (1-\rho)G(x)$ and here

$$(2.34) \qquad F'(x) = \frac{s_3}{\sqrt{2\pi}}\exp\{-(x-\theta)^2/2 - a\,\Phi^{-1}(H(x))/\rho\}$$

$$(2.35) \qquad G'(x) = \frac{s_4}{\sqrt{2\pi}}\exp\{-x^2/2 + a\,\Phi^{-1}(H(x))/\rho\} \ .$$

3. The Chernoff efficiency of the Wilcoxon test relative to the two-sample t-test. Before going further, we give some general remarks on the relationship of Chernoff efficiency and the information numbers I_H and I_A as defined in section 2. Let $\{T_N\}$, N=1,2,--- be a sequence of test statistics for testing H against A. Let P_H and P_A be the probability measures associated with H and A and such that

$$(3.1) \qquad P_H\{T_N \to \mu_0\} = 1, \quad P_A\{T_N \to \mu_1\} = 1$$

where $\mu_0 < \mu_1$. We further assume that for $\mu_0 < c < \mu_1$,

$$(3.2) \qquad \alpha_N = P_H\{T_N > c\} = \exp\{-N[I_H(c) + o(1)]\} \text{ as } N \to \infty$$

and

$$(3.3) \qquad \beta_N = P_A\{T_N \leq c\} = \exp\{-N[I_A(c) + o(1)]\} \text{ as } N \to \infty$$

where $I_H(c)$ and $I_A(c)$ are respectively the information numbers under P_H and P_A, and $0 < I_H(c), I_A(c) < \infty$. That is, the error probabilities for the test, based on T_N and using an approximate large sample cut-off point $c \in (\mu_0, \mu_1)$, converges to zero exponentially. Let

$$(3.4) \qquad I = \sup_{\mu_0 \leq c \leq \mu_1} \{\min[I_H(c), I_A(c)]\}.$$

"I" will be called the information index determined by T_N. Thus if we achieve the same level of significance and same power at a fixed alternative, for the statistic T_1 (with information index $I_{(1)}$) at sample size N_1 that the statistic T_2 (with information index $I_{(2)}$), requires with sample size N_2 then Chernoff defines the asymptotic relative efficiency of T_1 relative T_2 to be

$$(3.5) \qquad e^{(c)}_{T_1,T_2} = \lim_{N_1,N_2\to\infty} N_2/N_1 = I_{(1)}/I_{(2)}$$

provided for any fixed s; $0 < s < \infty$,

$$(3.6) \qquad \inf_{\mu_0 \leq c \leq \mu_1} \left\{\beta_{N_i} + s\,\alpha_{N_i}\right\} = \exp\left\{-N\left[I_{(i)} + o(1)\right]\right\}$$

as $N\to\infty$, i=1,2. In order to calculate the information index determined by the type B sequence of Chernoff-Savage statistics $T_N(F_{0m},G_{0n})$ defined as in (1.3) we must show (3.1) holds for $T_N(F_{om},G_{on})$. More generally, we prove

<u>Theorem 3.1.</u> $T_N(F_{0m},G_{0n}) \to \int_{-\infty}^{\infty} J(H_0(x))dF_0(x)$ <u>almost surely as</u> $N\to\infty$ <u>for</u> $J(.)$ <u>satisfying Property B where</u> $H_0(x) = \rho F_0(x) + (1-\rho)G_0(x)$.

<u>Proof:</u>

$$\left|\int_{-\infty}^{\infty} J_N(NH_{0N}(x)/N+1)dF_{0m}(x) - \int_{-\infty}^{\infty} J(H_0(x))dF_0(x)\right|$$

$$= \left|\int_{-\infty}^{\infty}\left\{J_N(NH_{0N}(x)/N+1) - J(NH_{0n}(x)/N+1)\right\}dF_{0m}(x)\right.$$

$$+ \int_{-\infty}^{\infty}\{J(NH_{0N}(x)/N+1)-J^{(M)}(\rho F_{0m}(x)+(1-\rho)G_{0n}(x))\}dF_{0m}(x)$$

$$+ \int_{-\infty}^{\infty}\{J^{(M)}(\rho F_{0m}(x)+(1-\rho)G_{0n}(x))-J^{(M)}(H_0(x))\}dF_{0m}(x)$$

$$+ \int_{-\infty}^{\infty}J^{(M)}(H_0(x))d\{F_{0m}(x)-F_0(x)\}$$

$$+ \int_{-\infty}^{\infty}\{J^{(M)}(H_0(x)) - J(H_0(x))\}dF_0(x)\Big|$$

$$\leq \frac{N+1}{m}\sum_{i=1}^{N}\left|J_N(i/N+1) - J(i/N+1)\right|\frac{1}{N+1}$$

$$+ \left|\int_{-\infty}^{\infty}J(NH_{0N}(x)/N+1)dF_{0m}(x) - T^{(M)}(F_{0m},G_{0n})\right|$$

$$+ \int_{-\infty}^{\infty}\left|J^{(M)}(\rho F_{0m}(x)+(1-\rho)G_{0n}(x)) - J^{(M)}(H_0(x))\right|dF_{0m}(x)$$

$$+ \left|\int_{-\infty}^{\infty}J^{(M)}(H_0(x))d\{F_{0m}(x)-F_0(x)\}\right| + \left|T^{(M)}(Q_0)-T(Q_0)\right| .$$

By part (ii) of Property B, the first term goes to zero as $N\to\infty$. By lemma 1.9, for given $\varepsilon>0$ there exist $N_0(\varepsilon)$ and $M_0(\varepsilon)$ such that the second term is bounded by ε for all $N\geq N_0(\varepsilon)$ and all $M\geq M_0(\varepsilon)$. By the Glivenko-Cantelli theorem and the uniform continuity of $J^{(M)}$, for given $\varepsilon>0$ there exists an integer $N_0(\varepsilon)$ such that

$$\left|J^{(M)}(\rho F_{0m}(x)+(1-\rho)G_{0n}(x)) - J^{(M)}(H_0(x))\right| < \varepsilon$$

almost surely for all $N\geq N_0(\varepsilon)$ and all $x\in(-\infty,\infty)$. Thus the third term will be almost surely bounded by ε for sufficiently large N. The fourth term is a.s. less than $\mathcal{E}$, using the Helly-Bray lemma (p.180 of [19]) for sufficiently large N. By lemma 1.5, the fifth term will be less than $\mathcal{E}$, for sufficiently large M. Put everything together, we complete the proof of Theorem 3.1.

Theorem 1.10 and Theorem 1.11 show that (3.2) and (3.3) will be satisfied for the type B sequence of Chernoff-Savage statistics. In order to verify (3.6) we need the following modification of Chernoff [4](Theorem 2).

Theorem 3.2. Let α_N, β_N, I_H and I_A be defined as in (3.2) and (3.3) for a type B sequence of Chernoff-Savage statistics with information index I. Then for given $\mathcal{E} > 0$ and s; $0 < s < \infty$,

$$\inf_c \, (\beta_N + s\alpha_N)/\exp(-NI + \mathcal{E}) \longrightarrow 0 \quad \text{as } N \longrightarrow \infty \tag{3.10}$$

and

$$\inf_c \, (\beta_N + s\alpha_N)/\exp(-NI - \mathcal{E}) \longrightarrow \infty \quad \text{as } N \longrightarrow \infty. \tag{3.11}$$

Proof: For given $\mathcal{E} > 0$, by the definition of supremum there exists a value c* of c such that $I(c^*) = I - \mathcal{E}/N$ where

$I(c) = \min[I_H(c), I_A(c)]$. Thus

$$\inf_c \, (\beta_N + s\alpha_N)/\exp(-NI + \mathcal{E})$$

$$= \inf_{c} (\beta_N + s\alpha_N) / \exp(-NI(c^*))$$

$$= \exp\{-N[\sup_{c} I_A(c) - I(c^*) + o(1)]\}$$

$$+ s \exp\{-N[\sup_{c} I_H(c) - I(c^*) + o(1)]\}$$

$$\leq \exp\{-N[I_A(c^*) - I(c^*) + o(1)]\}$$

$$+ s \exp\{-N[I_H(c^*) - I(c^*) + o(1)]\}$$

$$\to 0 \quad \text{as } N \to \infty .$$

Thus (3.10) is established. Since $I_H(\mu_0) = 0$, $I_A(\mu_0) = \infty$ and $I_H(\mu_1) = \infty$, $I_A(\mu_1) = 0$, and I_H , I_A are respectively non-decreasing and non-increasing continuous over $[\mu_0, \mu_1]$ then there exists c** in $[\mu_0, \mu_1]$ such that $I = I_H(c^{**}) = I_A(c^{**})$. Suppose that the infimum of $\beta_N + s\alpha_N$ falls in $[\mu_0, \mu_1]$ then

$$\inf_{\mu_0 \leq c \leq c^{**}} (\beta_N + s\alpha_N) / \exp(-NI-\varepsilon)$$

$$\geq \inf_{\mu_0 \leq c \leq c^{**}} s\, P_H\{T_N(F_{0m}, G_{0n}) \geq c\} / \exp(-NI-N\varepsilon)$$

$$\geq s\, P_H\{T_N(F_{0m}, G_{0n}) \geq c^{**}\} / \exp\{-NI_H(c^{**})-N\varepsilon\}$$

$$= s \exp\{-N[I_H(c^{**}) + o(1)]\} / \exp\{-NI_H(c^{**})-N\varepsilon\}$$

$$= s \exp\{N(\varepsilon + o(1))\} \to \infty \quad \text{as } N \to \infty \ .$$

Similarly we can prove (3.11) for the case of infimum of $\beta_N + s\alpha_N$ falling in $[c^{**}, \mu_1]$. Thus we established this theorem.

