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DESIGN OF CUMULATIVE SUM CONTROL CHARTS BASED ON MINIMUM COST CRITERION

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ABSTRACT

A cost model is derived for the design of cumulative sum control charts using the Average Run Length to calculate the average time of out-of-control operation. The minimum cost design is obtained by employing the "pattern search" technique. The nature of the loss-cost surface and the effect of the design variables on the loss-cost surface are investigated. The effects of the shift parameter (δ), the average time for an assignable cause to occur ($1/\lambda$) and the cost factors b and c on the optimum designs and the loss-cost surfaces are also evaluated by numerical examples.

1. INTRODUCTION

The operation of a cumulative sum (Cusum) control chart for controlling positive deviations in the mean of a process consists of taking samples of size n at regular intervals of s hours and plotting the cumulative sums $S_r = \sum_{j=1}^{r} (x_j - k)$ versus sample number r.* If a plotted point rises a distance h or more above the lowest previous point, it is assumed that a shift in the process mean has occurred. Thus the sample size n, sampling interval s, reference value k and decision interval h are the parameters needed to operate a one-sided Cusum chart. If both positive and negative deviations are to be controlled, two one-sided charts with reference values k_1, k_2 ($k_1 > k_2$) and respective decision intervals h and -h or a V-mask with lead distance d and half angle ϕ can be employed.

In order to determine the parameters of a Cusum chart, the acceptable and rejectable quality levels along with the respective Average Run Lengths are generally specified. The design values of the parameters are then determined to approximately satisfy these requirements. This approach of designing the Cusum charts based on the Average Run Length (A.R.L.) criterion does not take into consideration the cost aspects of the process and also the sampling interval, s, has to be selected arbitrarily. Taylor [6] has recently given a method for obtaining the

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*See Glossary for explanation of symbols.

parameters of a V-mask using the average cost criterion under the assumption that the sample size and the sampling interval are known from other considerations.

The objective of this paper is to develop a procedure for the design of Cusum charts based on minimum cost criterion. In a recent paper [2], we discussed the economic design of $\bar{X}$-charts based on Duncan's model [1]. A similar approach is adopted in this paper to derive a cost model that gives the loss-cost for the process as a function of the parameters of the Cusum chart and the cost and risk factors associated with the process.

An explicit solution of the cost model to obtain the optimum values of the parameters does not seem possible, therefore a numerical method is employed to find the parameters that minimize the loss-cost for the process. The loss-cost surfaces are analyzed and the effects of the parameters δ and λ and cost factors b and c are investigated.

2. DERIVATION OF THE COST MODEL

A cost model is derived to measure the loss-cost from the process as a function of the design variables of the Cusum chart and the various cost and risk factors associated with the process being controlled. The derivation closely follows the approach adopted by Duncan for the design of $\bar{X}$-charts except that the Average Run Length of the Cusum Chart is used for the development of the model.

It is assumed that the process starts in a state of control at time

$t = 0$ with a mean value μ_a and a known constant variance σ^2. A single assignable cause occurs at random and causes a shift in the process mean of a known magnitude $\delta\sigma$ so that its new value is $\mu_r = \mu_a \pm \delta\sigma$. The time between occurrences is assumed to be exponentially distributed with mean $1/\lambda$ and the process stays at the new level until a lack of control is indicated by the Cusum chart and adjustments are made to bring it back to the control level μ_a. No assignable cause is assumed to occur during the taking of a sample, and the process is not shut down while a search is being made for the assignable cause. Also the costs of repairs and bringing the process back to control are not charged against the control chart procedure.

The behavior of the process can be best explained by a diagram shown in Figure 1. At the starting point, O, of the cycle the process is in control at level μ_a and stays at this level until an assignable cause, which occurs at E, changes it to μ_r. The assignable cause is removed at G and the process level is brought back to μ_a. The quantities OE and EG denoted by t_a and t_r are respectively the average lengths of time the process is in-control and out-of-control. The distance OG denoted by t_c is the average length of one cycle of the process.

The average net income per hour from the above process can be written as:

Average net income = Income - Cost.

The hourly income is divided into two parts:

(a) income, when the process is in-control, ωI_a, and

(b) income, when the process is out-of-control, γI_r.

Similarly, the cost per hour is divided into three parts:

(a) cost of looking for an assignable cause when none exists, $a_f T$

(b) cost of looking for an assignable cause when one exists, ϵW, and

(c) cost of maintaining the control chart, C_m,

where ω and γ are respectively the proportions of the time the process is in-control and out-of-control in the long run, and I_a and I_r $(I_a > I_r)$ are the respective average incomes per hour from the process. The average number of false alarms per hour is a_f and the average number of times the process actually goes out-of-control is ϵ. T is the cost of looking for an assignable cause when none exists and W is the cost of looking for an assignable cause when one exists.

The proportions ω and γ are equal to $t_a/(t_a + t_r)$ and $t_r/(t_a + t_r)$, respectively, where, under the assumption that the time between assignable causes is exponential, the quantity t_a is $1/\lambda$. To obtain an expression for t_r we note that it is made up of three quantities t_2, t_3 and t_4 as shown in Figure 1. The time t_2 is $\left\{ \frac{s}{(1-e^{-\lambda s})} - \frac{1}{\lambda} \right\}$ and t_4 is $(D + en)$ as shown by Duncan [1]. The quantity t_3 is obtained from the Average Run Length of the Cusum chart as explained below.

When the quality remains constant, the Average Run Length of a scheme is defined as the average number of samples obtained before action is taken. The A. R. L. of a one-sided Cusum chart with horizontal boundaries at (0, h) is given by [5]

$$\text{A.R.L.} = \frac{N(0)}{1 - P(0)} \tag{1}$$

where P(0) is the probability that a test that starts on the lower boundary will end on or below it, and

N(0) is the unconditional average sample number of the test.

For the case when the sample average, x_j, is normally distributed with mean μ and variance unity, P(0) and N(0) satisfy the following integral equations:

$$P(0) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{h} P(x)\cdot f(x)\cdot dx \tag{2}$$

and

$$N(0) = 1 + \int_{0}^{h} N(x)\cdot f(x)\cdot dx \tag{3}$$

where

$$f(x) = \frac{1}{(2\pi)^{1/2}} \exp[-\frac{1}{2}(x-\mu)^2] \tag{4}$$

Thus the A. R. L. of a Cusum chart depends on the chart parameters n, h, k and the process level μ. When the process is in-control, i.e., $\mu = \mu_a$, the A. R. L. is denoted by L_a and when the process is out-of-control with $\mu = \mu_r$, the A. R. L. is represented by L_r. If we calculate f(x) from equation (4) by substituting $\mu = \mu_a$ and obtain the values of P(0) and N(0) from equations (2) and (3) for this value of f(x), then L_a is given by equation (1). Similarly, L_r is obtained from these equations by substituting $\mu = \mu_r$ in equation (4) to get the value of f(x).

The quantity t_3 as seen in Figure 1 is the average time elapsed between the first sample after the occurrence of the assignable cause and its detection. Since L_r gives the average number of samples obtained when the process is out-of-control, therefore t_3 is equal to $(L_r - 1)s$ hours.

To determine a_f, we note that L_a is the average number of samples taken before a false signal is given by the chart. In other words, on the average there will be one false alarm after each $L_a \cdot s$ hours. Hence the average number of false alarms per t_a hours of in-control operation will be $t_a/L_a \cdot s$ or equivalently this will be the average number of false alarms per t_c hours of operation. Therefore, a_f is equal to $t_a/L_a s\, t_c$.

The process is out-of-control once in t_c hours. Therefore, ϵ, the average number of times the process actually goes out-of-control per hour is equal $1/t_c$.

The cost C_m of maintaining the control chart per hour of operations is $(b + cn)/s$, where c is the cost per unit of inspection and b is the cost per sample of sampling and plotting. [1]

In summary, the average net income from the process per hour of operation is:

$$I_n = (\omega I_a + \gamma I_r) - (a_f T + \epsilon w + C_m)$$

$$= I_a - (\gamma M + a_f T + \epsilon w + C_m)$$

$$= I_a - C$$

where $M = I_a - I_r$ and $C = (\gamma M + a_f T + \epsilon w + C_m)$.

