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TOPICS IN CONTROL

2. CONSTRAINED FEEDBACK CONTROL OF LINEAR DYNAMIC-STOCHASTIC SYSTEMS

by

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CHAPTER 2. CONSTRAINED FEEDBACK CONTROL OF LINEAR DYNAMIC-STOCHASTIC SYSTEMS

This chapter is concerned with the design of feedback controllers for dynamic-stochastic systems when, in addition to optimizing some quadratic function of the outputs, it is desired to constrain the variances of the manipulated or control variables. There are two very powerful methods for solving this problem, one employing the minimum mean square theory of Wiener [70] and in particular the Wiener-Hopf technique, and the other employing Bellman's theory of dynamic programming [12,13]. These two methods are presented and their relative merits discussed. (A third approach to the problem using non-linear programming theory is treated by Tabak and Kuo [61] and will not be discussed.)

2.1 Minimum Mean Square Error Control

The classical theory of control [40,45,55] as widely taught in engineering curricula, and still the mainstay of industrial control practice, ignores the stochastic characteristics of the underlying disturbances affecting the system. In general the control system is designed to give a satisfactory response to some deterministic upset, usually a step or sinusoidal function, with the implicit assumption that if it can adequately respond to these, then it will adequately respond to most stochastic disturbances encountered in practice. There is a certain vagueness in the criterion of performance used in designing these controllers. What is usually involved is a mutual compromise of various indirect measures of control loop performance and stability.

On the other hand, in most modern control theory the actual disturbance upsetting the system is modelled by some suitable stochastic model and the controller designed to give a satisfactory response to this. It is also usual to define a single criterion of control system performance often subject to constraints, whose value is to be minimized by the choice of control. Wiener's theory of minimum M.S.E. prediction led to the development of the first such controllers [49]. More recently Box and Jenkins [19-24] and Aström [2,3] have further extended the theory of M.M.S.E. control by working in the discrete time domain with transfer function dynamic models and ARIMA disturbance models. They showed that the optimal controller consists of two parts: one a predictor which predicts the effect of the disturbance on the output, and the other a regulator which computes the control signal necessary to make the predicted output equal to the desired value. This separation of the problem into two independent parts is often referred to as the separation theorem (or in economics as the certainty equivalence principle).

Aström showed that these optimal controllers can be very sensitive to parameter errors, particularly so when the zeros of the transfer function (i.e. the roots of $\omega(B)$) or the roots of the moving average part of the disturbance model ($\theta(B)$) lie near the unit circle. He suggests using a suboptimal strategy in these situations which employs the common practice in classical control theory of shifting these troublesome roots to a more suitable location away from the unit circle. This results in controllers which have slightly larger output variance but are quite insensitive to parameter errors. (In addition these controllers result in a greatly reduced manipulated variable variance.)

This raises the point of the present chapter; that in some situations it is not possible to tolerate the large variance of the manipulated variable required to accomplish M.M.S.E. control, and even if it can be tolerated the properties of the controller other than optimal regulation are often enhanced by its restriction. These problems of large variance in the manipulated variable arise whenever the dynamic or stochastic parts of the model lie near their stability (stationarity) or invertibility regions. Further as Box and Jenkins [24] have shown and will be shown below, it is usually possible to make very large reductions in the manipulated variable variance by allowing only a very small increase in the output error variance.

2.2 Transfer-Function Models and the Wiener-Hopf Technique

Although in some cases it may be desirable to specify lower and upper limits on the range of the manipulated variable u_t (corresponding possibly to physical limitations) this type of restriction leads to a highly non-linear problem that is very difficult to solve. A criterion for designing a constrained controller which yields a much more tractable problem is to minimize the variance of the output deviation from target ($V(\varepsilon_t)$) subject to a restriction on the variance of the input ($V(u_t)$). That is the controller is to be chosen to yield the unrestricted minimum of $\{V(\varepsilon_t) + \lambda V(u_t)\}$ where λ is an undetermined multiplier which can be chosen finally so that the variances of u_t and ε_t are jointly acceptable.

Newton [49] appears to have been the first to solve this problem using the powerful Wiener-Hopf technique. Following Whittle's treatment of this problem [68], Wilson [71] extended it to cover the entire range of discrete transfer function-ARIMA models.

2.2.1 General Forms for Constrained Controllers

It is of some interest to see how a minimum variance controller is modified when we place a constraint on the variance of the manipulated variable. Consider the general transfer function-ARIMA model:

$$Y_t = \frac{\omega_s(B)}{\delta_r(B)} u_{t-f-1} + \frac{\theta_q(B)}{\nabla^d \phi_p(B)} a_t \qquad (2.1)$$

where f represents the number of sampling periods of pure delay and the subscripts s, r, q, p, and d represent the order of the polynomials in B.

It is easily shown from the minimum variance control theory of Box and Jenkins that the minimum variance feedback controller for this general model is of the form

$$(1-\psi_1 B-\ldots-\psi_f B^f)\omega_s(B)\phi_p(B)\nabla^d u_t =$$

$$= -(\alpha_0-\alpha_1 B-\ldots-\alpha_k B^k)\delta_r(B)\varepsilon_t \qquad (2.2)$$

where $k = \max\{p+d-1; q-f-1\}$, and the parameters ψ_i and α_i are functions of the parameters in (2.1). From the Wiener-Hopf solution to the constrained control problem [71] it can further be shown that the constrained feedback controller for this general model will be of the form

$$(1+c_1B+\ldots+c_hB^h)\nabla^d u_t = -(b_0-b_1B-\ldots-b_kB^k)\delta_r(B)\varepsilon_t \qquad (2.3)$$

where

$h = \mathrm{Max}\{f+s+k+1-d;\ \ell+q-d\}$

k is as defined above in (2.2)

$\ell = \mathrm{Max}\{r+d;\ s\}$

and the parameters c_i and b_i are functions of the parameters in (2.1) and of the amount of constraining desired (i.e. the value of λ).