This theorem says that the minimum of $\ln(\beta_N + s\alpha_N)$ is roughly about $-NI$. Equating (2.8) and (2.13), we obtain the information index determined by $T_N(F_{0m}, G_{0n})$ as follows:

$$I = I_H(c^*; J)$$

where c^* is the unique solution of $I_H(c;J) = I_A(c;J)$, and simple calculations give

$$(3.12) \qquad c^* = \frac{1}{2\rho}\int_0^1 J(u)du + \frac{1}{2(h+a)}\{\rho \ln(s_3/s_1) + (1-\rho)\ln(s_4/s_2)\}$$

and

$$(3.13) \qquad I = \frac{1}{(h+a)}\{\rho[h \ln(s_3) + a \ln(s_1)] + (1-\rho)[h \ln(s_4) + a \ln(s_2)]\}$$

$$- \frac{h}{2\rho}\int_0^1 J(u)du$$

where (h, s_1, s_2) and (a, s_3, s_4) satisfy (2.9) and (2.14), (2.15) and (2.16) respectively. By equating (2.19) and (2.22), the information index of the Wilcoxon test can be presented as follows:

(3.14) I(Wilcoxon)

$$= \frac{1}{4(a+h)}\{4h\rho\ln(s_3)+4h(1-\rho)\ln(s_4)+2a\rho\ln(K(\rho))\}$$

$$- \frac{1}{4(a+h)}\{a\ln[\rho^2+(1-\rho)^2K^2(\rho)+\rho(1-\rho)K(\rho)K^*(\rho)]\}$$

where $K^*(\rho) = \exp(h/\rho) + \exp(-h/(1-\rho))$ and (h,a,s_3,s_4) satisfy (2.20) and (2.23),(2.24),(2.25) but replacing c by

(3.15) $$c^* = \frac{1}{4(a+h)}\{2\rho\ln(s_3)+2(1-\rho)\ln(s_4)-\rho\ln(K(\rho))\}$$

$$+ \frac{1}{4(a+h)}\ln\{\rho^2+(1-\rho)^2K^2(\rho)+\rho(1-\rho)K(\rho)K^*(\rho)\} .$$

The information indices and critical values are tabulated in Tabe 3.1 for $\rho = 0.5, 0.25$ and $\theta = 0.1, 0.25, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0$.

Table 3.1.

Information index and critical value of the two-sample Wilcoxon test under normal shift alternative: $F_0(x) = \Phi(x-\theta)$ and $G_0(x) = \Phi(x)$.

=	0.5		0.25	
0	c*	I	c*	I
0.10	0.51410	0.00029	0.51411	0.00022
0.25	0.53522	0.00186	0.53521	0.00139
0.50	0.57015	0.00741	0.57013	0.00556
0.75	0.60453	0.01655	0.60449	0.01241
1.00	0.63810	0.02912	0.63799	0.02183
1.50	0.70185	0.06355	0.70146	0.04763
2.00	0.75972	0.10827	0.75871	0.08102
2.50	0.81045	0.16015	0.80845	0.11952
3.00	0.85348	0.21580	0.84969	0.16009

In order to calculate the Chernoff efficiency of the Wilcoxon relative to the two-sample t-test, we compute an explicit expression for the information index of the two-sample t-test. Consider " Student" statistic

$$T_N^{(2)} = (mn/NS_N)^{\frac{1}{2}}(\overline{X} - \overline{Y})$$

where $(N-2)S_N^2 = \sum_{i=1}^{m}(X_i - \overline{X})^2 + \sum_{j=1}^{n}(Y_j - \overline{Y})^2$ and $\overline{X}$, $\overline{Y}$ are the two sample means. In [1] , Abrahamson derived following result

(3.16) $$\lim_{N\to\infty} -N^{-1} \; P_H\left\{T_N^{(2)} \geq cN^{\frac{1}{2}}\right\} = \tfrac{1}{2}\ln(1+c^2).$$

Consider

$$\beta_N$$

$$= P_A\left\{T_N^{(2)} \leq cN^{\frac{1}{2}}\right\}$$

$$= P_H\left\{T_N^{(2)} \leq N^{\frac{1}{2}}\left[c-\theta(mn/S_N N^2)^{\frac{1}{2}}\right]\right\}.$$

By a convergence theorem of Cramer(p. 254 of [6]), for every $\varepsilon > 0$ there exists an integer $N_0(\varepsilon)$ such that for all $N \geq N_0(\varepsilon)$

$$P_H\left\{T_N^{(2)} \leq N^{\frac{1}{2}}\left[c-\theta(mn/(1-\varepsilon)N^2)^{\frac{1}{2}}\right]\right\}$$

$$\leq P_H\left\{T_N^{(2)} \leq N^{\frac{1}{2}}\left[c-\theta(mn/S_N N^2)^{\frac{1}{2}}\right]\right\}$$

$$\leq P_H\left\{T_N^{(2)} \leq N^{\frac{1}{2}}\left[c-\theta(mn/(1+\varepsilon)N^2)^{\frac{1}{2}}\right]\right\}.$$

The property of symmetry of the central t gives, for all $N \geq N_0(\varepsilon)$

$$- N^{-1} \ln P_H\left\{T_N^{(2)} \geq N^{\frac{1}{2}}\left[\theta(mn/(1+\varepsilon)N^2)^{\frac{1}{2}} - c\right]\right\}$$

$$\leq -N^{-1} \ln P_H \left\{ T_N^{(2)} \geq N^{\frac{1}{2}} [\theta(mn/S_N N2)^{\frac{1}{2}} - c] \right\}$$

$$\leq -N^{-1} \ln P_H \left\{ T_N^{(2)} \geq N^{\frac{1}{2}} [\theta(mn/(1-\varepsilon)N^2)^{\frac{1}{2}} - c] \right\} .$$

Thus (3.16) yields

$$(3.17) \qquad \lim_{N\to\infty} -N^{-1} \ln(\beta_N) = \tfrac{1}{2} \ln\{1+[\theta(\rho(1-\rho))^{\frac{1}{2}}-c]^2\}$$

where $m/N \to \rho$ and $n/N \to 1-\rho$ as $N \to \infty$. Equating (3.16) and (3.17) we obtain $c^* = \frac{1}{2}\theta(\rho(1-\rho))^{\frac{1}{2}}$ and

$$(3.18) \qquad I_{(2)} = \tfrac{1}{2} \ln\{1 + \rho(1-\rho)\theta^2/4\} .$$

Thus the Chernoff efficiency of the Wilcoxon test relative to the two-sample t-test is

$$(3.19) \qquad e^{(c)}_{w,t} = 2I(\text{Wilcoxon})/\ln\{1 + \rho(1-\rho)\theta^2/4\}$$

where I(Wilcoxon) is given in (3.14). The values of $e^{(c)}_{w,t}$ corresponding to various normal shift alternatives are given in Table 4.3 and Figure 4.4.

4. <u>Small sample comparisons.</u> The Hodges-Lehmann small sample efficiency defined in [16] has been further discussed by Milton for the Wilcoxon and other rank tests (p.29 of [20]). Values are given on page 37 of [20] for the Wilcoxon test under various normal shift alternatives relative to the two-sample t-test at sample size $m = n = 7$. Table 4.1 reproduces these values for ease in reference. We note the nearly constant values. An additional set of finite sample($m = n = 7$) Hodges-Lehmann efficiency values were computed using Milton's tables of rank order probabilities(table c-1, pp.273-274). Comparisons are given in Table 4.2 with type I and type II errors chosen to be roughly equal as in the Chernoff limit. Again the flat shape of the finite sample curve is in agreement with the Chernoff curve and the efficiency values appear to increase toward the limiting value as the error probabilities get small.