On substituting the expressions for various quantities derived above, we get

$$C = \frac{\left[\left\{\frac{s}{1-e^{-\lambda s}} - \frac{1}{\lambda}\right\} + (L_r - 1)s + D + en\right] M + \frac{T}{\lambda L_a s} + w}{\left\{\frac{1}{\lambda} + \left[\frac{s}{1-e^{-\lambda s}} - \frac{1}{\lambda}\right] + (L_r - 1)s + D + en\right\}} + \left(\frac{b + cn}{s}\right) \qquad (5)$$

The quantity C is termed the loss-cost. Since I_a is independent of the parameter of the chart, I_n will be maximum when C is minimum. Therefore, minimizing C is equivalent to maximizing I_n.

Equation (5) is the cost model that relates the design variables of the Cusum chart and the cost and risk factors of the process. The values n, h, k and s that minimize C are the optimum values of the design variables.

Note that if an $\bar{X}$-chart with control limits at $\mu_a \pm B\sigma/\sqrt{n}$ is used for the control of the process, the Average Run Lengths L_a and L_r can be shown to be:

$$L_a = \frac{1}{\alpha}$$

and

$$L_r = \frac{1}{P}.$$

On substituting these values of L_a and L_r in equation (5), the cost model reduces to the form given by Duncan and discussed by the authors in [2].

3. NUMERICAL METHOD FOR THE DETERMINATION OF OPTIMUM DESIGN PARAMETERS—AN EXAMPLE

The optimum design parameters are those values of n, h, k and s that minimize the loss-cost, C, given by equation (5). An explicit solution for these parameters does not appear to be feasible because the Average Run Lengths, L_a and L_r, are complicated functions of n, h, and k. Therefore we employ the "pattern search" technique (7) to obtain the design values of the parameters.

Pattern search is a numerical method for optimizing a function S(ξ) of several variables ξ. The argument ξ is varied until the optimum of S(ξ) is obtained. The pattern search routine determines the sequence of values ξ; an independent routine computes the functional values of S(ξ). This technique has been used by the authors to determine the optimum design parameters of $\bar{X}$-charts. It was found that the optimum design values for the $\bar{X}$-charts were same to third decimal place or better as compared with those shown in [2].

In the case of a Cusum chart, the function S(ξ) is C and the variables are n, h, k and s. If we use a central reference value, k, then the only parameters to be determined are n, h, and s. Since the sample size has to be an integer, a search is made for the values of h and s that minimize C for a given integer value of n with known cost and risk factors.

For a given integer n, the search starts with a local exploration in the $h-s$ plane around a starting point arbitrarily chosen. If the loss-cost reduces during local exploration, the step size grows; if not, the step size reduces. If a change of direction is necessary, the method starts over again with a new pattern. The search is terminated when the step size reduces to a specified value or when the number of iterations equals a predetermined value, whichever occurs first. This search process is repeated for other values of n until an overall minimum loss-cost is obtained.

For illustration, we consider the design of two one-sided cumulative sum charts using the pattern search method and the model given by equation (5). Suppose a process is to operate at an acceptable quality level, $\mu_a = 10$ units and the rejectable quality levels are $\mu_{r_1} = 12$ and $\mu_{r_2} = 8$ units. The standard deviation of the process is assumed to be one unit. Other cost and risk factors, etc., are:

$M = \$100,\qquad T = \$50,\qquad W = \$25,\qquad b = \$0.50$

$\lambda = 0.01,\qquad D = 2.00,\qquad e = 0.05,\qquad c = \$0.10.$

The parameters to be determined are the sample size n, the sampling interval s and the decision intervals h and $-h$. The central reference values for these charts are $k_1 = 11$ and $k_2 = 9$. In order to calculate the loss-cost C from equation (5), the Average Run Lengths L_a and L_r for a two-sided chart are obtained from the Average Run Lengths of the two equivalent one-sided charts by using the relationship given by Kemp [4].

To obtain the values of h and s, assume an initial value $n_1 = 1$. Suppose the search starts at $h = 2.0$ and $s = 0.2$ as shown by A in figure 2. The loss-cost at the basepoint A for 100 hours of operation is $756.66. The first local exploration in two dimensions, using a step size of 0.10, yields B as the second base point with a loss-cost equal to $594.44. A and B establish the first pattern. Since similar exploration around B may be expected to produce the same results, local exploration is skipped and the arrow from A to B is extended to C where C is 2B - A in vector notations. Local exploration at C gives D as the best point which becomes the third base point. Now, the arrow from B to D is extended to E where E = 2D - B. The point F gives the minimum loss-cost on a local exploration around E. The arrow from D to F is extended to E where E = 2F - D and a local exploration again gives F as the best point. At F, the step size is reduced. After a few trials and step size reductions, the pattern moves through G, H, I, J and K, and stops at X, the minimum loss-cost point. The coordinates 2.51, 0.54 of X are the optimum values of h and s respectively.

Thus, for n = 1, the optimum values of h, s and C for 100 hours are 2.51, 0.54, and \$501.52, respectively. Proceeding similarly for values of n from 2 to 10, the optimal values of h, s and C are obtained as listed in Table 1. The values of the design parameters n, h, and s that yield the overall minimum loss-cost C = \$400.93 are:

$$n = 5, \quad h = 0.39, \quad \text{and} \quad s = 1.40.$$

If a V-mask is to be employed instead, the values of the design parameters d and ϕ using a scale factor w can be determined from the relationships given by Kemp as follows:

$$\tan\phi = \frac{|\mu_a - \mu_{r_1}|}{2w} = 1 \text{ if } w = 1 \text{ and}$$

$$d = \frac{h}{\tan\phi} = 0.39.$$

Note that the quantity "Percentage Increase in Loss-Cost, PIL," is also listed in the table for comparison purposes where PIL is the percentage increase in loss-cost compared to the overall minimum [2]. For example, for n = 4, the loss-cost is \$402.32 and since the overall minimum is C = \$400.93,

$$\text{PIL} = \frac{402.32 - 400.93}{400.93} \times 100 = 0.35.$$

4. ANALYSIS OF CUSUM DESIGNS BASED ON COST CRITERION

It has been shown that for a given set of cost and risk factors and for a given n, there is a set of values of h and s that yield the minimum loss-cost for that n. These values of h, s and C for various n provide useful information to evaluate the flexibility in the choice of the design variables. Also, an investigation of the loss-cost contours enables us to evaluate the effect of n, s and h on the nature of the loss-cost surfaces.

To illustrate the flexibility in the choice of h and s for various n, consider the results shown in Table 1. The values of h, s and PIL for n from 1 to 10 are plotted in Figure 3. It can be seen that the sampling interval s increases while the decision interval h decreases with increasing sample size. Also the rate of change of s and h is higher for small n than for large n. For example, s increases by 0.86 hours when n is changed from 1 to 5 while the increase is only 0.39 hours when n is increased from 5 to 10. Similarly, h decreases by 2.12 units for a change in n from 1 to 5 and by 0.22 units when n is increased from 5 to 10. This implies that sampling frequency should be cut down and a smaller decision interval should be used as the sample size is increased and vice versa.

For the case when a V-mask is employed, the variations in the values of s, d and tan ϕ are illustrated in Figure 4. The changes in d and h in figures 3 and 4 follow an almost identical pattern due to the fact that both h and d control the frequency of false alarms.

The variation in loss-cost with n is exhibited by the PIL curve in figure 3. For small deviations from the optimum sample size, the value of PIL is relatively small. However, for a large deviation, the PIL is generally greater when n is reduced from the optimum value compared to the PIL when n is increased by the same amount. For example, if $n = 2$ is used instead of $n = 5$, the value of PIL is 9.44, while for $n = 8$ the "Percentage Increase is Loss-cost" is only 4.61.

To investigate the nature of the loss-cost surface as a function of the variables n, h and s, the loss-cost is calculated for a two-dimensional grid of the values of h and s and for a given n from equation (5). The loss-cost contours are obtained by parabolic interpolation on a digital computer. Such contours in the h-s plane for n from 4 to 6 are shown in figure 5(a). The range of h is from 0.25 to 0.75 and s is varied from 0.8 to 1.8. Each of these three sets of contours represents a section of the loss-cost surface at the given value of n for the above set of data with $\delta = 2.0$.