By comparing these general forms (2.2) and (2.3) for the minimum variance and the constrained controller respectively, a few important points can be noted. Firstly, the order of the right hand side of the equation operating on the output deviation does not change, only its parameters. Secondly, for sufficiently large f ($f \geq \max(q-p-d;\ \ell-s+q-p-d)$) the orders of both sides will be the same, only the parameters being different. These general forms (2.2) and (2.3) could be of value when the trial-and-error approach often used in industry of implementing a given controller form and then adjusting the constants to give satisfactory control is attempted.

2.2.2 Example: Delayed First Order Dynamics with IMA(1,1) Noise

To illustrate the effect on control of constraining the variance of the manipulated variable, consider as an example the following system with delayed first order dynamics and an IMA(1,1) disturbance process,

$$Y_t = \frac{g(1-\delta)\{(1-\nu)+\nu B\}}{(1-\delta B)} u_{t-f-1} + \frac{(1-\theta B)}{(1-B)} a_t \qquad (2.4)$$

From the previous section we can write the form of the optimal constrained controller for $f = 1$ as

$$(1+c_1B+c_2B^2)\nabla u_t = -b_0(1-\delta B)\varepsilon_t$$

where following Wilson's generalized Wiener-Hopf approach [71] it can be shown after a considerable amount of algebra that the parameters are given by

$$c_1 = \gamma_0^{-1}\{\gamma_1 + (1-\theta)\gamma_0\}$$

$$c_2 = \gamma_0^{-1}\{(1-\theta)(1-\delta)\nu + \theta\gamma_2\}$$

$$b_0 = \gamma_0^{-1}g^{-1}(1-\theta)$$

where

$$\gamma_0 + \gamma_1 + \gamma_2 = (1-\delta)$$

$$\gamma_0\gamma_1 + \gamma_1\gamma_2 = \nu(1-\nu)(1-\delta)^2 - (1+\delta)^2\lambda$$

$$\gamma_0\gamma_2 = \delta\lambda$$

where λ is an undetermined multiplier. The variance of ∇u_t and Y_t are given by

$$\frac{V(\nabla u_t)}{\sigma_a^2} = \frac{(1-\theta)^2\{(\gamma_0+\gamma_2)(1+\delta^2) + 2\gamma_1\delta\}}{g^2(\gamma_0-\gamma_2)\{(\gamma_0+\gamma_2)^2 - \gamma_1^2\}}$$

$$\frac{V(\varepsilon_t)}{\sigma_a^2} = 1 + \frac{(1-\theta)^2\{(\gamma_0+\gamma_2)[\gamma_0^2+(\nu-\nu\delta-\gamma_2)^2] - 2\gamma_0\gamma_1(\nu-\nu\delta-\gamma_2)\}}{(\gamma_0-\gamma_2)\{(\gamma_0+\gamma_2)^2 - \gamma_1^2\}}$$

To illustrate the above scheme, calculations were made for the case

$$g = 1 \qquad \nu = 0.4$$

$$f = 1 \qquad \theta = 0.6$$

for both $\delta = 0.5$ and $\delta = 0.9$. The characteristics of the unconstrained schemes are as follows. For $\delta = 0.5$

$$-\nabla u_t = 1.07\ \nabla u_{t-1} + 0.27\ \nabla u_{t-2} + 1.33(\varepsilon_t - 0.5\varepsilon_{t-1})$$

$$\frac{V(\varepsilon_t)}{\sigma_a^2} = 1.16 \qquad \frac{V(\nabla u_t)}{\sigma_a^2} = 6.13$$

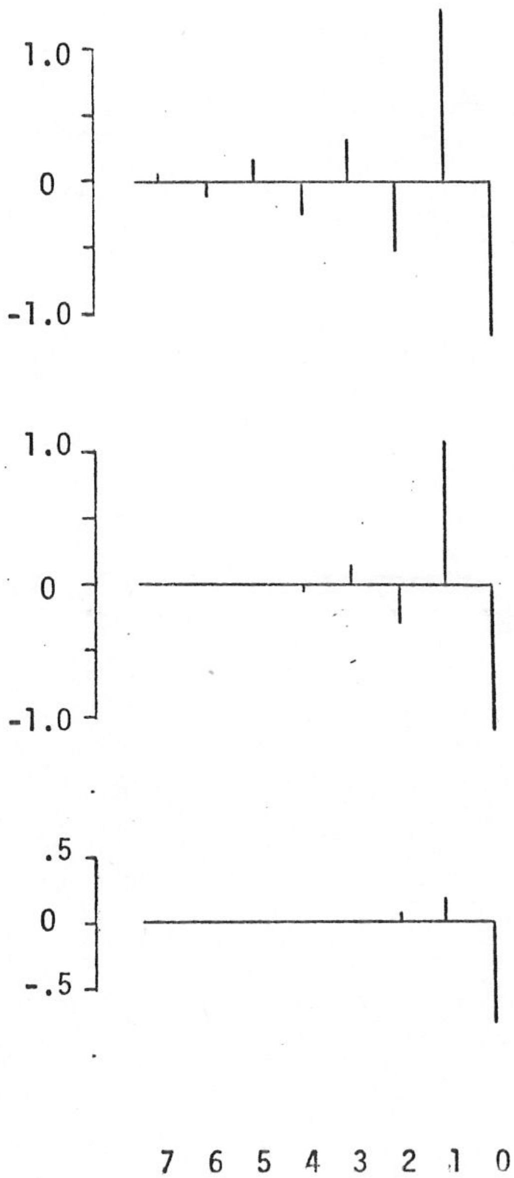
and for $\delta = 0.9$

$$-\nabla u_t = 1.07\ \nabla u_{t-1} + 0.27\ \nabla u_{t-2} + 1.33(\varepsilon_t - 0.9\varepsilon_{t-1})$$

$$\frac{V(\varepsilon_t)}{\sigma_a^2} = 1.16 \qquad \frac{V(\nabla u_t)}{\sigma_a^2} = 9.63$$