Table 4.1.

Hodges-Lehmann efficiency of the one-sided Wilcoxon test relative to the two-sample t-test.

θ	type I error	power	efficiency
0.20	0.01	0.022801	0.9657
0.40	0.01	0.047134	0.9637
0.60	0.01	0.088646	0.9618
0.80	0.01	0.152294	0.9602
1.00	0.01	0.240128	0.9587
1.50	0.01	0.535000	0.9558
2.00	0.01	0.807526	0.9544
3.00	0.01	0.991453	0.9552
0.20	0.05	0.096541	0.9508
0.40	0.05	0.168787	0.9495
0.60	0.05	0.268527	0.9484
0.80	0.05	0.391083	0.9476
1.00	0.05	0.525236	0.9470
1.50	0.05	0.818054	0.9466
2.00	0.05	0.959479	0.9476
3.00	0.05	0.999646	0.9528

Table 4.2.

Hodges-Lehmann efficiency of the one-sided Wilcoxon test relative to the two-sample t-test with type I and type II errors chosen to be roughly equal.

test	m=n	θ	type I error	type II error	efficiency
Wilcoxon	7	0.20	0.450760	0.407194	0.9203
t	6	0.20	0.450760	0.411949	
t	7	0.20	0.450760	0.401177	
Wilcoxon	7	0.40	0.355189	0.365334	0.9221
t	6	0.40	0.355189	0.374854	
t	7	0.40	0.355189	0.353909	
Wilcoxon	7	0.60	0.310026	0.282588	0.9236
t	6	0.60	0.310026	0.295671	
t	7	0.60	0.310026	0.267353	
Wilcoxon	7	0.80	0.267485	0.210797	0.9251
t	6	0.80	0.267485	0.226187	
t	7	0.80	0.267485	0.193820	
Wilcoxon	7	1.00	0.191436	0.185279	0.9298
t	6	1.00	0.191436	0.204583	
t	7	1.00	0.191436	0.166609	
Wilcoxon	7	1.50	0.104314	0.088269	0.9392
t	6	1.50	0.104314	0.108686	
t	7	1.50	0.104314	0.073151	

Wilcoxon	7	2.00	0.048661	0.041630	0.9494
t	6	2.00	0.048661	0.059969	
t	7	2.00	0.048661	0.031581	
Wilcoxon	7	3.00	0.008741	0.009914	0.9601
t	6	3.00	0.008741	0.020928	
t	7	3.00	0.008741	0.005764	

Table 4.3.

Chernoff efficiency of the Wilcoxon one-sided test of fit relative to the two-sample t-test under normal shift alternatives: $F_0(x) = \Phi(x-\theta)$ and $G_0(x) = \Phi(x)$.

θ \ ρ	0.5	0.25
0.10	0.955	0.957
0.25	0.955	0.954
0.50	0.956	0.954
0.75	0.958	0.954
1.00	0.960	0.953
1.50	0.965	0.950
2.00	0.970	0.943
2.50	0.971	0.930
3.00	0.967	0.909

It is conjectured in the absence of a proof that $\lim_{\theta \to 0} e_{w,t}^{(c)}(\theta) = 3/\pi$ as it is for the Bahadur efficiency. Comparing Tables 4.1, 4.2 and 4.3, we note that Chernoff efficiency is in reasonable agreement with the finite sample results. The shape of the bold curves presented in Figure 4.4 are nearly flat as in the finite sample comparisons in contrast to the Bahadur efficiency curves(faint curves in the same figure, see Figure 8.1 of [15]) which decreases more rapidly. We note also that the performance of the Wilcoxon test decreases with unequal sample size for the Chernoff efficiency criterion as it does in the Bahadur efficiency.

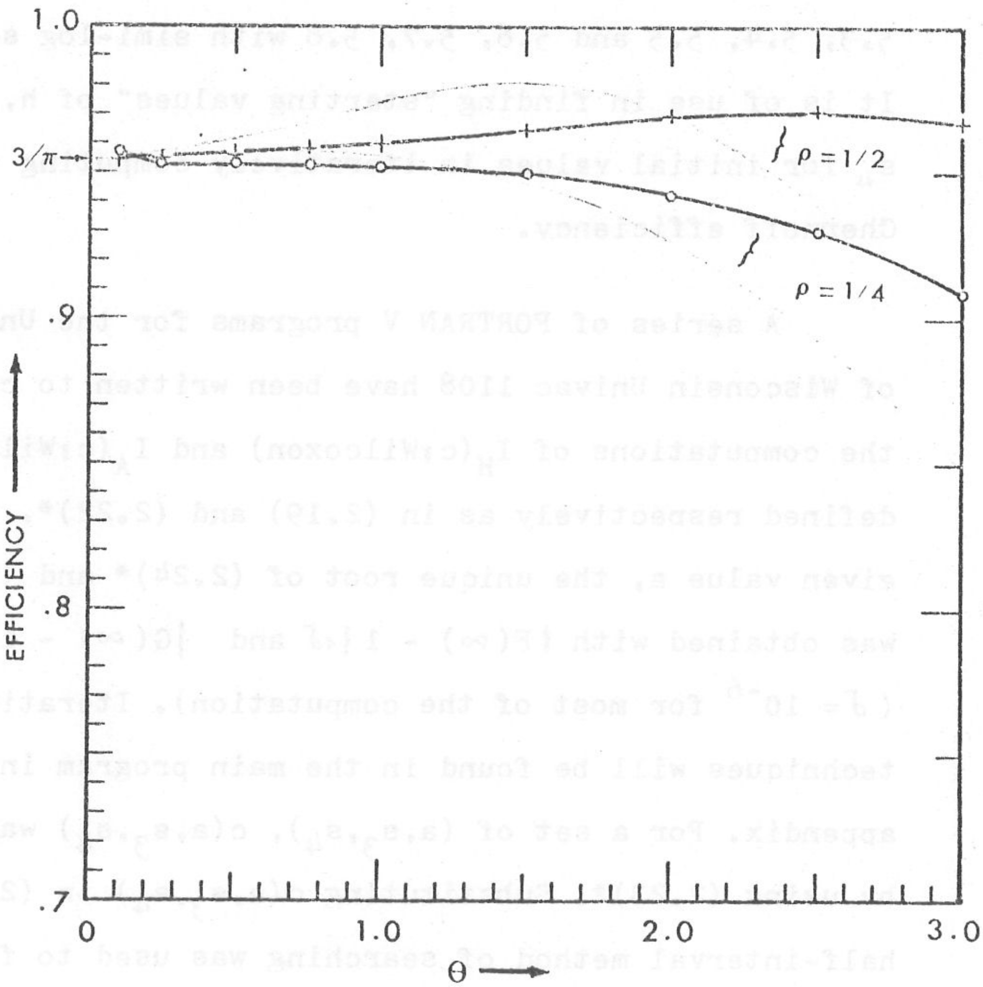


Fig. 4.4. Chernoff Efficiency (bold curve)

Bahadur Efficiency (faint curve).

5. Computations. As the Chernoff efficiency computations were nontrivial, a detailed description may be of interest. Values and curves for h, a, s_3, s_4 corresponding to various alternatives and $\rho = 0.5$, 0.25 are presented in Tables 5.1 and 5.2 and Figures 5.3, 5.4, 5.5 and 5.6, 5.7, 5.8 with simi-log scale. It is of use in finding "starting values" of h, a, s_3, s_4 for initial values in iteratively computing the Chernoff efficiency.

A series of FORTRAN V programs for the University of Wisconsin Univac 1108 have been written to carry out the computations of I_H(c;Wilcoxon) and I_A(c;Wilcoxon) defined respectively as in (2.19) and (2.22)*. For a given value a, the unique root of (2.24)* and (2.25)* was obtained with $|F(\infty) - 1| < \delta$ and $|G(\infty) - 1| < \delta$. ($\delta = 10^{-6}$ for most of the computation). Iteration techniques will be found in the main program in the appendix. For a set of (a, s_3, s_4), $c(a, s_3, s_4)$ was found by using (2.23)*. Substituting $c(a, s_3, s_4)$ in (2.20) a half-interval method of searching was used to find the unique root h of (2.20) employing strict monotonicity of c(h). If $I_H \neq I_A$ for this set of (h, a, s_3, s_4) then by monotonicity of $I_H(c(h))$ and $I_A(c(a))$ we can suitably change a and obtain a different set of (h, a, s_3, s_4);

repeating the procedure until $|I_H - I_A| \leq \varepsilon I_t$ ($\varepsilon = 0.0005$) where I_t is the information index determined by the two-sample t-test. Integrals were evaluated by Gauss-Legendre integration(NIQUAD subroutine). Errors in this subroutine were detected and a corrected version was checked for accuracy using a known integral and by comparing results using 25,50, and 100 points in the integrations. Differences were detected in the sixth digit only of c using 25 points and 100 points in the integration. Numerical solutions of the coupled differential equations (2.26)* and (2.27)* were computed by repeated applications of a fifth-order predictor -corrector scheme developed by Hamming [12] using the DEPC subroutine. The method requires the approximate solution at points $x-3\Delta$, $x-2\Delta$, $x-\Delta$, and x to compute an approximation to the solution at the $x+\Delta$. The required starting points were computed using an error-controlled fourth-order Runge-Kutta scheme. The accuracy of the completed solution using the DEPC subroutine is at least 10^{-6} . With efficiency values presented in Table 4.3 believed accurate to one unit in the third decimal place.