These loss-cost contours appear to be distorted concentric ellipses, the center being the point of minimum loss-cost. The constant loss-cost surfaces can be compared to a set of footballs, one inside another, somewhat twisted in a counter-clockwise direction along the axis of increasing sample size. As an example, let us consider the surface for C = \$415 and examine its sections at n = 4, 5 and 6 as represented by the \$415 contours in figure 5(a). We see that the contour is somewhat elliptical at n = 4, bulges to a larger size at n = 5 and narrows down at n = 6. During this process of bulging and then narrowing, the surface slowly twists in the counter-clockwise direction. The phenomenon of "twisting" is directly related to the change in the optimum values of h and s for various n. As n increases from 4 to 6, the optimum values of (h, s) change from (0.51, 1.27) to (0.32, 1.50) going through the overall optimum (0.39, 1.40) at n = 5. This increase in s and decrease in h is similar to that seen in figure 3.

The loss-cost contours in figure 5(a) also show that if it is not feasible to operate the chart at the optimum point, i.e., at n = 5, h = 0.39 and s = 1.40, there is a large choice of the parameter values to choose from. Thus, for n = 5, any combination of h and s within the contour \$405 can be used for an additional 1.25% cost or less. Also, for a given C and a fixed s, there are, in general, two values of h to choose from and vice versa. For example, if s = 1.50 hours, any value of h between 0.260 and 0.515 will yield C less than or equal to \$405. If h is fixed at 0.45, any value of s between 1.10 and 1.66 will result in a loss-cost of \$405 or less. While keeping C fixed at \$405, there is a choice of various values of s and h for n = 4 or n = 6 as well.

5. EFFECT OF SHIFT PARAMETER, δ

The shift parameter δ is related to the change, $\delta\sigma$, in the process mean that the Cusum Chart is designed to detect. The effect of variations in δ on the design variables and the loss-cost surfaces are illustrated by numerical examples.

Consider three values of δ equal to 2.0, 1.0 and 0.50 with the respective loss rates M as \$100, \$12.87 and \$2.25. These loss rates have been calculated on the assumption that $\pm 3\sigma$ specification limits are used on the process and M is proportional to the increase in percent defective. M has been arbitrarily taken equal to \$100 for $\delta = 2.0$. Let the remaining cost and risk factors be the same as in the previous example. The results for the case when $\delta = 2.0$ and for n from 1 to 10 were given in Table 1 and figure 3. The optimum values of s, h and C for $\delta = 1.0$ and 0.50 and various values of n are listed in Table 2.

The optimum design parameters for $\delta = 2.0$, 1.0 and 0.50 are as follows:

δ	M	n	s	h	C
2.0	100	5	1.4	0.39	400.93
1.0	12.87	14	5.4	0.23	141.28
0.5	2.25	37	22.3	0.12	83.39

It can be seen that the optimum sample size n and sampling interval s increase and the decision interval h decreases with decreasing δ. In other words, as the shift to be detected decreases, larger samples should be taken less often and a smaller decision interval should be used. Also, the loss-cost decreases with δ due to a decrease in the loss rate M.

The variations in the parameters of the Cusum charts with changes in sample size are illustrated in figures 3, 6 and 7 for $\delta = 2.0$, 1.0 and 0.50, respectively. Note that these three graphs are drawn on different scales to accommodate the relevant values of the variables. As seen in these graphs, the effect of deviating from the optimum sample size on the "Percentage Increase in Loss-cost," is maximum for $\delta = 2$ and minimum for $\delta = 0.50$. As an example, for $\delta = 0.50$, when n is reduced by 4 units from 37 to 33, PIL is only 0.077, while PIL is 25.09 when n is reduced from 5 to 1 for $\delta = 2.0$. Therefore, deviations from the optimum sample size are more critical for higher values of δ than for lower values. The variation in s for all three cases follows a somewhat similar pattern, rising gradually as the sample size increases. The decision interval h decreases with increasing n, the rate of decrease being maximum for $\delta = 2.0$ and minimum for $\delta = 0.5$.

To study the nature of the surfaces in the h-s plane, nine sets of loss-cost contours are given in figures 5(a), (b) and (c) for δ equal to 2.0, 1.0 and 0.5 respectively. Three different scales are used to be able to show the surfaces in the vicinity of the minimum loss-cost point. The center point of each surface represents the overall minimum loss-cost and occurs at $n = 5$, $s = 1.4$ and $h = 0.39$ for $\delta = 2.0$; at $n = 14$, $s = 5.4$, and $h = 0.23$ for $\delta = 1.0$; and at $n = 37$, $s = 22.3$ and $h = 0.12$ for $\delta = 0.50$. The loss-cost contours tend to become elliptical and the ratio of the major to minor axis increases as δ decreases as shown in figure 5.

In order to explore the slope of the surfaces around the minimum loss-cost points, consider arbitrary areas, one unit high and 0.05 units wide, as shown shaded in figures 5(a), (b) and (c) for n equal to 5, 14, and 37, respectively. Within these shaded areas, the loss-cost changes respectively by about \$25, \$0.70 and \$0.31 for δ equal to 2.0, 1.0 and 0.5. This implies that the loss-cost surface is more steep for $\delta = 2.0$ compared to the surface for $\delta = 0.50$. Due to this difference in the slopes of the loss-cost surfaces, there is less flexibility in the choice of h and s for higher values of δ than for lower values.

6. EFFECT OF COST FACTORS b AND c AND PARAMETER $1/\lambda$

The cost factors b and c determine the cost of maintaining the control chart which is equal to $(b + cn)$ per sample, where b is the cost of sampling and plotting and cn is the cost of sampling, plotting and computation. To evaluate the effect of these factors on the optimum designs and the loss-costs, consider the data for the previous example as the base level and four additional sets of data with b and c increasing in steps of 50% of the base level. The loss-costs for n from 1 to 10 are given in Table 3 along with the values of b and c. The variation in "Percentage Increase in Loss-cost," with n is illustrated in figure 8.

It can be seen that the general shape of the PIL vs. n curves is the same for all five levels of b and c. However the PIL for equal deviations from the optimum n depends on whether the deviation is negative or positive and on the values of b and c. To illustrate, let us consider the following results taken from Table 3.

Data	b	c	n*	n^*-2	PIL	n^*+2	PIL
A	0.50	0.10	5	3	2.92	7	2.57
B	0.75	0.15	5	3	2.80	7	2.87
C	1.00	0.20	5	3	2.59	7	3.13
D	1.25	0.25	4	2	9.68	6	1.38
E	1.50	0.50	4	2	9.54	6	1.61

(n* is the overall optimum sample size.)

These values indicate that in some cases (A, B and C), a reduction of sample size by two units results in an approximately 3% additional loss-cost while in others (D, E), the additional loss-cost is about 10%.

The optimum values of the design variables for these five sets of data are shown in Table 4. It is seen that the only design variable significantly affected by the values of b and c is the sampling interval s, which increases from 1.40 hours to 2.31 hours when b and c increase by 200%. This means that the samples should be taken less often as the costs of sampling, etc., go up. When the increase in sampling costs is as high as 200%, a slight reduction in the sample size is called for. It should be noted that the above general conclusions are based on the five sets of data considered. The same approach can be used to analyze the results for other costs and risk factors.

The average number of assignable causes per hour is equal to λ and an increase in λ is equivalent to a decrease in the average time for an assignable cause to occur. To evaluate its significance, consider three values of λ equal to 0.01, 0.02 and 0.03 other cost and risk factors being the same as in the previous examples. The loss-cost values, calculated as above, are listed in Table 5 and the PIL vs. n curves are shown in figure 9. It is seen that the minimum loss-cost increases from \$400.93 to \$957.32 as λ changes from 0.01 to 0.03. This addition in loss-cost is due to the increased frequency of occurrence of the assignable causes. The effect of λ on the nature of the PIL curve appears to be relatively insignificant.

7. CONCLUSION

1. A cost model is derived for the design of Cusum charts similar to that of the design of $\bar{X}$-charts. The Average Run Length of the Cusum chart is used to calculate the average number of false alarms and the average time of out-of-control operation. This cost model gives the loss-cost from the process as a function of the design variables of the chart and the cost and risk factors associated with the process.

2. The "pattern search" technique is used to obtain the optimum values of sampling interval (s) and the decision interval (h) for a given integer n. The values of n, s and h that minimize the loss-cost are the design parameters of the Cusum chart. This procedure gives not only the optimum design but also provides additional information which enables us to analyze the loss-cost surfaces.