δ	% increase in $V(\varepsilon_t)/\sigma_a^2$	% decrease in $V(\nabla u_t)/\sigma_a^2$	Controller Parameters c_1	c_2	b_0
0.5	0	0	(unconstrained scheme) 1.07	0.27	1.33
	0.2	49.1	0.88	0.24	1.19
	0.7	69.8	0.69	0.23	1.06
	2.8	90.0	0.24	0.20	0.74
0.9	0	0	(unconstrained scheme) 1.07	0.27	1.33
	0.1	40.5	0.93	0.25	1.23
	0.7	71.7	0.68	0.23	1.05
	2.4	89.4	0.29	0.21	0.78

Table 2.1: Comparison of constrained and unconstrained feedback schemes.



	% Increase in $V(\varepsilon_t)/\sigma_a^2$	% Decrease in $V(\nabla u_t)/\sigma_a^2$
Unconstrained		
	.23%	49.1%
	2.8%	90.%

Figure 2.1: Controller weights $\nabla u_t = L(B)a_t = (\ell_0+\ell_1 B+\ldots)a_t$ for some of the controllers in Table 2.1 ($\delta = .5$)

Various optimal constrained schemes are shown in Table 2.1. It will be noticed from this table that very dramatic reductions are possible in the adjustment variance $V(\nabla u_t)$ with very little increase in the output variance $V(\varepsilon_t)$.

It has been noted [56] that in the minimum variance control situations where the variance of the manipulated variable is unduely large, the manipulated variable behaves in a highly oscillatory manner, its setting at time t to a great extent cancelling out the effect of its setting at time t-1. Some insight into this effect and how constraining the manipulated variance affects it can be seen in Figure 2.1 where the weights of the minimum variance and various constrained controllers taken from Table 2.1 in the form $\nabla u_t = L(B)a_t$ are plotted. It can be seen from this Figure that the constraining of $V(\nabla u_t)$ has the effect of removing this large alternating effect of the unconstrained scheme and appears to be tending in the limit to put all the weight on the present shock only.

2.3 State-Variable Models and Dynamic Programming

The use of Dynamic Programming or the Calculus of Variations [2,27,35,44,51,52,57] with discrete state variable dynamic-stochastic models provides another very powerful procedure for designing constrained feedback controllers. Dynamic Programming essentially involves the application of a recurrence relationship to a staged process involving successive transitions from one state to another, the solution to the problem being derived iteratively by working backwards from the final state. Although a numerical solution can theoretically, at least, be obtained for any model and any performance criterion, an analytical solution in closed form is feasible only for models linear in the variables and for a quadratic performance criteria.

2.3.1 Optimal Controller Algorithm

Consider the linear state system of equation (1.20)

$$\begin{aligned} x_{t+1} &= Ax_t + Gu_t + w_t \\ y_t &= Hx_t + v_t \end{aligned} \qquad (1.20)$$

and suppose it is desired to find the multi-stage control policy $u_t, u_{t+1}, \ldots, u_N$ which will optimize the quadratic performance criterion

$$\underset{u_t, u_{t+1}, \ldots, u_N}{\text{Minimize}} \quad E\{x_N' Q_0 x_N + \sum_{s=t}^{N-1} (x_s' Q_1 x_s + u_s' Q_2 u_s)\} \qquad (2.5)$$

where Q_0, Q_1, and Q_2 are symmetric positive semi-definite matrices. Such a performance criterion is quite general. It can be seen to include the previously discussed case of univariate minimum mean square error control subject to a constraint on the variance of the manipulated variable if we choose $Q_0 = Q_1 = H'H$ (where H is now a $(1\times n)$ vector) and $Q_2 = \lambda$ a Lagrangian multiplier. The use and specification of such a criterion will become more apparent in subsequent sections and in particular section 2.3.5 where an example is treated.

It has been shown [2,51,57,65] (Aström [2] gives perhaps the most general treatment) that the solution to this optimal control problem is given by

$$\underset{\sim}{u}_t = -\underset{\sim}{L}_t \hat{\underset{\sim}{x}}_{t|\tau} \tag{2.6}$$

where $\underset{\sim}{u}_t$ is the optimal control setting to be applied at time t, and $\hat{\underset{\sim}{x}}_{t|\tau}$ is the conditional expectation of the state vector $E(\underset{\sim}{x}_t|\underset{\sim}{Y}_\tau)$ where $\underset{\sim}{Y}_\tau = (\underset{\sim}{y}_\tau, \underset{\sim}{y}_{\tau-1}, \ldots, \underset{\sim}{y}_0)$ represents the data which is available for determining the control action. $\underset{\sim}{L}_t$ is an (m×n) matrix of constants given by

$$\underset{\sim}{L}_t = [\underset{\sim}{Q}_2 + \underset{\sim}{G}'\underset{\sim}{S}_{t+1}\underset{\sim}{G}]^{-1}\underset{\sim}{G}'\underset{\sim}{S}_{t+1}\underset{\sim}{A} \tag{2.7}$$

where

$$\underset{\sim}{S}_t = \underset{\sim}{A}'\underset{\sim}{S}_{t+1}\underset{\sim}{A} + \underset{\sim}{Q}_1 - \underset{\sim}{A}'\underset{\sim}{S}_{t+1}\underset{\sim}{G}[\underset{\sim}{Q}_2 + \underset{\sim}{G}'\underset{\sim}{S}_{t+1}\underset{\sim}{G}]^{-1}\underset{\sim}{G}'\underset{\sim}{S}_{t+1}\underset{\sim}{A} \tag{2.8}$$

with the initial condition

$$\underset{\sim}{S}_N = \underset{\sim}{Q}_0 \tag{2.9}$$

The state estimator $\hat{\underset{\sim}{x}}_{t|\tau}$ can be obtained from the appropriate Kalman filter or predictor of section 1.3.