Table 5.1.

Values of h, a, s_3, s_4 for $\rho = \frac{1}{2}$

θ	h	a	s_3	s_4
0.10	0.042338	0.042323	1.044790	0.959992
0.25	0.105898	0.106178	1.122457	0.907704
0.50	0.212357	0.214841	1.287111	0.837543
0.75	0.319982	0.328516	1.512490	0.784056
1.00	0.429359	0.449930	1.828194	0.743392
1.50	0.655878	0.728325	2.962126	0.690220
2.00	0.897148	1.079062	5.741553	0.663512
2.50	1.157656	1.543400	14.323060	0.653815
3.00	1.444089	2.176562	50.964966	0.655735

Table 5.2.

Values of h, a, s_3 , s_4 for $\rho = \frac{1}{4}$

θ	h	a	s_3	s_4
0.10	0.021186	0.031720	1.068122	0.979734
0.25	0.052945	0.079650	1.190902	0.952281
0.50	0.106226	0.161111	1.468323	0.913481
0.75	0.160195	0.246233	1.882705	0.881849
1.00	0.215150	0.337063	2.524250	0.855988
1.50	0.329578	0.544506	5.323727	0.817618
2.00	0.452261	0.803750	14.702151	0.791803
2.50	0.585930	1.139375	57.954823	0.773562
3.00	0.731770	1.579181	361.178593	0.758027