3. Effects of the shift-parameter (δ), average time for an assignable cause to occur ($1/\lambda$) and the cost factors b and c on the loss-cost surfaces and the optimum design are evaluated by numerical examples. The loss-cost surface in the h-s plane was found to become steeper as the value of δ is increased. Effect of b and c and $1/\lambda$ on the "Percentage Increase in Loss-cost" was relatively insignificant.

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GLOSSARY

n	sample size
s	sampling interval, hours
h	decision interval
k	reference value
d	lead distance in terms of a unit distance on the vertical scale
ϕ	half angle of the V-mask
B	control limit factor of an $\bar{X}$-chart
μ	process mean
σ	process standard deviation
x_j	j^{th} sample mean
r	sample number
S_r	cumulative sum at the r^{th} sample
L_a	average run length, when the process is in-control at $\mu = \mu_a$
L_r	average run length when the process is out-of-control at $\mu = \mu_r$
t_a	average time the process is in-control in one cycle
t_r	average time the process is out-of-control in one cycle
I_a	income per hour when the process is in-control
I_r	income per hour when the process is out-of-control
P	probability that an assignable cause will be detected by an $\bar{X}$-chart
α	probability of looking for an assignable cause when it does not exist
δ	shift in the process mean is $\delta\sigma$
$1/\lambda$	average time for an assignable cause to occur

M	Loss rate equals $I_a - I_r$
e	delay factor
D	average time taken to find an assignable cause
T	cost of looking for an assignable cause when none exists
W	cost of looking for an assignable cause when one exists
b	cost per sample of sampling and plotting
c	cost per unit of sampling, testing and computation
ω	proportion of the time the process will be in-control in many repetitions
γ	proportion of the time the process will be out-of-control in many repetitions
a_f	average number of false alarms per hour
ϵ	average number of times per hour that the process actually goes out-of-control

LIST OF TABLES

Table 1

Values of the Parameters of Two-sided Cusum Charts for n from 1 to 10 (δ=2.0)

		Two one-sided charts	V-mask, $w = 2/\sqrt{n}$		C	
n	s	h	d	tan ϕ	(for 100 hrs.)	PIL
1	0.54	2.51	5.02	0.500	501.52	25.09
2	0.84	1.16	1.64	0.707	438.77	9.44
3	1.09	0.71	0.83	0.866	412.65	2.92
4	1.27	0.51	0.51	1.000	402.32	0.35
5	1.40	0.39	0.35	1.118	400.93	0.00
6	1.50	0.32	0.26	1.225	404.64	0.93
7	1.58	0.27	0.20	1.323	411.23	2.57
8	1.66	0.23	0.16	1.414	419.41	4.61
9	1.72	0.20	0.13	1.500	428.44	6.86
10	1.79	0.17	0.11	1.581	437.89	9.21

Note: Underlined values denote the overall optimum.

Table 2

Values of the Parameters of Two-Sided Cusum Charts (δ=1.0 and 0.50).

δ = 1.0					δ = 0.50				
n	s	h	C (for 100 hrs.)	PIL	n	s	h	C (for 100 hrs.)	PIL
11	4.63	0.32	142.59	0.93	31	19.74	0.157	83.675	0.342
12	4.90	0.28	141.80	0.37	32	20.23	0.150	83.582	0.231
13	5.16	0.25	141.38	0.07	33	20.67	0.143	83.509	0.143
14	5.40	0.23	141.28	0.00	34	21.07	0.138	83.454	0.077
15	5.62	0.21	141.45	0.12	35	21.50	0.133	83.415	0.031
16	5.81	0.20	141.84	0.40	36	21.87	0.128	83.395	0.006
17	6.00	0.18	142.44	0.82	37	22.29	0.123	83.389	0.000
18	6.18	0.17	143.21	1.37	38	22.66	0.118	83.399	0.011
19	6.36	0.16	144.12	2.01	39	23.02	0.114	83.424	0.043
20	6.52	0.15	145.16	2.75	40	23.38	0.111	83.462	0.086

Note: Underlined values denote the overall optimum.

Table 3

Loss-Cost and PIL for Evaluation of Effect of b and c

	A		B		C		D		E	
	b=0.50, c=0.10		b=0.75, c=0.15		b=1.00, c=0.20		b=1.25, c=0.25		b=1.50, c=0.30	
n	C	PIL	C	PIL	C	PIL	C	PIL	C	PIL
1	501.52	25.09	550.28	27.14	588.99	28.18	621.40	28.75	649.41	29.09
2	438.77	9.44	475.30	9.81	504.54	9.80	529.32	9.68	551.05	9.54
3	412.65	2.92	444.95	2.80	471.38	2.59	494.19	2.40	514.51	2.28
4	402.32	0.35	433.76	0.22	459.86	0.08	482.63	0.00	503.06	0.00
5	400.93	0.00	432.82	0.00	459.50	0.00	482.87	0.05	503.92	0.17
6	404.64	0.94	437.50	1.08	465.09	1.22	489.31	1.38	511.16	1.61
7	411.23	2.57	445.26	2.87	473.87	3.13	499.02	3.40	521.73	3.71
8	419.41	4.61	454.67	5.05	484.34	5.41	510.44	5.76	534.01	6.15
9	428.44	6.86	464.94	7.42	495.66	7.87	522.70	8.30	547.11	8.76
10	437.89	9.21	475.61	9.89	507.37	10.42	535.31	10.92	560.55	11.43

Note: Underlined values denote the overall optimum.

Table 4

Optimum Designs for Different Values of b and c

Data	Optimum Design by Cost Criterion						Other cost and risk factors, etc.
	b	c	n	h	s	C (for 100 hrs.)	
A	0.50	0.10	5	0.394	1.40	400.93	$\delta = 2.0$
B	0.75	0.15	5	0.349	1.73	432.82	M = \$100
							D = 2.0
C	1.00	0.20	5	0.318	2.01	459.50	$\lambda = 0.01$
D	1.25	0.25	4	0.385	2.10	482.63	e = 0.05
E	1.50	0.30	4	0.360	2.31	503.06	T = \$ 50
							W = \$ 25