This result implies that the optimal strategy can be separated into two parts: a state estimator for obtaining the best estimate of the state variables, and a linear feedback law which operates on the estimated state. The feedback matrix $\underset{\sim}{L}_t$ depends only on the system dynamics ($\underset{\sim}{A}$ and

G matrices) and on the parameters of the loss function (Q_0, Q_1, and Q_2), but not on the covariance matrices (R_1 and R_2) of the noise processes w_t and v_t. Therefore, the optimal control strategy for the state model (1.20) will be identical to that derived from the equivalent deterministic system where $w_t = v_t = 0$. In addition the state estimator $\hat{x}_{t|\tau}$ given by the Kalman filters in section 1.3 is unaffected by the feedback control law. This important result is referred to as the separation theorem or certainty equivalence principle, and results because we have assumed a linear system with Gaussian inputs and have minimized a quadratic cost function.

In general, it is usually of more interest to find out what the form of the steady-state optimal feedback control law is, rather than the more general time-varying form given above. As the terminal time for control $N \to \infty$ then for a completely controllable system the value of S_t in the recursive equation (1.46) will tend to a constant matrix S_∞ [35,37,41] with corresponding steady-state feedback gain given by

$$L_\infty = [Q_2 + G'S_\infty G]^{-1} G'S_\infty A \qquad (2.10)$$

This asymptotic solution L_∞ together with the steady-state Kalman filtering solution for $\hat{x}_{t|\tau}$ provide the optimum control policy, $u_t = -L_\infty \hat{x}_{t|\tau}$.

2.3.2 The Special Case of Univariate Unconstrained Control

Consider a system described by the state equations (1.20) with a single manipulated variable u_t ($r = 1$), and suppose that we are interested in obtaining the unconstrained control ($Q_2 = 0$) which minimizes the mean square error of a single output variable y_{1t}. That is, we want the asymptotic control policy which gives (for $N \to \infty$)

$$\underset{u_t, \ldots, u_N}{\text{Min}} \; E \sum_{s=t}^{N} y_{1s}^2$$

Since in (1.20) v_t is uncorrelated with x_t, this is equivalent to that which gives (for $N \to \infty$)

$$\underset{u_t, \ldots, u_N}{\text{Min}} \; E \sum_{s=t}^{N} x_s'(h_1 h_1')x_s$$

where h_1' is the row of the H matrix corresponding to y_{1t}. This is now in the form of (2.5) but with $Q_2 = 0$ and $Q_0 = Q_1 = (h_1 h_1')$.

In this above special case the optimal asymptotic control vector L_∞ is given explicitly in terms of the system parameters by

$$L_\infty = [G'Q_1G]^{-1}G'Q_1A \tag{2.11}$$

To prove this let us first consider the following lemma.

Lemma: $$SG[G'SG]^{-1}G'S = S \tag{2.12}$$

where S is a symmetric $(n \times n)$ matrix of rank 1

and G is an $(n \times 1)$ vector

Proof: Since $\underset{\sim}{S}$ is a symmetric (n×n) matrix of rank one it can be written as

$$\underset{\sim}{S} = \lambda_1 \underset{\sim}{Z}_1 \underset{\sim}{Z}_1'$$

where λ_1 is the single non-zero eigenvalue of $\underset{\sim}{S}$ and $\underset{\sim}{Z}_1$ is the corresponding eigenvector. The left hand side of (2.12) can then be written

$$SG[G'SG]^{-1}G'S = \lambda_1 \underset{\sim}{Z}_1 \underset{\sim}{Z}_1' \underset{\sim}{G}[\lambda_1 \underset{\sim}{G}' \underset{\sim}{Z}_1 \underset{\sim}{Z}_1' \underset{\sim}{G}]^{-1} \underset{\sim}{G}' \underset{\sim}{Z}_1 \underset{\sim}{Z}_1' \lambda_1$$

$$= \lambda_1 \frac{\underset{\sim}{Z}_1 (\underset{\sim}{Z}_1' \underset{\sim}{G})(\underset{\sim}{G}' \underset{\sim}{Z}_1) \underset{\sim}{Z}_1'}{(\underset{\sim}{G}' \underset{\sim}{Z}_1)(\underset{\sim}{Z}_1' \underset{\sim}{G})}$$

$$= \lambda_1 \underset{\sim}{Z}_1 \underset{\sim}{Z}_1' = \underset{\sim}{S} \qquad \text{Q.E.D.}$$

Applying this result in (2.8) yields $\underset{\sim}{S}_t = \underset{\sim}{Q}_1$ for all time t. The relationship (2.11) then follows directly upon setting $\underset{\sim}{S}_\infty = \underset{\sim}{Q}_1$ in (2.10).

This relationship (2.11) considerably simplifies the design of univariate unconstrained minimum mean square error controllers since it expresses the control vector $\underset{\sim}{L}_\infty$ explicitly in terms of the system parameters, whereas ordinarily it would have to be obtained numerically by iterating on equation (2.8) until convergence.