Fig. 5.3. Curves of a and h for $\rho = \frac{1}{2}$

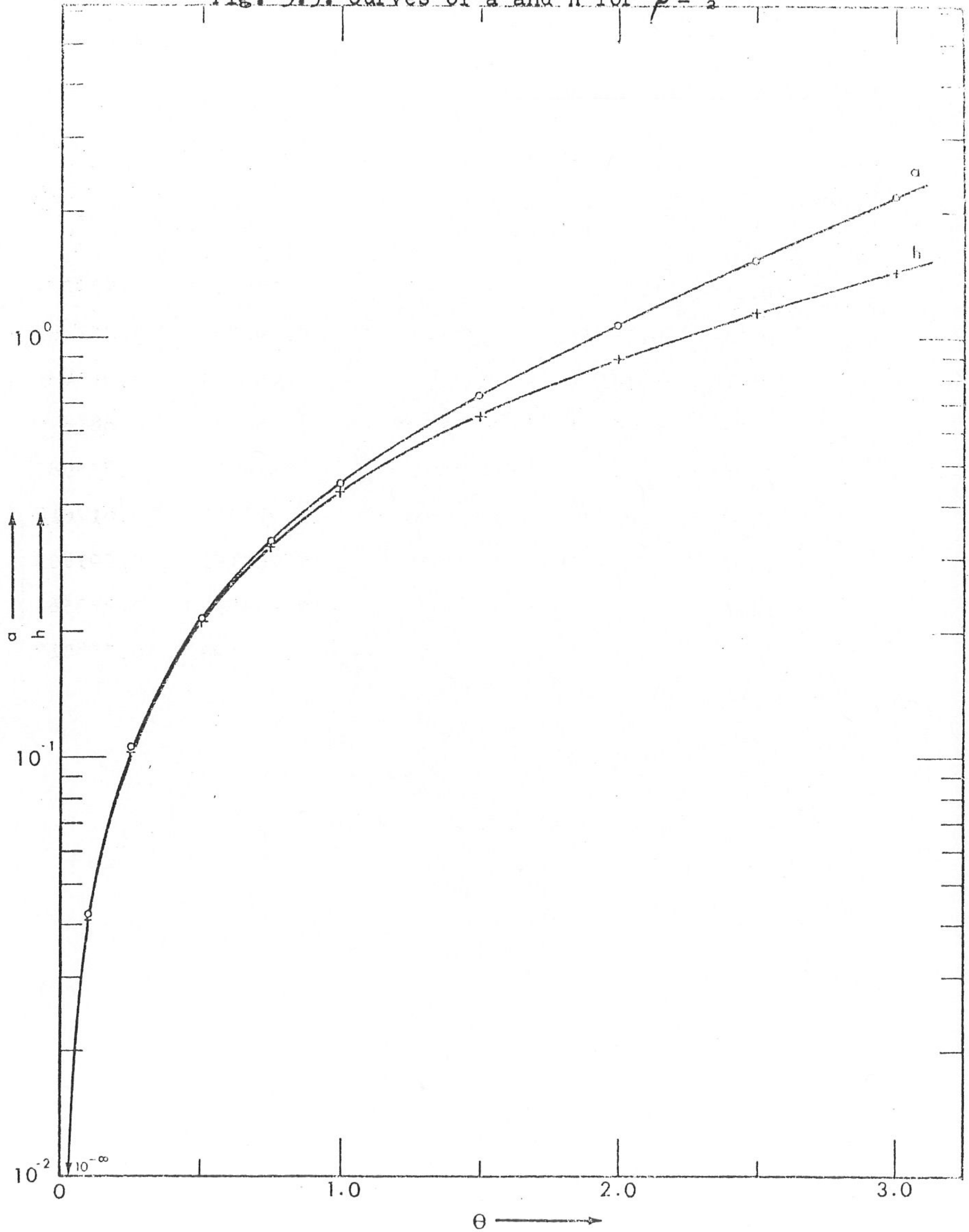


Fig. 5.4. Curves of a and h for $\rho = \frac{1}{4}$

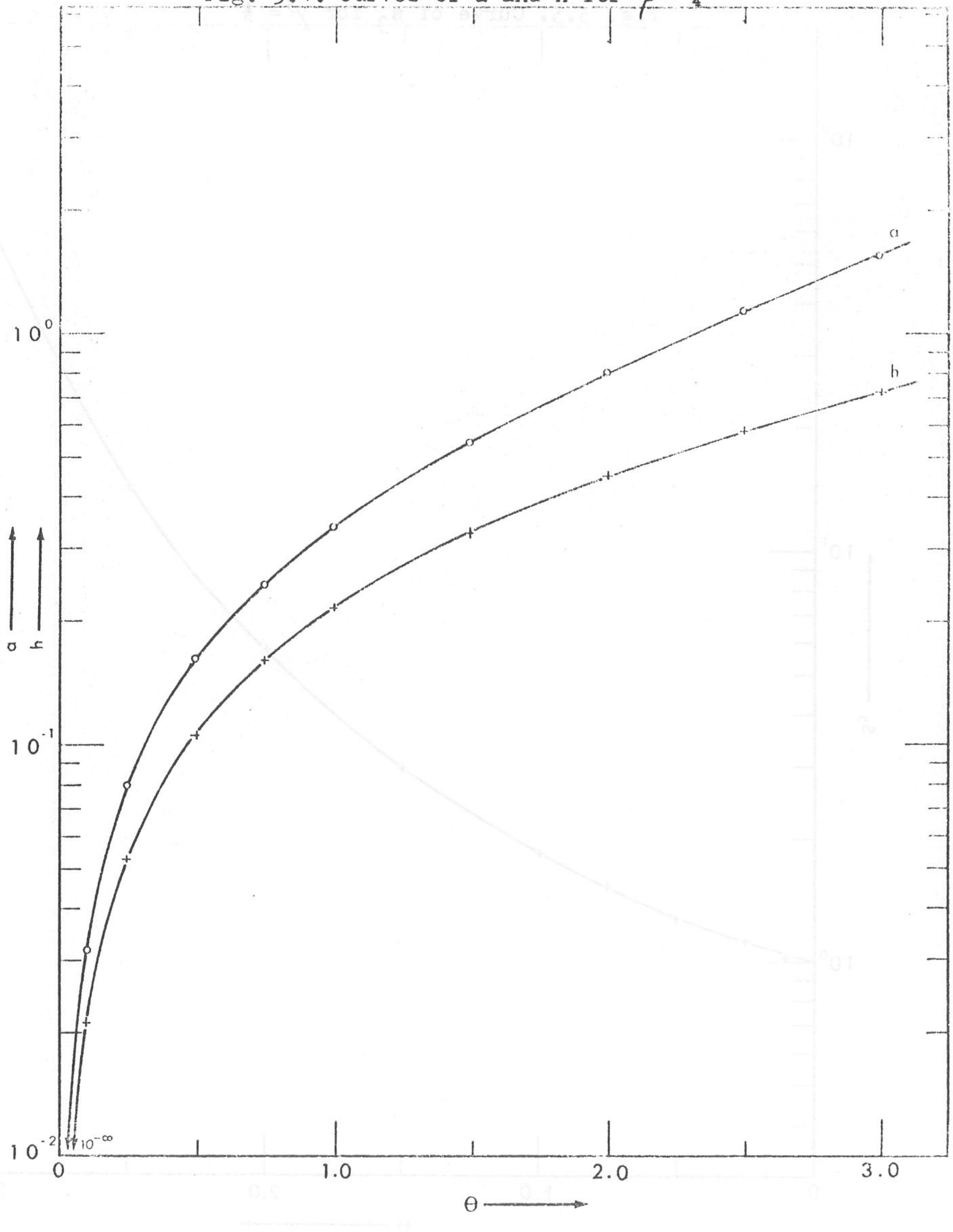


Fig. 5.5. Curve of s_3 for $\rho = \frac{1}{2}$

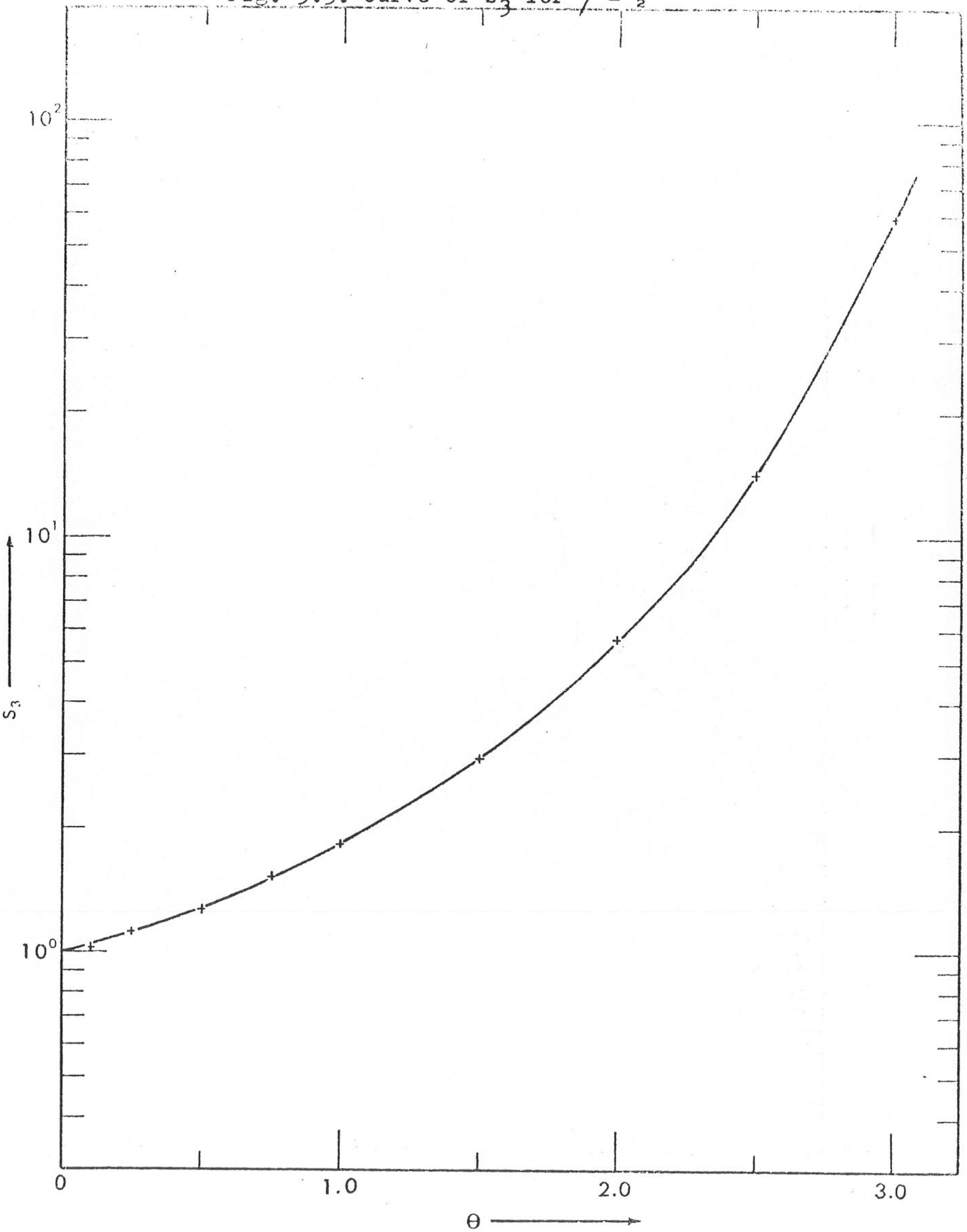
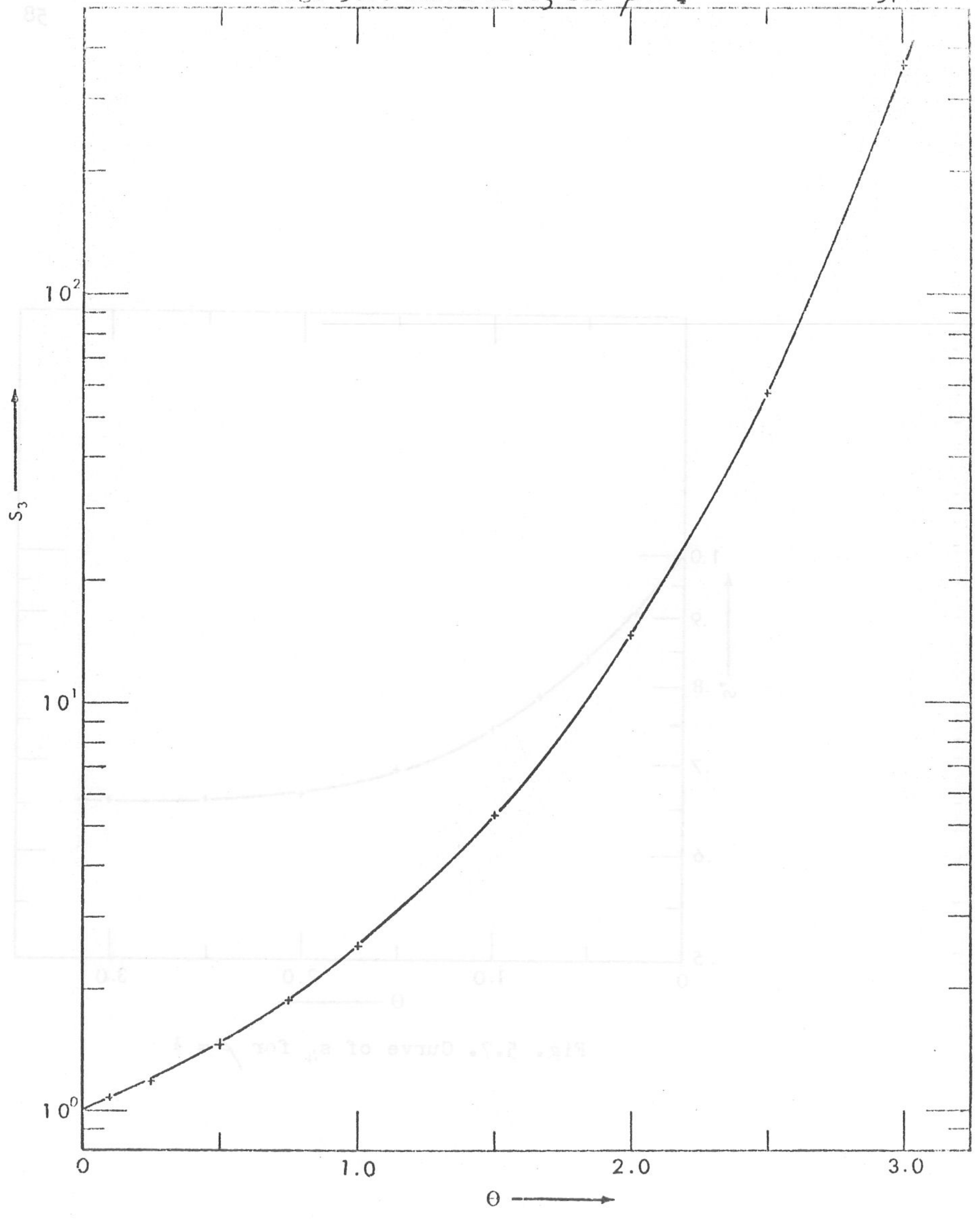