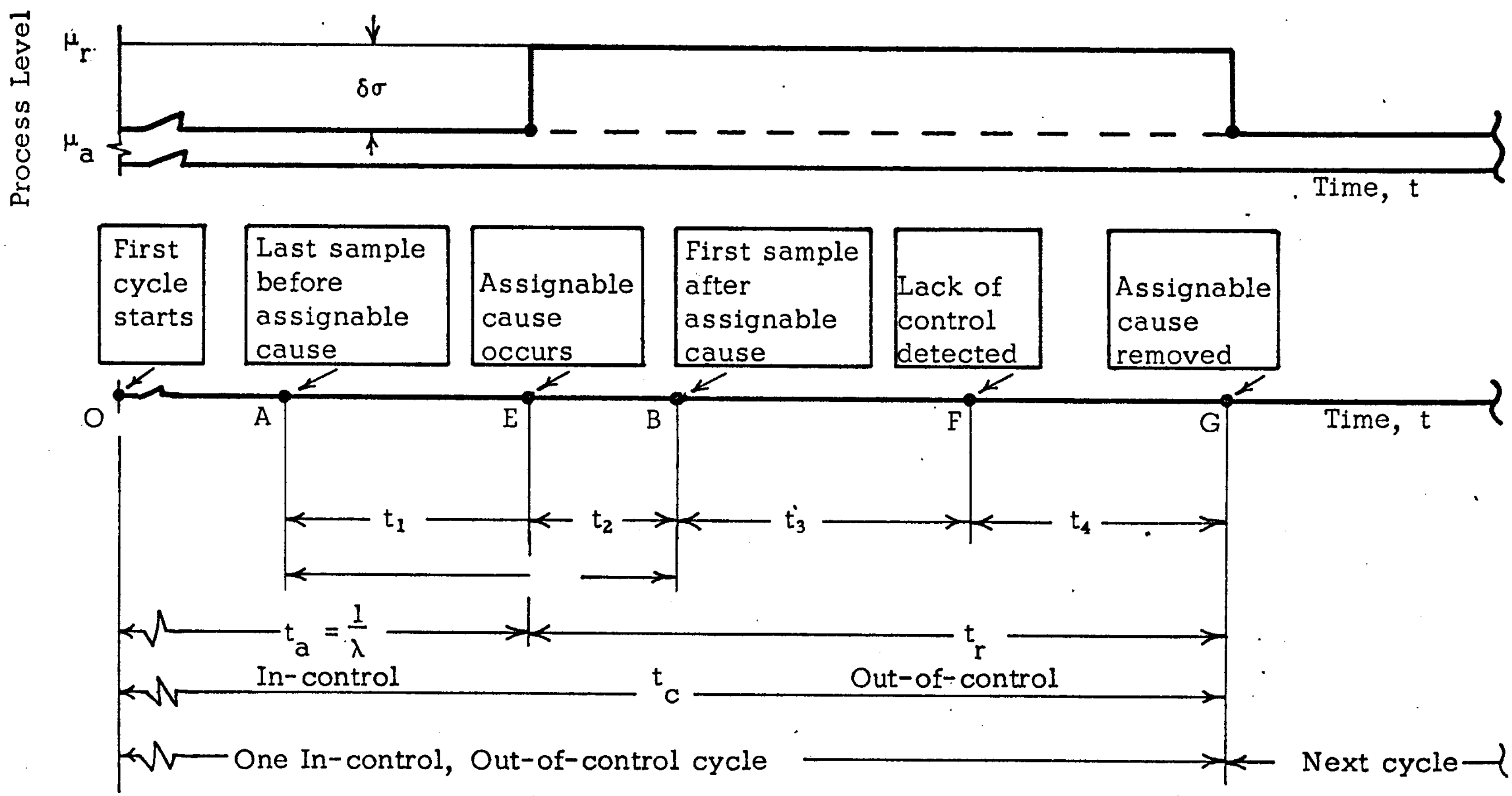
Table 5

Loss-Cost and PIL for Evaluation of Effect of $1/\lambda$

n	$\lambda = 0.01$		$\lambda = 0.02$		$\lambda = 0.03$	
	C	PIL	C	PIL	C	PIL
1	501.52	25.09	822.22	18.56	1104.25	15.35
2	438.77	9.44	738.46	6.48	1006.50	5.14
3	412.65	2.92	705.10	1.67	968.92	1.21
4	402.32	0.35	693.50	0.00	957.33	0.00
5	400.93	0.00	694.16	0.10	960.35	0.32
6	404.64	0.93	701.81	1.20	971.65	1.50
7	411.23	2.57	713.37	2.87	987.58	3.16
8	419.41	4.61	727.09	4.84	1006.03	5.09
9	428.44	6.86	741.95	6.99	1025.81	7.15
10	437.89	9.21	757.36	9.21	1046.21	9.28

Note: Underlined values denote the overall optimum.

LIST OF FIGURES



$$t_1 = \left\{ \frac{1 - (1+\lambda s)e^{-\lambda s}}{\lambda(1-e^{-\lambda s})} \right\}$$