2.3.3 Duality

A very interesting and useful duality between the state estimation problem of section 1.3 and the optimal control problem of this section was pointed out by Kalman [34]. He showed that these two problems were equivalent mathematically and the resulting iterative equations (1.28), (1.30), (1.25) for the Kalman filter and (2.8), (2.9) for the optimal controller are identical if one makes the following correspondence between the matrices:

Optimal Control	State Estimation
$\tilde{S}_t$	$\tilde{P}_{t\|t-1}$
$\tilde{Q}_0$	$\tilde{R}_0$
$\tilde{Q}_1$	$\tilde{R}_1$
$\tilde{Q}_2$	$\tilde{R}_2$
$\tilde{A}$	$\tilde{A}'$
$\tilde{G}$	$\tilde{H}'$
$\tilde{L}_t'$	$\tilde{A}\tilde{K}_t$

Making use of this principle, whatever results are obtained for the linear regulator problem with a quadratic cost function can be translated into the corresponding results for the linear filtering problem with minimum mean square error criterion, or vice-versa. This fact is very useful in proving the convergence, uniqueness, and stability of the corresponding sets of iterative equations [41]. It also simplifies the numerical solution of these problems since it allows one to use the same iterative computer program for both.

2.3.4 Structure of the Optimal Controller

Suppose for the moment that the system is a deterministic one in which the true value of the state vector is known at every time instant t.

The optimal controller is then

$$u_t = -L_\infty x_t$$

In this deterministic case the state vector x_t may be precomputed for all time and hence there is no difference between a control program operating on open-loop and a feedback control law operating on closed-loop. A difference between the open-loop and closed-loop control laws will occur only when stochastic disturbances occur in the system, and hence a state estimator is necessary. It is the state estimator which gives rise to dynamics in the optimal feedback law.

To see this consider the case in which the state is estimated by the Kalman filter (1.26), (1.28)

$$\hat{x}_{t|t} = (A - K_\infty HA)\hat{x}_{t-1|t-1} + (G - K_\infty HG)u_t + K_\infty y_t$$

and the optimal controller given by

$$u_t = -L_\infty \hat{x}_{t|t}$$

Substituting this optimal control law into the state estimator gives

$$\hat{x}_{t|t} = (I - K_\infty H)(A - GL_\infty)\hat{x}_{t-1|t-1} + K_\infty y_t$$

as the closed-loop estimator. This can be rewritten in the form

$$\hat{x}_{t|t} = (I - CB)^{-1} K_\infty y_t$$

$$= K_\infty y_t + \sum_{j=1}^{\infty} C^j K_\infty y_{t-j}$$

Thus the optimal feedback controller becomes

$$u_t = -L_\infty K_\infty y_t - L_\infty \sum_{j=1}^{\infty} C^j K_\infty y_{t-j}$$

which is a multivariate proportional-integral controller where the first term gives the control proportional to the current output deviations and the second term expresses the control due to the summation (integral) of the past output deviations. The relative amount of proportional and integral control depends on the matrix $C = (I - K_\infty H)(A - GL_\infty)$ the dynamic matrix of the state estimator.

2.3.5 Variance Formulas for Closed-Loop System

In the engineering literature on optimal control the matrices Q_0, Q_1, and Q_2 in the loss function (2.5) have always been assumed known. Whereas the specific control desired will usually imply values for Q_0 and Q_1 (and possibly Q_2 in electrical engineering situations), this is not usually true in the process industries for Q_2, the constraint matrix. As previously mentioned one will usually want to choose Q_2 by iterating on it until the variances of the outputs and inputs are jointly acceptable. Therefore in this section we develop the formulas required to calculate these variances in any given control situation.

For the simultaneous estimation case (that is, where the state estimate at time t is based on information up to and including time t) consider the steady-state closed-loop system characterized by the following equations:

$$\begin{aligned} x_{t+1} &= Ax_t + Gu_t + w_t \\ y_t &= Hx_t + v_t \end{aligned} \tag{1.20}$$

$$u_t = -L_\infty \hat{x}_{t|t} \tag{2.13}$$

$$\hat{x}_{t|t} = (I - K_\infty H)(A - GL_\infty)\hat{x}_{t-1|t-1} + K_\infty y_t \tag{2.14}$$

Write

$$x_t = \hat{x}_{t|t} + \tilde{x}_{t|t} \tag{2.15}$$

where $\tilde{x}_{t|t}$ is the estimation error, independent of the estimate $\hat{x}_{t|t}$. Then by using (2.15) and subtracting (2.14) from (1.20) the following expression results for the estimation error under closed-loop control:

$$\tilde{x}_{t+1|t+1} = (I - K_\infty H)A\ \tilde{x}_{t|t} + (I - K_\infty H)w_t \tag{2.16}$$

where $(I - K_\infty H)A$ has all eigenvalues inside the unit circle. We have already shown in (1.28), (1.30) that this has the conditional covariance matrix

$$P^{\infty}_{t+1|t+1} = (I - K_{\infty}H)A\, P^{\infty}_{t|t}\, A'(I - K_{\infty}H)' + (I - K_{\infty}H)R_1(I - K_{\infty}H)'$$

(This also follows directly by noting that (2.16) is a multivariate AR(1) process).

Also, under closed-loop control, equation (1.20) can be rewritten as

$$\begin{aligned} x_{t+1} &= Ax_t - GL_{\infty}\hat{x}_{t|t} + w_t \\ &= (A - GL_{\infty})x_t + GL_{\infty}\tilde{x}_{t|t} + w_t \end{aligned} \qquad (2.17)$$

where $(A - GL_{\infty})$ also is assumed to have all its eigenvalues inside the unit circle. Thus the overall closed-loop system plus estimator given by equations (2.16) and (2.17) is stable and of order 2n.