Fig. 5.6. Curve of s_3 for $\rho = \frac{1}{4}$



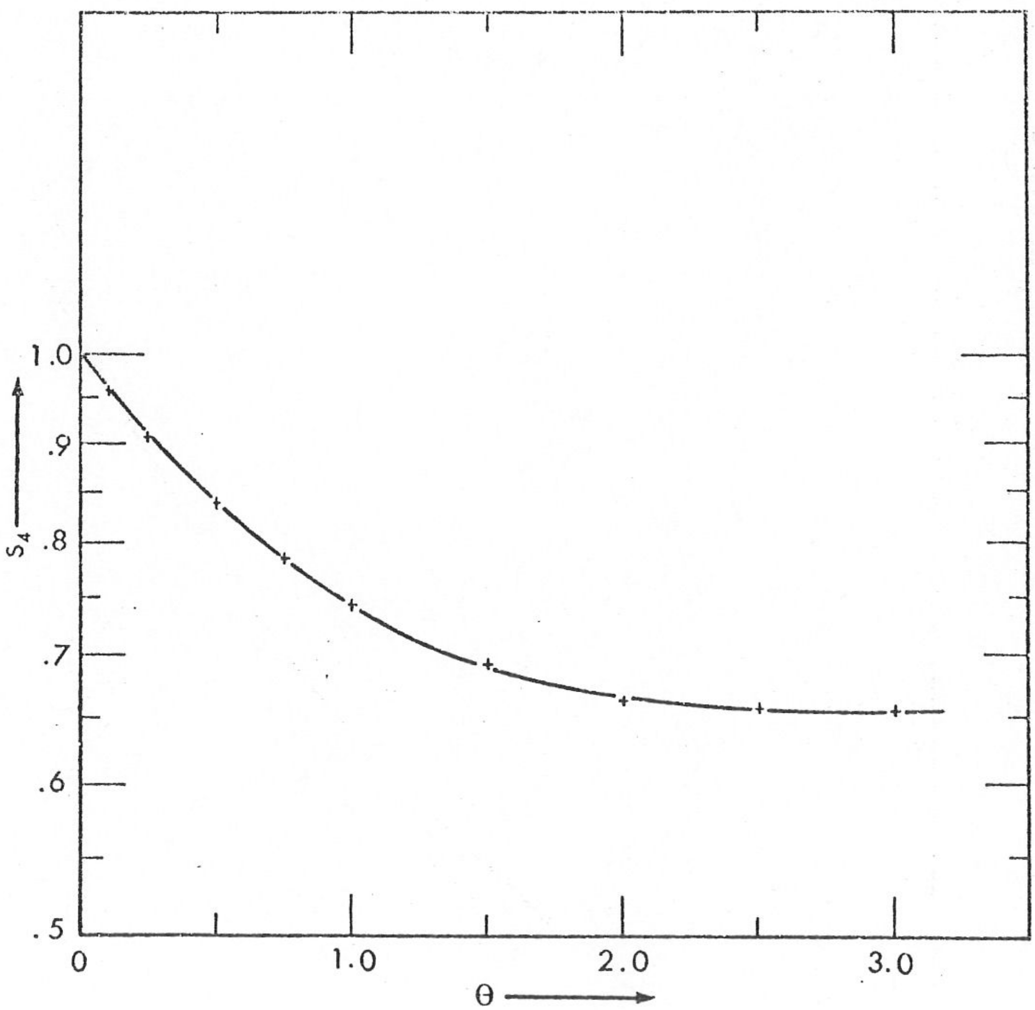


Fig. 5.7. Curve of s_4 for $\rho = \frac{1}{2}$

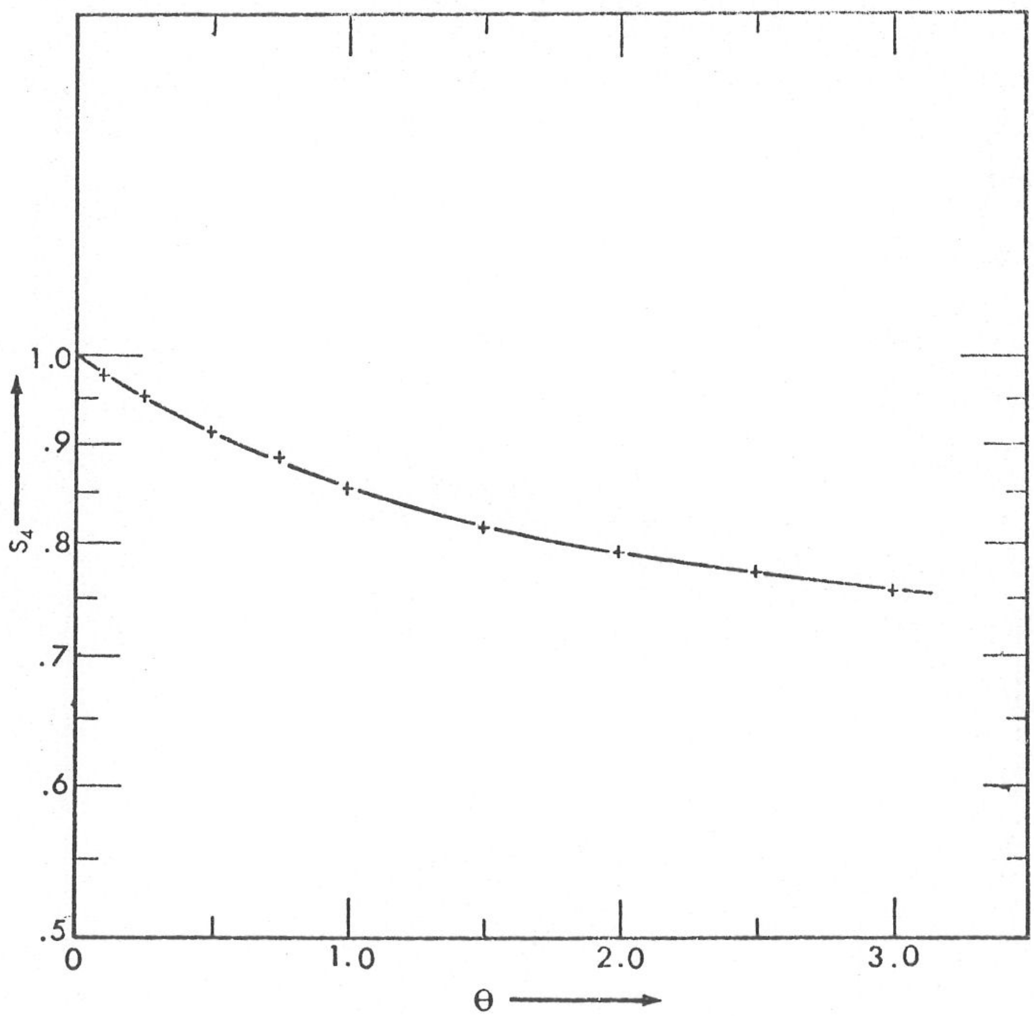


Fig. 5.8. Curve of s_4 for $\rho = \frac{1}{4}$

ACKNOWLEDGEMENTS

I wish to express my great appreciation for the guidance and advice of Professor Jerome Klotz.

I am also greatly indebted to Professor Gouri K. Bhattacharyya and Professor Richard A. Johnson for their careful reading, criticism, and assistance which greatly improved the thesis.

I would also like to thank Harry Hull and Jane Henkel and Professor J.M. Yohe for advice in computing and Dr. W. Yang for drawing pictures.

I am also thankful to the Department of Statistics for arranging financial assistance for the period September, 1968 to August, 1972. Research was supported in part by the National Science Foundation under grant G.P. 12093, the Air Force under grant AF-AFOSR 69-1803 and the Office of Naval Research under grant ONR N00014 -67-A-0128-0017.

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APPENDIX:

Theorem 2.2. If J_N satisfies part (i) of Property B and converges in the first mean to J which satisfies part (iii) of Property B then part (ii) of Property B is satisfied.