$t_2 = s - t_1$

$t_3 = (t_r - 1)s$

$t_4 = D + en$

$t_a = 1/\lambda$

$t_r = t_2 + t_3 + t_4$

$t_c = t_a + t_r$

$\omega = t_a/t_c$

$\gamma = t_r/t_c$

$a_f = \omega/s\ La$

$\epsilon = 1/t_c$

Figure 1. Diagrammatic Explanation of the Process Behavior

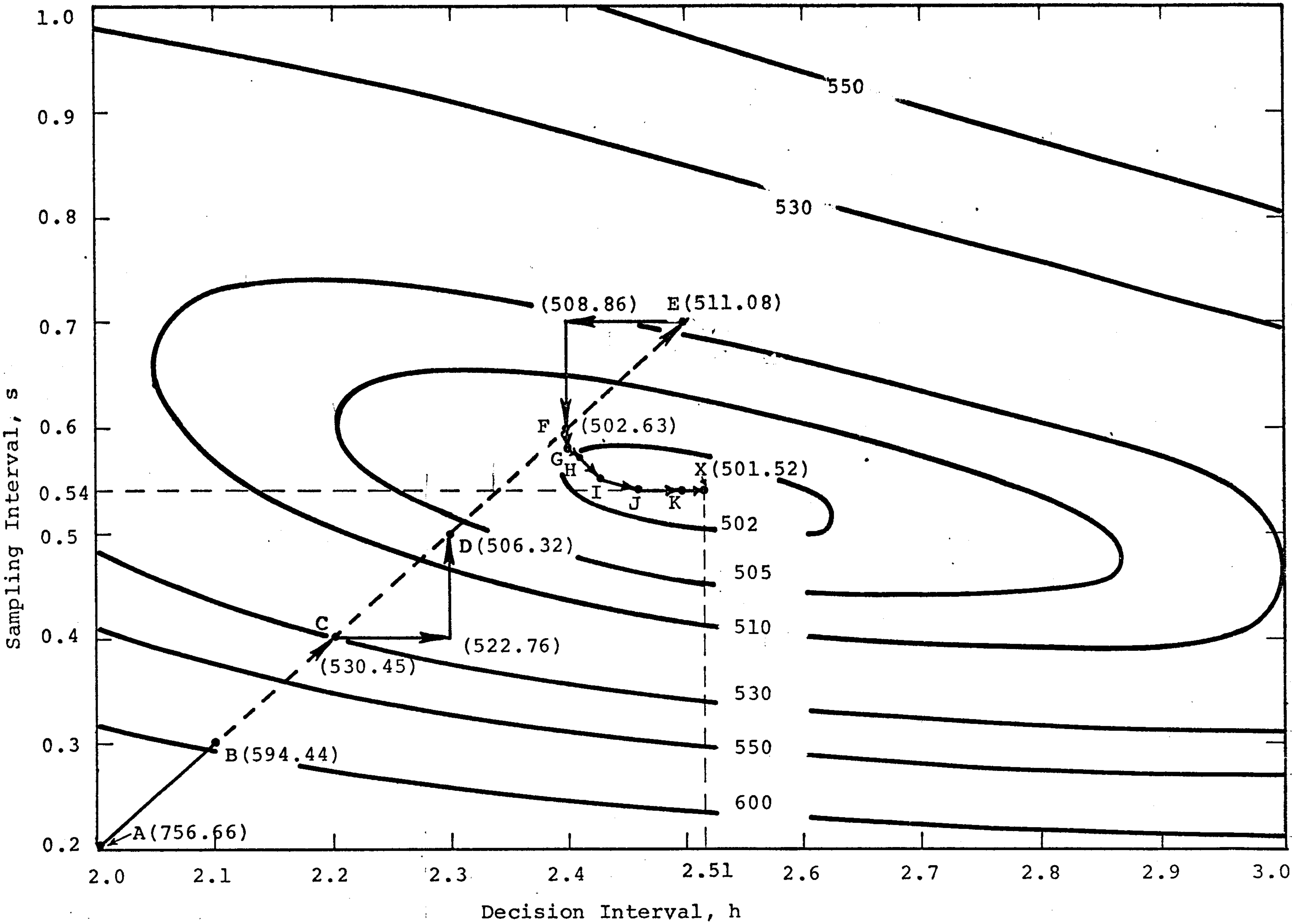


Figure 2. Search for the Minimum Loss-Cost Point by the Pattern Search Technique

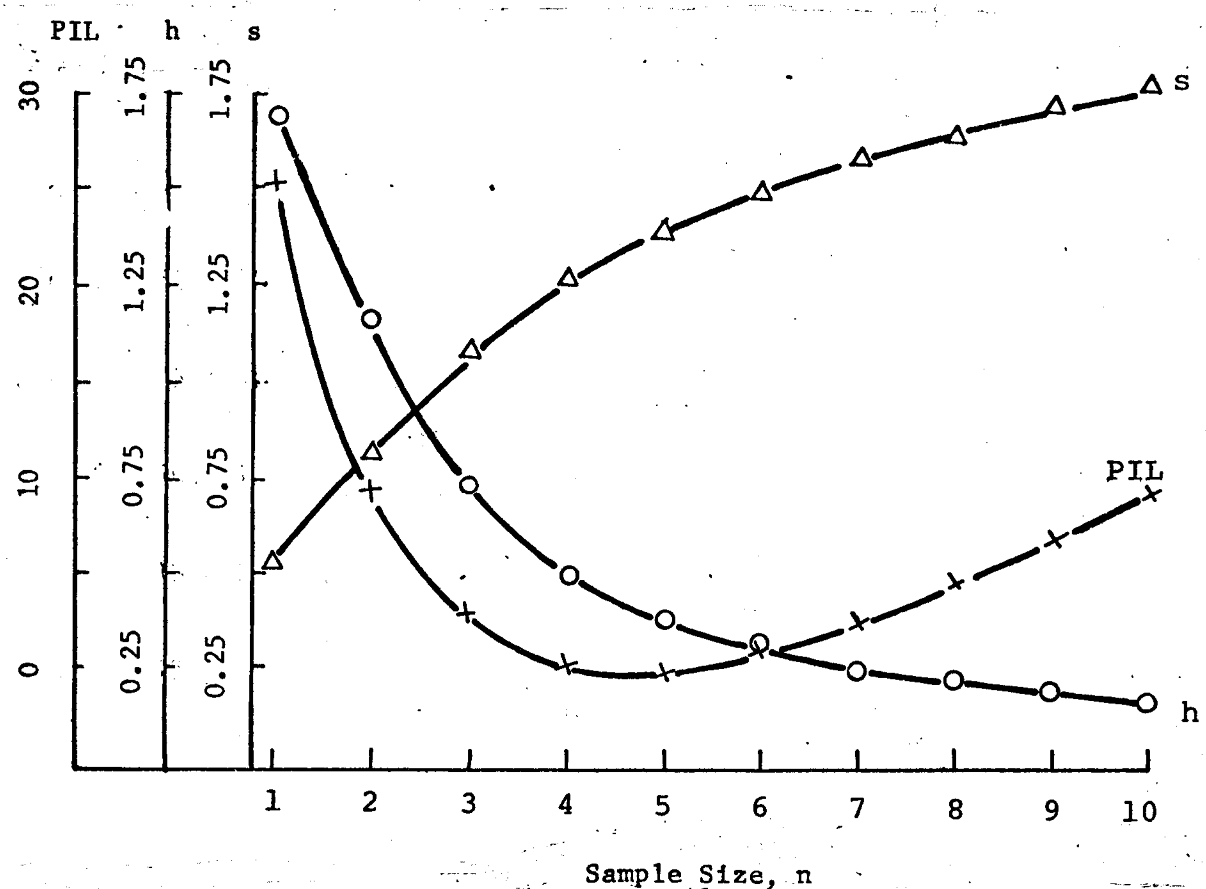


Figure 3. Variation in s, h and PIL for n from 1 to 10 (δ=2.0).

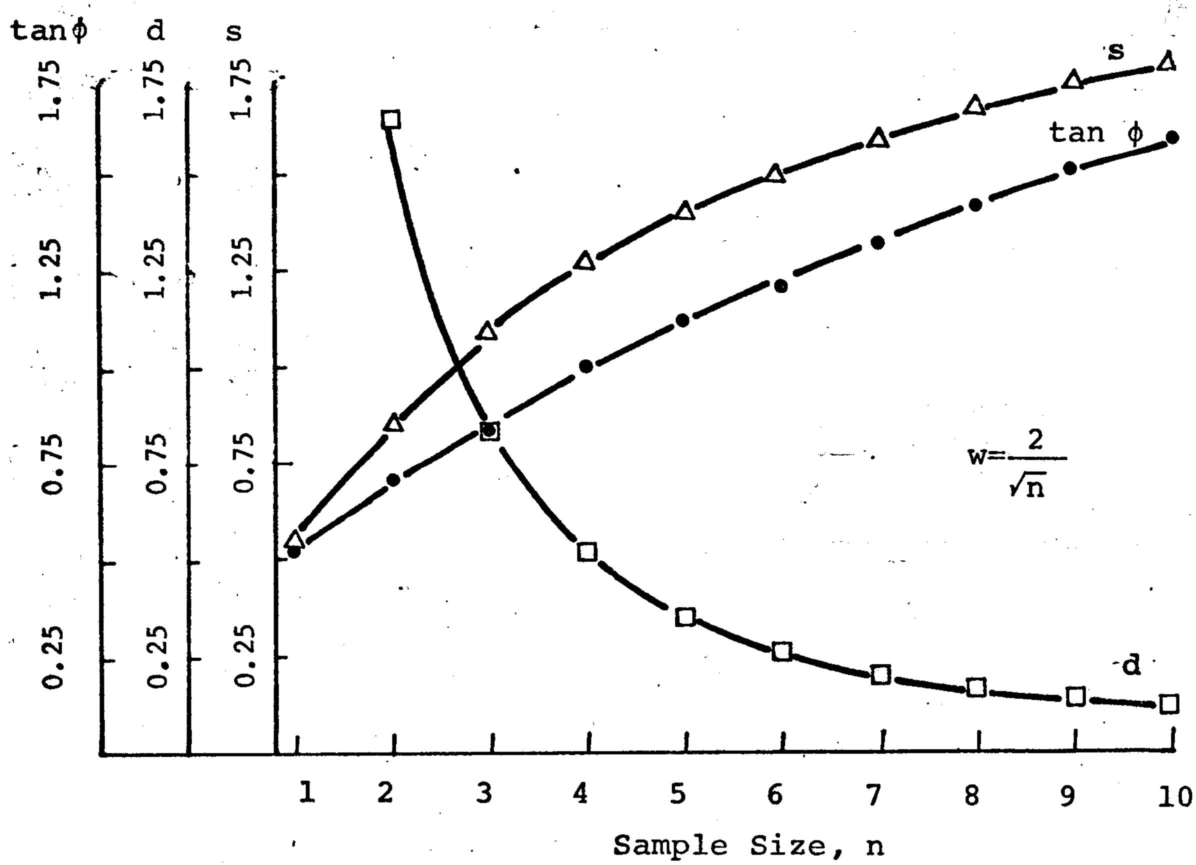


Figure 4. Variation in s, d and tanϕ for n from 1 to 10 (δ=2.0)