The covariance matrix $\Gamma_x(0)$ of the state vector x_t can be obtained from (2.17) as

$$\begin{aligned} \Gamma_x(0) = E(x_{t+1}x'_{t+1}) &= (A - GL_{\infty})E(x_t x'_t)(A - GL_{\infty}) \\ &+ (A - GL_{\infty})E(x_t\tilde{x}'_{t|t})L'_{\infty}G' + GL_{\infty}E(\tilde{x}_{t|t}x'_t)(A - GL_{\infty})' \\ &+ GL_{\infty}E(\tilde{x}_{t|t}\tilde{x}'_{t|t})L'_{\infty}G' + E(w_t w'_t) \end{aligned}$$

or

$$\begin{aligned} \Gamma_x(0) &= (A - GL_{\infty})\Gamma_x(0)(A - GL_{\infty})' + (A - GL_{\infty})P^{\infty}_{t|t}\, L'_{\infty}G' \\ &+ GL_{\infty}\, P^{\infty}_{t|t}(A - GL_{\infty})' + GL_{\infty}\, P^{\infty}_{t|t}\, L'_{\infty}G' + R_1 \end{aligned} \qquad (2.18)$$

where the fact has been used that

$$E(\underset{\sim}{x}_t \underset{\sim}{w}_t') = E(\underset{\sim}{\tilde{x}}_{t|t}\ \underset{\sim}{w}_t') = \underline{0}$$

and

$$E(\underset{\sim}{x}_t\ \underset{\sim}{\tilde{x}}_{t|t}') = E(\underset{\sim}{\tilde{x}}_{t|t}\ \underset{\sim}{\tilde{x}}_{t|t}') = \underset{\sim}{P}^{\infty}_{t|t}$$

Now from (2.18) the covariance matrix of the output $\underset{\sim}{y}_t$ and of the input $\underset{\sim}{u}_t$ can be obtained. Since

$$\underset{\sim}{y}_t = \underset{\sim}{H}\underset{\sim}{x}_t + \underset{\sim}{v}_t$$

then

$$E(\underset{\sim}{y}_t \underset{\sim}{y}_t') = \underset{\sim}{H}\underset{\sim}{\Gamma}_x(0)\underset{\sim}{H}' + \underset{\sim}{R}_2 \tag{2.19}$$

and since

$$\underset{\sim}{u}_t = -\underset{\sim}{L}_{\infty}\ \underset{\sim}{\hat{x}}_{t|t}$$

then

$$\begin{aligned} E(\underset{\sim}{u}_t \underset{\sim}{u}_t') &= \underset{\sim}{L}_{\infty} E(\underset{\sim}{\hat{x}}_{t|t}\ \underset{\sim}{\hat{x}}_{t|t}')\underset{\sim}{L}_{\infty}' \\ &= \underset{\sim}{L}_{\infty}(\underset{\sim}{\Gamma}_x(0) - \underset{\sim}{P}^{\infty}_{t|t})\underset{\sim}{L}_{\infty}' \end{aligned} \tag{2.20}$$

Consider now the system representation of (1.21) or (1.31), (1.33) whose steady-state closed-loop behavior is given by the equations

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t-1} + Gu_{t-1} + \Gamma a_t$$

$$= (A - GL_\infty)\hat{x}_{t|t-1} + \Gamma a_t$$

$$y_t = H\hat{x}_{t|t-1} + a_t$$

$$u_t = -L_\infty \hat{x}_{t+1|t}$$

It is easily shown that the covariance matrix of the state vector $\hat{x}_{t+1|t}$ is given by

$$\Gamma_{\hat{x}}(0) = (A - GL_\infty)\Gamma_{\hat{x}}(0)(A - GL_\infty)' + \Gamma\Sigma\Gamma' \qquad (2.21)$$

and hence

$$E(u_t u_t') = L_\infty \Gamma_{\hat{x}}(0)L_\infty' \qquad (2.22)$$

and

$$E(y_t y_t') = H\Gamma_{\hat{x}}(0)H' + \Sigma + H\Gamma\Sigma + \Sigma\Gamma'H' \qquad (2.23)$$

These equations will enable one to calculate the covariance matrices for the output y_t and input u_t under closed-loop conditions for any of the state model forms considered in Chapter 1.

2.3.6 Optimal Control with Delayed Dynamics

Very often one has to deal with processes in which there is a time lag or transport delay of f sampling intervals between a change in the input and its effect on the output. The incorporation of this delay into the design of the optimal controllers currently being discussed can be done in either of two ways or a combination of them.

The first possibility is to incorporate the delay into the state model by the addition of extra state variables. This may easily be done for example for model (1.24) by the addition of f new state variables of the form

$$\begin{aligned} x_{n+2,t+1} &= x_{1,t} \\ x_{n+3,t+1} &= x_{n+2,t} \\ &\vdots \\ x_{n+f+1,t+1} &= x_{n+f,t} \end{aligned} \qquad (2.24)$$

and the altering of the output equation to

$$y_t = [0 \ \dots \ 0 \quad 1]\underset{\sim}{x}_t$$

The second possibility does not involve altering any of the state equation forms. Suppose the state model is

$$\underset{\sim}{x}_{t+1} = A\underset{\sim}{x}_t + \underset{\sim}{G}\underset{\sim}{u}_{t-f} + \underset{\sim}{w}_t$$