Proof: By Lemma 1.1, for given $\varepsilon > 0$, there exist L, U; $0 < L < U < 1$, and $0 < M < \infty$ depending on ε such that

$$\text{(A.1)} \qquad \int_0^L |J(u)|\, du + \int_U^1 |J(u)|\, du < \varepsilon$$

and $|J(u)| \leq M$ for $u \in [L, U]$ and hence $J(.)$ is uniformly continuous over $[L, U]$. Since $J_N(u)$ converges to $J(u)$ in the first mean, then for this particular ε there exists an integer $N_0(\varepsilon)$ such that for all $N \geq N_0(\varepsilon)$

$$\text{(A.2)} \qquad \int_0^1 |J_N(u) - J(u)|\, du < \varepsilon \,.$$

And (A.1) and (A.2) give

$$\text{(A.3)} \qquad \int_0^L |J_N(u)|\, du + \int_U^1 |J_N(u)|\, du < \varepsilon \,.$$

We choose a, b; $0 < a < L < U < b < 1$ with $\min(L-a, b-U) > 2/(N_1+1)$ for some $N_1 \geq N_0(\varepsilon)$. For $N \geq N_0(\varepsilon)$, there exist integer j_N, k_N such that $j_N/(N+1) \leq L < (j_N+1)/(N+1)$ and $k_N/(N+1) > U \geq (k_N-1)/(N+1)$. Then $\min(L-a, b-U) > 2/(N_1+1)$ implies

(A.4) $$a \leq (j_N-1)/(N+1) < j_N/(N+1) \leq L$$

and

(A.5) $$U \leq k_N/(N+1) < (k_N+1)/(N+1) \leq b \; .$$

Thus $$\frac{1}{N+1}\sum_{i=1}^{N}\left|J_N(i/N+1) - J(i/N+1)\right|$$

$$= \frac{1}{N+1}\sum_{i=1}^{j_N}\left|J_N(i/N+1) - J(i/N+1)\right|$$

$$+ \frac{1}{N+1}\sum_{i=j_N+1}^{k_N}\left|J_N(i/N+1) - J(i/N+1)\right|$$

$$+ \frac{1}{N+1}\sum_{i=k_N+1}^{N}\left|J_N(i/N+1) - J(i/N+1)\right|$$

$$\leq \frac{1}{N+1}\sum_{i=1}^{[(N+1)L]}\left\{\left|J_N(i/N+1)\right| + \left|J(i/N+1)\right|\right\}$$

$$+ \frac{1}{N+1}\sum_{i=j_N+1}^{k_N}\left|J_N(i/N+1) - J(i/N+1)\right|$$

$$+ \frac{1}{N+1}\sum_{i=[(N+1)U]+1}^{N}\left\{\left|J_N(i/N+1)\right| + \left|J(i/N+1)\right|\right\}$$

$$\leq \int_0^{[(N+1)L]/(N+1)} \left|J_N(u)\right| du + \int_0^{[(N+1)L]/(N+1)} \left|J(u)\right| du$$

$$+ \frac{1}{N+1} \sum_{i=j_N+1}^{k_N} \left| J_N(i/N+1) - J(i/N+1) \right|$$

$$+ \int_{1+[(N+1)U]/(N+1)}^{1} \left| J_N(u) \right| du + \int_{1+[(N+1)U]/(N+1)}^{1} \left| J(u) \right| du$$

$$\leq \int_0^L \left| J_N(u) \right| du + \int_0^L \left| J(u) \right| du + \int_U^1 \left| J_N(u) \right| du + \int_U^1 \left| J(u) \right| du$$

$$+ \frac{1}{N+1} \sum_{i=j_N+1}^{k_N} \left| J_N(i/N+1) - J(i/N+1) \right|$$

$$< 2\varepsilon + \frac{1}{N+1} \sum_{i=j_N+1}^{k_N} \left| J_N(i/N+1) - J(i/N+1) \right| .$$

The last strict inequality follows from (A.1),(A.3). Next we show the second term of last inequality is less than ε for $N \geq N_3(\varepsilon)$. Define $J_{Ni}(u) = J_N(i/N+1)$ on I_{Ni} and

$$\alpha_{Ni} = \min_{u \in I_{Ni}} \left| J_{Ni}(u) - J(u) \right| , \qquad = \max_{u \in I_{Ni}} \left| J_{Ni}(u) - J(u) \right|$$

and

$$M_{Ni} = \max_{u \in I_{Ni}} J(u) \quad , \quad m_{Ni} = \min_{u \in I_{Ni}} J(u)$$

where $I_{Ni} = [i/(N+1), (i+1)/(N+1)]$ for $i = j_N - 1, ---, k_N - 1$. It is easy to see that

$$\text{(A.6)} \qquad \frac{\alpha_{Ni}}{N+1} \leqq \int_{I_{Ni}} |J_{Ni}(u) - J(u)|\, du \leqq \frac{\beta_{Ni}}{N+1}$$

and

$$\text{(A.7)} \qquad \frac{\alpha_{Ni}}{N+1} \leqq \frac{1}{N+1} \left| J_{Ni}((i+1)/(N+1)) - J((i+1)/(N+1)) \right| \leqq \frac{\beta_{Ni}}{N+1}$$

Summing on each side of (A.6) and (A.7) from j_N-1 to k_N-1, then subtracting the one of resulting inequalities from the other we have

$$\text{(A.8)} \qquad \left| \frac{1}{N+1} \sum_{i=j_N}^{k_N} |J_N(i/N+1) - J(i/N+1)| - \int_{(j_N-1)/(N+1)}^{k_N/(N+1)} |J_N(u) - J(u)|\, du \right|$$

$$\leqq \frac{1}{N+1} \sum_{i=j_N-1}^{k_N-1} (\beta_{Ni} - \alpha_{Ni})$$

$$= \frac{1}{N+1} \sum_{i=j_N-1}^{k_N-1} \left\{ |J_N(i/N+1) - J(y_{Ni})| - |J_N(i/N+1) - J(w_{Ni})| \right\}$$

$$= \frac{1}{N+1} \sum_{i=j_N-1}^{k_N-1} \left\{ J(y_{Ni}) - J(w_{Ni}) \right\}$$

$$\leqq \frac{1}{N+1} \sum_{i=j_N-1}^{k_N-1} (M_{Ni} - m_{Ni}).$$

Where y_{Ni}, w_{Ni} are chosen such that $|J_N(i/N+1) - J(y_{Ni})| = \max_{I_{Ni}} |J_{Ni}(u) - J(u)|$ and $|J_N(i/N+1) - J(w_{Ni})| = \min_{I_{Ni}} |J_{Ni}(u) - J(u)|$

respectively. The last equality holds if $J(w_{Ni}) \leq J_N(i/N+1) \leq J(y_{Ni})$. Similar argument can be applied to the another three cases. Thus we have by using (A.8)

$$\text{(A.9)} \qquad \frac{1}{N+1} \sum_{i=j_N}^{k_N} \left| J_N(i/N+1) - J(i/N+1) \right|$$

$$\leq \frac{1}{N+1} \sum_{i=j_N-1}^{k_N-1} (M_{Ni} - m_{Ni}) + \int_0^1 \left| J_N(u) - J(u) \right| du.$$

Since $J(u)$ is uniformly continuous over $[a,b]$, there exists an integer $N_2(\varepsilon)$ such that for all $N \geq N_2(\varepsilon)$

$$\text{(A.10)} \qquad M_{Ni} - m_{Ni} < \varepsilon, \quad i = j_N-1, ---, k_N-1.$$

Thus for all $N \geq N_3(\varepsilon) = \max(N_0(\varepsilon), N_2(\varepsilon))$, (A.2) and (A.10) give

$$\frac{1}{N+1} \sum_{i=j_N}^{k_N} \left| J_N(i/N+1) - J(i/N+1) \right| < 2\varepsilon.$$