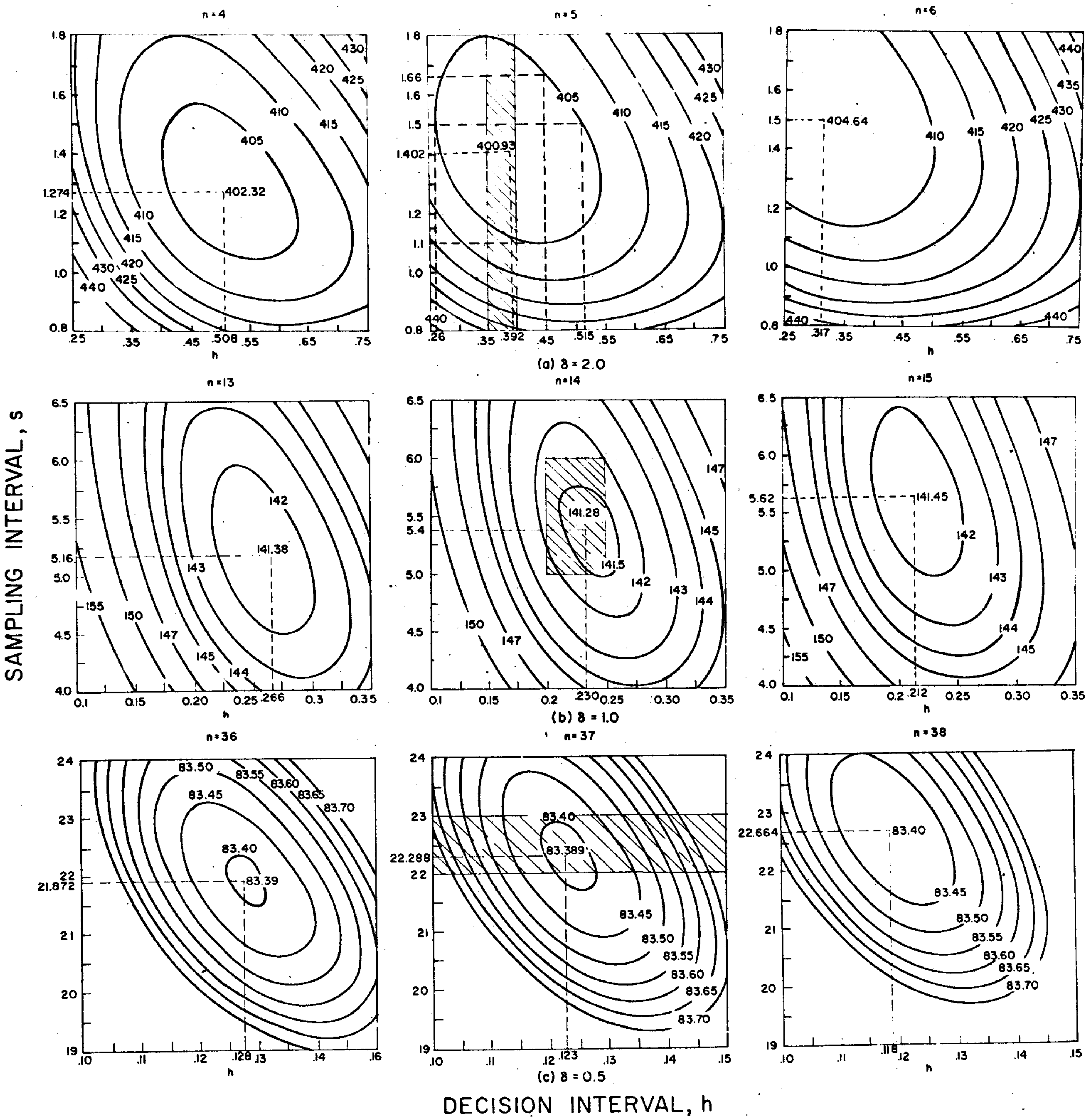

n=4
n=5
n=6
(a) δ = 2.0
n=13
n=14
n=15
(b) δ = 1.0
n=36
n=37
n=38
(c) δ = 0.5
SAMPLING INTERVAL, s
DECISION INTERVAL, h
402.32
400.93
404.64
141.38
141.28
141.5
141.45
83.39
83.389
83.40

Figure 5

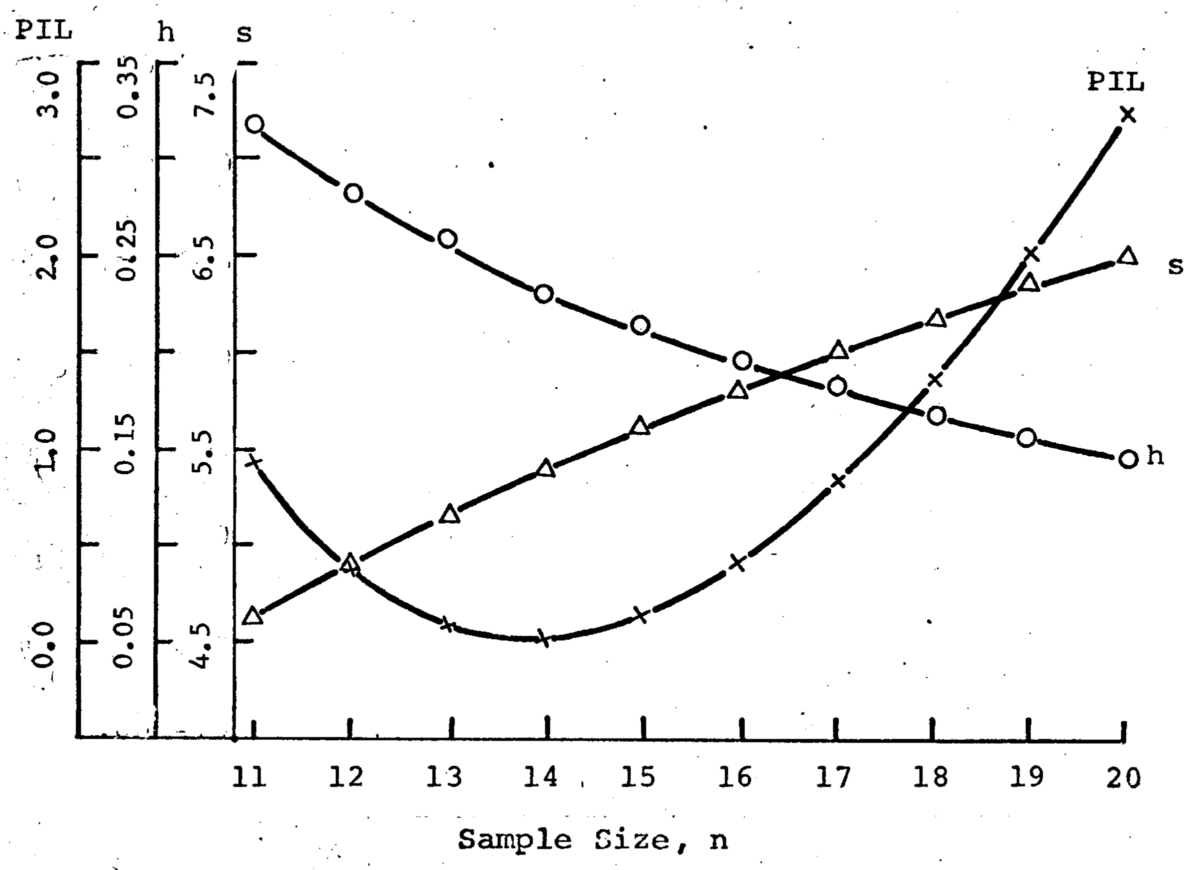


Figure 6. Variation in s, h and PIL for n from 11 to 20 (δ=1.0).

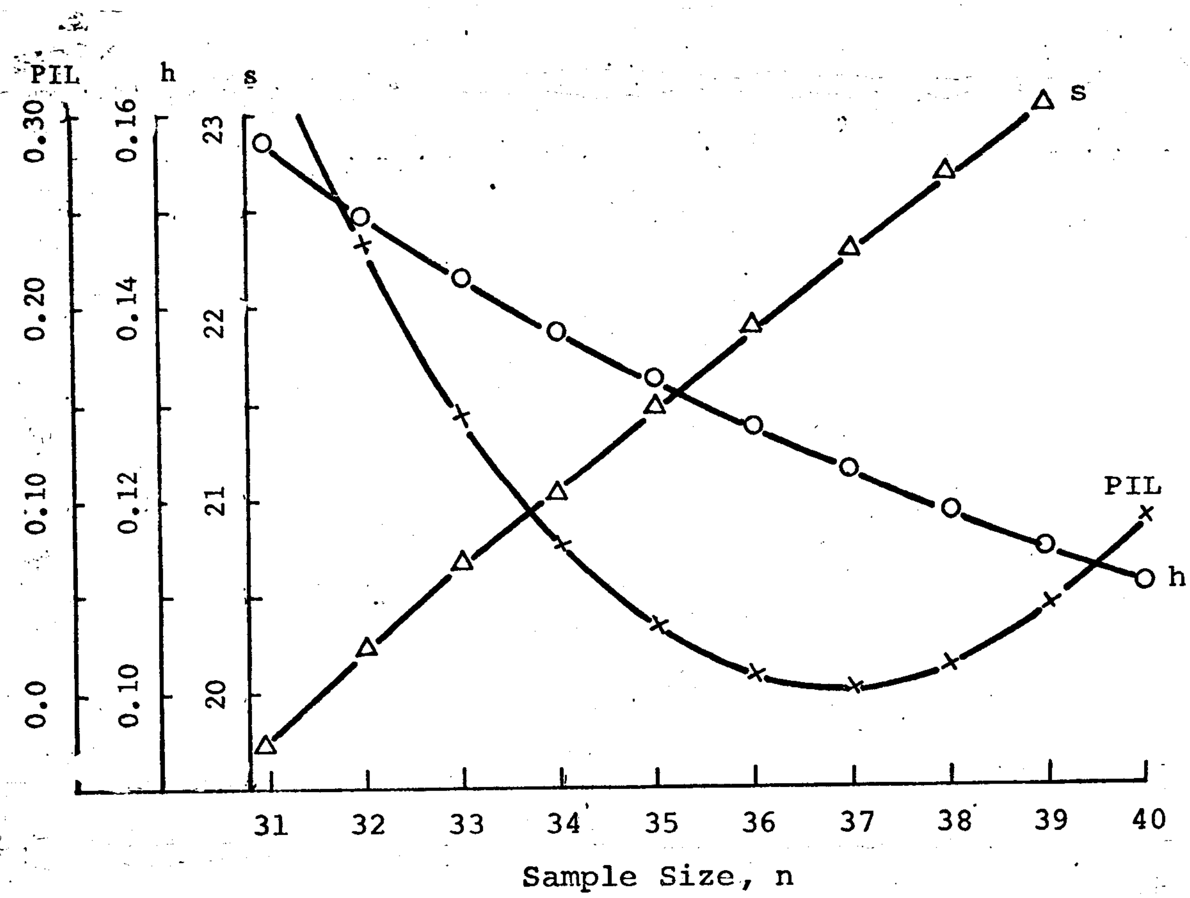


Figure 7. Variation in s, h and PIL for n from 31 to 40 (δ=0.50).