We can obtain the optimal controller directly as

$$\underset{\sim}{u}_t = -\underset{\sim}{L}_\infty \hat{\underset{\sim}{x}}_{t+f|t} \tag{2.25}$$

where the f-step ahead forecast from (1.34) is given by

$$\begin{aligned}\hat{\underset{\sim}{x}}_{t+f|t} &= \underset{\sim}{A}^{f-1}\hat{\underset{\sim}{x}}_{t+1|t} + \underset{\sim}{A}^{f-2}\underset{\sim}{G}\underset{\sim}{u}_{t-f+1} + \ldots + \underset{\sim}{G}\underset{\sim}{u}_{t-1} \\ &= \underset{\sim}{A}^{f}\hat{\underset{\sim}{x}}_{t|t} + \underset{\sim}{A}^{f-1}\underset{\sim}{G}\underset{\sim}{u}_{t-f} + \ldots + \underset{\sim}{G}\underset{\sim}{u}_{t-1}\end{aligned}$$

and hence the optimal control is given by

$$\begin{aligned}\underset{\sim}{u}_t &= -\underset{\sim}{L}_\infty\{\underset{\sim}{A}^{f}\hat{\underset{\sim}{x}}_{t|t} + \underset{\sim}{A}^{f-1}\underset{\sim}{G}\underset{\sim}{u}_{t-f} + \ldots + \underset{\sim}{G}\underset{\sim}{u}_{t-1}\} \\ &= -\underset{\sim}{L}_\infty\{\underset{\sim}{A}^{f-1}\hat{\underset{\sim}{x}}_{t+1|t} + \underset{\sim}{A}^{f-2}\underset{\sim}{G}\underset{\sim}{u}_{t-f+1} + \ldots + \underset{\sim}{G}\underset{\sim}{u}_{t-1}\}\end{aligned} \tag{2.26}$$

However, in employing this equation it is important that $\hat{\underset{\sim}{x}}_{t|t}$ be evaluated using the Kalman filter of equations (1.27) and (1.29) and not that of (2.14) which is obtained from (1.27) and (1.29) by use of the relationship $\underset{\sim}{u}_t = -\underset{\sim}{L}_\infty\hat{\underset{\sim}{x}}_{t|t}$ which is no longer true in this instance. Lee [43] made this error and therefore obtained some confusing results.

2.3.7 An Example: First Order Delayed Dynamics with IMA(1,1) Noise

Consider again the example discussed in section 2.2.2 under the Wiener-Hopf technique and let us treat it now using the dynamic programming approach. Using the state model representation of (1.24) and accounting for the one unit of delay by means of an additional state variable as in (2.24) the state model of the system can be written as

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \\ x_{3t+1} \\ x_{4t+1} \end{bmatrix} = \begin{bmatrix} 1.5 & 1.0 & 0. & 0. \\ -0.5 & 0. & 1.0 & 0. \\ 0. & 0. & 0. & 0. \\ 1.0 & 0. & 0. & 0. \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \\ x_{4t} \end{bmatrix} + \begin{bmatrix} .3 \\ .2 \\ 0 \\ 0 \end{bmatrix} \nabla u_t + \begin{bmatrix} 1.0 \\ -1.1 \\ 0.3 \\ 0 \end{bmatrix} a_{t+1}$$

$$\underset{\sim}{y}_t = (0 \quad 0 \quad 0 \quad 1)\underset{\sim}{x}_t$$

with covariance matrices

$$\underset{\sim}{R}_1 = \underset{\sim}{\Gamma}\underset{\sim}{\Gamma}'\sigma_a^2 \qquad \text{where } \underset{\sim}{\Gamma}' = (1.0 \quad -1.1 \quad 0.3 \quad 0)$$

$$\underset{\sim}{R}_2 = 0$$

We want a controller of the form $\nabla u_t = -\underset{\sim}{L}_\infty \hat{\underset{\sim}{x}}_{t|t}$ which minimizes $E\{\Sigma y_{t+1}^2 + \lambda\ \Sigma u_t^2\}$ which can be done by choosing the loss function matrices as

$$\underset{\sim}{Q}_0 = \underset{\sim}{Q}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \underset{\sim}{Q}_2 = \lambda$$

Consider first the minimum variance case where $\lambda = 0$. An iterative computer program was used to obtain the asymptotic solution to the controller equations (2.7), (2.8), and (2.9), and to the Kalman filtering equations (1.27)-(1.30). Because of the duality between these two sets of equations (section 2.3.3) the same computer program was used for both problems. The converged values for the Kalman filter gain and the optimal control vector are respectively

$$\underset{\sim}{K}_{\infty} = \begin{bmatrix} 0.40 \\ -0.20 \\ 0.0 \\ 1.0 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{L}'_{\infty} = \begin{bmatrix} 5.0 \\ 3.33 \\ 0. \\ 0. \end{bmatrix}$$

Optimal control is therefore achieved by setting

$$\nabla u_t = -5.0\, \hat{x}_{1t|t} - 3.33\, \hat{x}_{2t|t}$$

where from (2.14)

$$\hat{\underset{\sim}{x}}_{t|t} = \begin{bmatrix} -0.40 & 0. & 0. & 0. \\ -1.30 & -.67 & 1.0 & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{bmatrix} \hat{\underset{\sim}{x}}_{t-1|t-1} + \begin{bmatrix} .40 \\ -.20 \\ 0. \\ 1.0 \end{bmatrix} y_t$$