Hence we have establied Theorem 2.2.

```
C       CALCULATE THE CHERNOFF EFFICIENCY OF THE WILCOXON
C       TEST RELATIVE TO THE TWO-SAMPLE T-TEST
'ASG,A NIQUAD*DATATABLE.,F2
'USE 8,NIQUAD*DATATABLE
'FOR,IS HWANG
        COMMON A,S3,S4,S,RHO,THI,PP
        DIMENSION R(150),W(150)
        DIMENSION YINIT(2),YFINAL(3),SAVE(200,3)
        EXTERNAL DERIVS
        EXTERNAL H1,H3
        D=-7.0
        B=8.0
        E=-0.000001
        EE=0.000001
        N=25
        M=25
        RHO=0.5
C       THI IS THE MEAN
        THI=2.5
  222   CONTINUE
        AL=1.5432
        AU=1.5436
        A=1.5434
  242   CONTINUE
        ALD=A-AL
        AUD=AU-A
        IF(AUD.LT.EE) GO TO 450
        IF(ALD.LT.EE) GO TO 450
        S3L=14.319781
        S3U=14.326437
        S3=14.323062
  111   CONTINUE
        S4L=0.653705
        S4U=0.653924
        S4=0.653814
  202   CONTINUE
        S4LD=S4-S4L
        S4UD=S4U-S4
        S3UD=S3U-S3
        S3LD=S3-S3L
        IF(S4UD.LT.EE) GO TO 904
        IF(S4LD.LT.EE) GO TO 908
        IF(S3UD.LT.EE) GO TO 904
        IF(S3LD.LT.EE) GO TO 908
        YINIT(1)=0.0
        YINIT(2)=0.0
        CALL DEPC(2,-7.0,YINIT,7.0,YFINAL,DERIVS,1.0E-6,1.0,
       *14.0E-5,.05,.02,2,1.0,'SOLVE TWO DIFFERENTIAL
       *EQUATIONS..',SAVE,200,3,NOPTS,NOTIFY,$110)
        FF=YFINAL(2)
        GG=YFINAL(1)
```

```
      FD=FF-1.
      GD=GG-1.
      IF(FD.GT.EE) GO TO 112
      IF(FD.LT.E) GO TO 113
      GO TO 133
  901 S4L=S4
      S4=(S4L+S4U)/2.
      GO TO 202
  801 S4U=S4
      S4=(S4L+S4U)/2.
      GO TO 202
  112 CONTINUE
      IF(FD.GT.EE) GO TO 607
      GO TO 801
  113 CONTINUE
      IF(FD.LT.E) GO TO 617
      GO TO 901
  133 CONTINUE
      IF(FD.GT.EE) GO TO  607
      IF(FD.LT.E) GO TO 801
      GO TO 600
  607 S3U=S3
      S3=(S3L+S3U)/2.
      GO TO 202
  617 S3L=S3
      S3=(S3L+S3U)/2.
      GO TO 202
  600 L=1
      CALL NIQUAD(D,B,H3,N,R,W,L,C,$100)
C     C IS THE CRITICAL POINT OF THE WILCOXON TEST UNDER
C     ALTERNATIVE HYPOTHESIS
      PRINT 408,A,S3,S4,C,FF,GG
  408 FORMAT(3X,'A=',F12.6,3X,'S3=',F12.6,3X,'S4=',F12.6,3X,
     *'C=',E17.6,3X,'FF=',E17.6,3X,'GG=',E17.6//)
      IF(C.LT.0.5) GO TO 904
C     FIA IS THE INFORMATION NUMBER OF THE WILCOXON TEST UNDER
C     THE ALTERNATIVE HYPOTHESIS
      FIA=RHO*LOG(S3)+(1.-RHO)*LOG(S4)-2.*A*C+A
      FITT=0.25*RHO*(1.-RHO)*THI**2
C     FIT IS INFORMATION INDEX OF TWO-SAMPLE T-TEST
      FIT=0.5*LOG(1.+FITT)
      CEOWT=FIA/FIT
      FIATT=CEOWT-1.
      IF(FIATT.GT.EE) GO TO 904
      IF(FIATT.LT.-0.3) GO TO 908
      S=1.155
      SL=1.14
      SU=1.160075
      L=1
  114 CONTINUE
      SLD=S-SL
      SUD=SU-S
      IF(SLD.LT.EE) GO TO 904
      IF(SUD.LT.EE) GO TO 908
```

```
      PP1=RHO*(EXP(2*S)-1.)
      PP2=EXP(2*S-2*S/(1.-RHO))
      PP3=(1.-RHO)*(PP2-1.)*EXP(S/(1.-RHO))
      PP=-PP1/PP3
      BBB=0
      CALL NIQUAD(BBB,1.,H1,M,R,W,L,CC,$100)
C     CC IS THE CRITICAL POINT OF THE WILCOXON TEST UNDER
C     THE NULL HYPOTHESIS
      CCC=RHO*(C-0.5)-CC
      IF(CCC.GT.EE) GO TO 650
      IF(CCC.GT.E) GO TO 550
      SU=S
      S=(SL+SU)/2.
      GO TO 114
  650 SL=S
      S=(SL+SU)/2.
      GO TO 114
  550 CONTINUE
      RUH=RHO*(1.-RHO)*PP*(EXP(S/(1.-RHO))+EXP(-S/(1.-RHO)))
      PPP=PP**2
      RHU=RHO**2+RUH+PPP*(1.-RHO)**2
      RU=LOG(RHU)
C     FIH IS THE INFORMATION NUMBER OF THE WILCOXON TEST UNDER
C     THE NULL HYPOTHESIS
      FIH=4.*CC*S+(1.-RHO)*LOG(PP)-0.5*RU
      CEOWTH=FIH/FIT
      PRINT 509,A,S3,S4,S,FIA,FIH
  509 FORMAT(1X,'A=',F12.6,2X,'S3=',F12.6,2X,'S4=',F12.6,
     *2X,'S=',F12.6,2X,'FIA=',E17.6,2X,'FIH=',E17.6//)
      RR=CEOWT-CEOWTH
      IF(RR.GT.0.0005) GO TO 904
      IF(RR.GT.-0.0005) GO TO 990
      GO TO 908
  904 AU=A
      A=(AL+AU)/2.
      GO TO 242
  908 AL=A
      A=(AL+AU)/2.
      GO TO 242
  990 PRINT 208,A,S3,S4,THI,CEOWTH,CEOWT
  208 FORMAT(1X,'A=',F12.6,2X,'S3=',F12.6,2X,'S4=',F12.6,
     *2X,'THI=',F12.6,2X,'CEOWTH=',E17.6,2X,'CEOWT=',E17.6//)
      GO TO 450
  110 PRINT 122,NOTIFY
  122 FORMAT(5X,'NOTIFY=',I5)
      GO TO 450
  100 PRINT 102
  102 FORMAT(20X,'INTEGRATION DIVERGES')
  450 END
```

```
'FOR,IS YUEH
      FUNCTION H1(Z)
      COMMON A,S3,S4,S,RHO,THI,PP
      PP4=EXP(2*S*(Z-.5)/(1.-RHO))
      PS=RHO+(1.-RHO)*PP*PP4
      H1=(Z-.5)*PP*PP4/PS
      RETURN
      END
'FOR,IS TYUAN
      FUNCTION H3(Z)
      COMMON A,S3,S4,S,RHO,THI,PP
      DIMENSION YINIT(2),YFINAL(3),SAVE(200,3)
      EXTERNAL DERIVS
      T=Z
      ZZZ=T
      PPP=ZZZ+7.0
      YINIT(1)=0.0
      YINIT(2)=0.0
      CALL DEPC(2,-7.0,YINIT,7.0,YFINAL,DERIVS,1.0E-6,1.0,
     *14.0E-5,0.2,.01,2,PPP,'SOLVE TWO DIFFERENTIAL
     *EQUATIONS..',SAVE,200,3,NOPTS,NOTIFY,$110)
      SS1=EXP(-(T-THI)**2/2.)
      SS2=-A*YFINAL(1)/RHO
      H3=0.39894228*S3*YFINAL(1)*SS1*EXP(SS2)
      RETURN
  110 PRINT 122,NOTIFY
  122 FORMAT(5X,'NOTIFY=',I5)
   45 END
'FOR,IS TEAY
      SUBROUTINE DERIVS(X,Y,DY,STORE,ITEST)
      COMMON A,S3,S4,S,RHO,THI,PP
      DIMENSION DY(1),Y(1),STORE(1)
      P2=-X**2/2.
      P22=-(X-THI)**2/2.
      P3=A*Y(2)/(1.-RHO)
      P4=EXP(P2+P3)
      P33=-A*Y(1)/RHO
      P5=EXP(P22+P33)
      DY(1)=.39894228*S4*P4
      DY(2)=.39894228*S3*P5
      STORE(1)=X
      STORE(2)=Y(1)
      STORE(3)=Y(2)
      RETURN
      END
'XQT
'FIN
```

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* University of Wisconsin Department of Statistics Madison, Wisconsin 53706	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Chernoff Efficiency of Linear Rank Statistics

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Scientific Interim

5. AUTHOR(S) *(First name, middle initial, last name)*

Tea-Yuan Hwang

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1972	71	27

8a. CONTRACT OR GRANT NO. NSF Grant G.P. 12093, XXXXXXXXX ONR N00014-67-A-0128-0017 AF-AFOSR 69-1803 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 318 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY AFOSR-1400 Wilson Blvd. Arlington, VA ONR-Washington, D.C.

13. ABSTRACT

A theorem of Hoadley used to compute large deviation probabilities for linear rank statistics with bounded score functions is extended to cover the unbounded case. The theorem is then applied to compute large deviation probabilities under alternatives in order to obtain the Chernoff efficiency of linear rank statistics. Numerical values are obtained under normal location alternatives for the two sample Wilcoxon and are shown to decrease more slowly than do corresponding Bahadur efficiency values with increasing location difference.

DD FORM 1 NOV 65 1473

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Large deviation probabilities						
Linear Rank Statistics						
Chernoff Efficiency						
Information						
Wilcoxon-Mann Whitey Test						