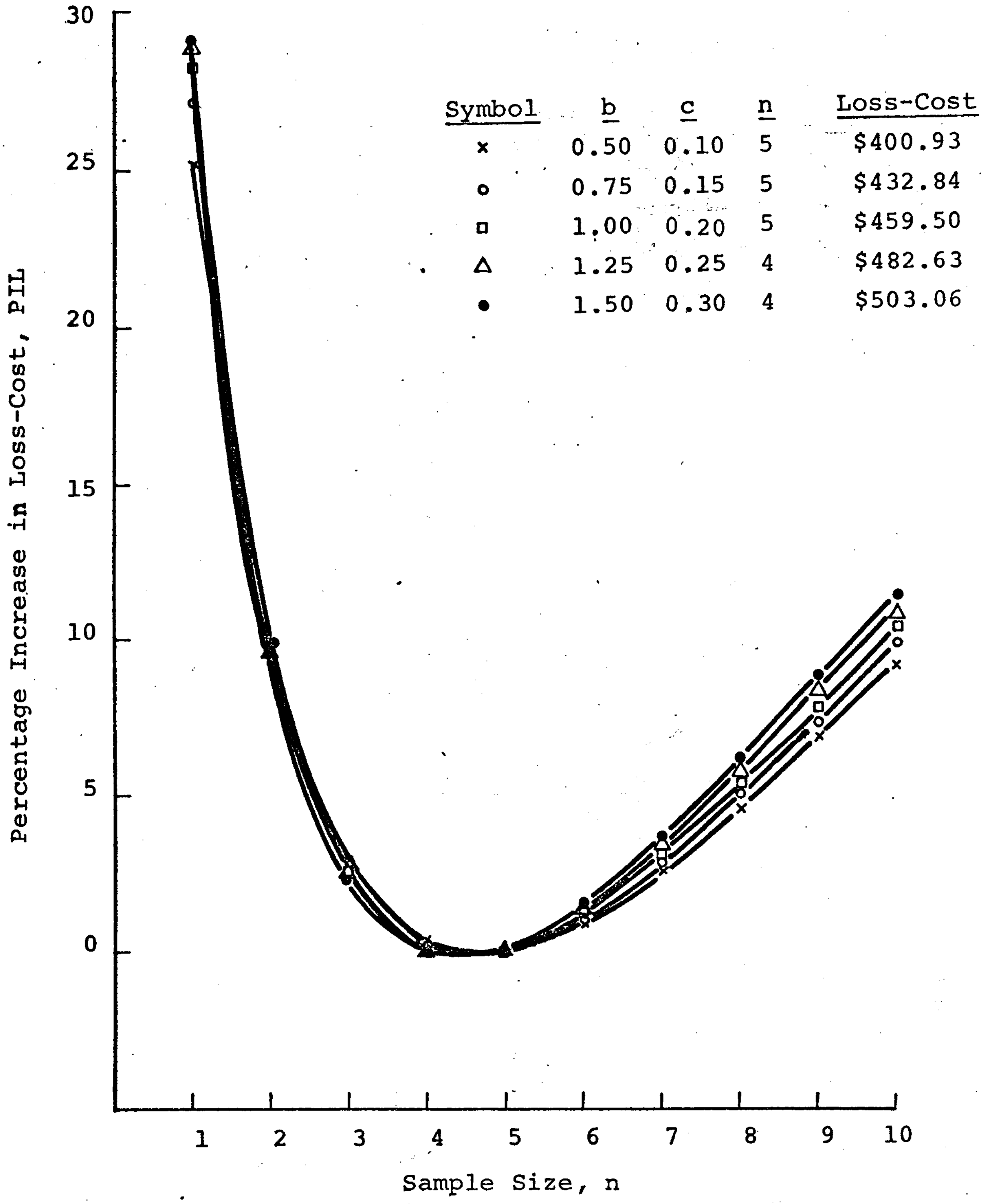


Figure 8. Effect of b and c on PIL.

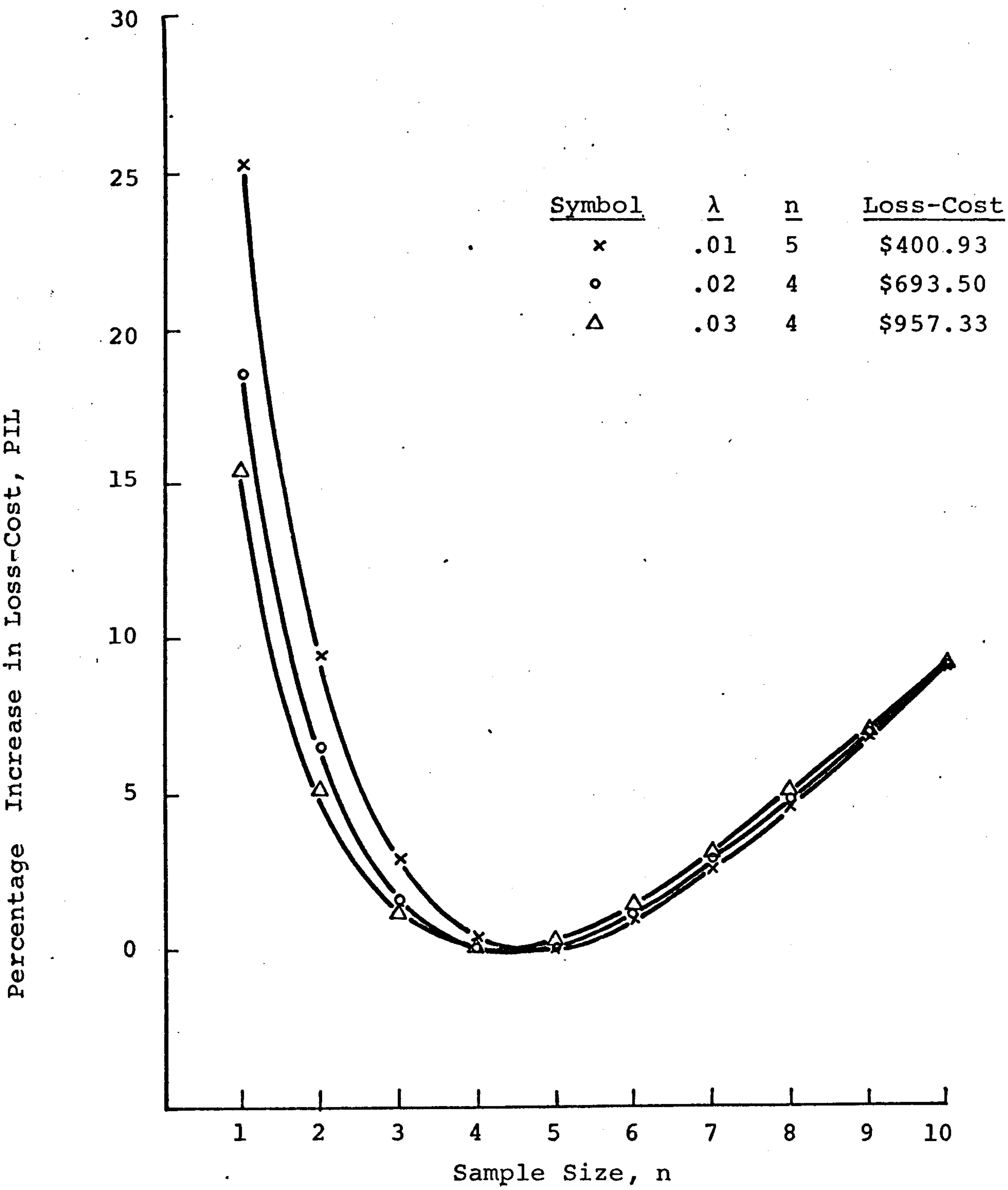


Figure 9. Effect of λ on PIL

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13. Abstract: A cost model is derived for the design of cumulative sum control charts using the Average Run Length to calculate the average time of out-of-control operation. The minimum cost design is obtained by employing the "pattern search" technique. The nature of the loss-cost surface and the effect of the design variables on the loss-cost surface are investigated. The effects of the shift parameter (δ), the average time for an assignable cause to occur ($1/\lambda$) and the cost factors b and c on the optimum designs and the loss-cost surfaces are also evaluated by numerical examples.

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