By substituting for $\hat{x}_{1t|t}$ and $\hat{x}_{2t|t}$ the control equation reduces to

$$(1+.67B)(1+.40B)\nabla u_t = -1.33(1-0.5B)y_t \qquad (2.26)$$

Using another iterative program to solve the covariance relationship (2.18) one gets for the covariance matrix of $\underset{\sim}{x}_t$

$$\underset{\sim}{\Gamma}_x(0) = \begin{bmatrix} 1.16 & -1.18 & .30 & .40 \\ -1.18 & 1.641 & -.33 & -.387 \\ .30 & -.33 & .09 & 0. \\ .40 & -.387 & 0. & 1.16 \end{bmatrix}$$

and hence from (2.19) and (2.20)

$$\mathrm{Var}(y_t) = 1.16\sigma_a^2$$

$$\mathrm{Var}(\nabla u_t) = 6.13\sigma_a^2$$

For the case of constrained control ($\lambda = .05$) the converged optimal control vector is

$$\underset{\sim}{L}_\infty = [2.96 \quad 2.24 \quad 0.80 \quad 0.0]$$

while the steady-state Kalman gain $\underset{\sim}{K}_\infty$ remains the same. The optimal control action is then

$$\nabla u_t = -2.96\ \hat{x}_{1t|t} - 2.24\ \hat{x}_{2t|t} - 0.80\ \hat{x}_{3t|t}$$

where from (2.14)

$$\hat{\underset{\sim}{x}}_{t|t} = \begin{bmatrix} .21 & .33 & -.24 & 0. \\ -.89 & -.45 & .84 & 0. \\ 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. \end{bmatrix} \hat{\underset{\sim}{x}}_{t-1|t-1} + \begin{bmatrix} .40 \\ -.20 \\ 0 \\ 1.0 \end{bmatrix} y_t$$

which reduces to

$$(1+.24B+.20B^2)\nabla u_t = -0.73(1-0.5B)y_t$$

The variance calculations give

$$\text{Var}(y_t) = 1.192 \quad \text{and} \quad \text{Var}(\nabla u_t) = 0.614$$

a 2.8% increase and a 90.0% reduction respectively.

(Comparing these schemes with those obtained in section 2.2.2 for the same system by using transfer-function and ARIMA models together with the methods of Box and Jenkins, and Wilson, one can see that they are identical.)

Although the above solution illustrates the general solution procedure it is not the most concise in this instance where there is one whole period of delay (f = 1). Using the parsimonious representation of (1.23) the state model can be written as

$$\begin{bmatrix} \hat{x}_{1t+1|t} \\ \hat{x}_{2t+1|t} \end{bmatrix} = \begin{bmatrix} 1.5 & 1. \\ -.5 & 0. \end{bmatrix} \begin{bmatrix} \hat{x}_{1t|t-1} \\ \hat{x}_{2t|t-1} \end{bmatrix} + \begin{bmatrix} .3 \\ .2 \end{bmatrix} \nabla u_{t-1} + \begin{bmatrix} .4 \\ -.2 \end{bmatrix} a_t \qquad (2.27)$$

$$y_t = (1 \quad 0)\hat{\underset{\sim}{x}}_{t|t-1} + a_t \tag{2.28}$$

The minimum variance controller can be calculated using (2.6) and (2.10) to give

$$\nabla u_t = -\underset{\sim}{L}_{\infty}\hat{\underset{\sim}{x}}_{t+1|t} = -[5.0 \quad 3.33]\hat{\underset{\sim}{x}}_{t+1|t} \tag{2.29}$$

and the state estimator ($\hat{\underset{\sim}{x}}_{t+1|t}$) can be obtained directly from the above state equations without having to solve the iterative Kalman filter equations (see section 1.3.1) by substituting in (2.27) for a_t and ∇u_t from (2.28) and (2.29) to give

$$\hat{\underset{\sim}{x}}_{t+1|t} = \begin{bmatrix} -.4 & 0. \\ -1.3 & -.67 \end{bmatrix} \hat{\underset{\sim}{x}}_{t|t-1} + \begin{bmatrix} .4 \\ -.2 \end{bmatrix} y_t$$

Upon reduction these equations again yield equation (2.26). The variances of the input and output can be calculated from equations (2.21), (2.22) and (2.23). The solution of the constrained controller is straightforward. For a case in which there is more than one whole period of delay ($f > 1$) equation (2.26) will yield the optimal controller directly in terms of that for $f = 1$ given above. This more concise form (2.27) can be used only in the case where the number of whole periods of delay (f) is greater than or equal to one since the state representation (1.21) yields only the delayed state estimator $\hat{\underset{\sim}{x}}_{t|t-1}$ and when $f = 0$ the simultaneous state estimator $\hat{\underset{\sim}{x}}_{t|t}$ is needed. In this latter situation the state representational form (1.24) as used in the first solution is very convenient.

2.4 Comparison of the Two Approaches

The two approaches just discussed for designing optimal controllers differ in two main aspects--the types of models used, and the method of solution. The Wiener-Hopf solution was used in conjunction with a transfer function and an ARIMA model representation of the dynamic-stochastic system, while a dynamic programming-Kalman filtering solution was used in conjunction with state variable representations.

The drawbacks of the former approach which seem to have led to its near abandonment in favour of the latter seem to be largely overexaggerated, although they do have some basis. A major drawback was believed to be the fact that non-stationary disturbances could not be handled using spectra or covariance generating functions. However, Yaglom [73] and Box and Jenkins [24] have shown that non-stationary disturbances can easily be incorporated into this framework if one works with a suitable difference of the original process. A second drawback is the difficulty, or more truly, the tediousness of the solution of the Wiener-Hopf equation by spectral factorization for higher order systems.

In the second approach using state variable models the Wiener-Hopf equation is in effect replaced by the first order matrix difference equations of the Kalman filter and of Dynamic Programming. These equations are not only very easily solved iteratively on a computer but the same set of equations hold for all linear dynamic-stochastic systems; only the parameters have to be changed. In addition, multivariate systems fit into exactly the same framework and so provide very little extra difficulty. It is this unity of representation and solution that accounts for the major advantage of this latter approach.

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