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TOLERANCE REGIONS
A SURVEY OF ITS LITERATURE

IV. Best Population Problems and Tolerance Regions

by

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5. Best Populations and Tolerance Regions

The framework of a "best" population problem consists (usually) of the following ingredients:

(1) there is a collection $\Pi = (\pi_1, \ldots, \pi_k)$ of k populations or processes, defined over the same sample space $\mathcal{X}(\mathcal{O}\!\mathcal{L})$;

(2) the population π_i is distributed with probability density function $f(x|\theta_i)$, where θ_i may be vector-valued;

(3) interest focuses on a specific criterion $h_i = g(\theta_i)$, where the functional form of g is known -- for example, $g(\theta_i)$ might be the population mean or the reciprocal of the variance of the i^{th} population, $i = 1, \ldots, k$;

(4) we wish to find (select, pick, estimate, etc.) that population which has the largest value amongst the h_i, $i = 1, \ldots, k$.

Using the above notation, we now can state the following definition.

Definition 5.1. A collection of populations $\Pi = (\pi_1, \ldots, \pi_k)$ contains a best population with respect to the criterion $h_i = g(\theta_i)$ if and only if there exists an ordering of the h_i such that

$$h_{[k]} > h_{[k-1]} \geq h_{[k-2]} \geq \cdots \geq h_{[1]} \tag{5.1}$$

We then say that the population corresponding to $h_{[k]}$ is the best population and we designate it by $\pi_{[k]}$.

where $\bar{X}$ is the sample mean computed from the best population, that is

$$P(CS) = \Pr(\bar{X}_{(1)} \geq \bar{X} - d_1) \qquad (5.5a)$$

$$= \int_{-\infty}^{\infty} \left\{ \prod_{i=2}^{k} [1 - G(\bar{x} - d_1; \mu_{[i]}, \sigma^2)] \right\} dG(\bar{x}; \mu_{[1]}, \sigma^2)$$

where $G(t; \mu, \sigma^2) = \int_{-\infty}^{t} \sqrt{n}\, (2\pi\sigma^2)^{-\frac{1}{2}} \exp\{-n(x-\mu)^2/2\sigma^2\}\, dx$. Hence we have that

$$\Pr(CS) = \left(\frac{n}{2\pi\sigma^2}\right)^{k/2} \int_{-\infty}^{\infty} \int_{\bar{x}-d_1}^{\infty} \cdots \int_{\bar{x}-d_1}^{\infty} \exp\{-\frac{n}{2\sigma^2}[\sum_{i=2}^{k} (\bar{x}_i - \mu_{[1]})^2 \qquad (5.5b)$$

$$+ (\bar{x} - \mu_{[1]})^2]\}\, d\bar{x}_2 \ldots d\bar{x}_k\, d\bar{x}.$$

If we set $t_i = \bar{x}_i - \mu_{[1]}$, $i = 2, \ldots, k$, and $t_1 = \bar{x} - \mu_{[1]}$, we then have that

$$\Pr(CS) = H_{d_1}(\mu_{[1]} - \mu_{[k]}, \ldots, \mu_{[1]} - \mu_{[2]}) \qquad (5.6)$$

where

$$H_{d_1}(\tau_1, \ldots, \tau_{k-1}) = \left(\frac{n}{2\pi\sigma^2}\right)^{k/2} \int_{-\infty}^{\infty} \int_{t_1-d_1+\tau_1}^{\infty} \cdots \int_{t_1-d_1+\tau_{k-1}}^{\infty} \qquad (5.6a)$$

$$\exp\{-\frac{n}{2\sigma^2} \sum_{1}^{k} t_i^2\}\, dt_2 \ldots dt_k\, dt_1$$

An examination of H_{d_1} shows it is a monotone decreasing function in its arguments. Further, if we fix $\mu_{[1]}$, and recalling (5.4), we note that

$$\bar{X}_i \leq \bar{X}_{(1)} + d_3 W$$

where d_3 is a constant satisfying

$$P^* = \int_{-\infty}^{\infty} [1 - T(t - \sqrt{n}\, d_3)]^{k-1} \, d\, T(t), \tag{5.10}$$

where T(t) is the cumulative distribution function of a Student-t variable with k(n-1) degrees of freedom. Table 5.2 gives some values of d_3 for selected P* and n, with k = 2, 3 and 4.

<u>Case N_4:</u> μ's known, with $\mu_i = \mu$, i=1, . . . , k; σ^2's unknown and variable.

This case splits itself into two cases.

<u>1. $\mu > a$.</u> Consulting (5. 3), the conditions defining this case imply (see 5. 3) that we are looking for that population with the largest of the σ_i. The parameter-free at level P* procedure is as follows:

<u>Procedure $N_{4,1}$:</u> Retain π_i in the subset if

$$V_i'^2 \geq d_{4,1} V_{(k)}'^2$$

where $V_i'^2 = n^{-1} \sum_{j=1}^{n} (X_{ij} - \mu)^2$, i = 1, . . . , k, $V_{(k)}'^2 = \max_{i=1}^{k} V_i'^2$, and $d_{4,1}$ satisfies

$$P^* = \int_0^{\infty} [F_n(u/d_{4,1})]^{k-1} \, d\, F_n(u) \tag{5.11}$$

where $F_n(u)$ is the cumulative distribution function of a Chi-Square variable with n degrees of freedom.

2. $\mu < a$. For this case, we are looking for that population with the least value of the σ_i. As is intuitively expected, the following procedure is parameter-free at level P^* for this situation.

Procedure $N_{4,2}$: Retain π_i in the subset if

$$V_i'^2 \leq d_{4,2} V_{(1)}'^2$$

where the $V_i'^2$'s are defined as in Procedure $N_{4,1}$ and where $d_{4,2}$ satisfies

$$P^* = \int_0^\infty [1 - F_n(v/d_{4,2})]^{k-1} \, d\, F_n(v) \tag{5.12}$$

where $F_n(v)$ is as defined in Procedure $N_{4,1}$.

We tabulate some values of $d_{4,1}$ and $d_{4,2}$ in Table 5.3 for selected P^* and n, with $k = 2, 3$ and 4.

Case N_5: μ's known, variable; σ's unknown and variable.

Bearing in mind the condition that defines Case N_5 and that we seek to find that population with least value of $(\mu_i - a)/\sigma_i$, we see immediately that Case N_5 splits into the following three cases.

Table 5.3

Values of the constant $d_{4,1}$ needed to make the P`cedure $N_{4,1}$ parameter free at level P^*.

	k = 2				k = 3				k = 4			
$n \backslash P^*$	.75	.90	.95	.99	.75	.90	.95	.99	.75	.90	.95	.99
2	.3333	.1111	.0526	.0101	.2078	.0723	.0347	.0067	.1663	.0587	.0283	.0055
4	.4845	.2435	.1565	.0626	.3505	.1830	.1194	.0485	.2997	.1586	.1041	.0426
6	.5611	.3274	.2334	.1181	.4313	.2601	.1881	.0969	.3792	.2315	.1683	.0873
8	.6099	.3862	.2909	.1659	.4856	.3166	.2416	.1403	.4340	.2862	.2195	.1284
10	.6446	.4305	.3358	.2062	.5256	.3605	.2846	.1778	.4750	.3293	.2612	.1644
12	.6711	.4657	.3722	.2407	.5568	.3960	.3202	.2105	.5074	.3644	.2961	.1960
14	.6921	.4945	.4026	.2704	.5821	.4255	.3503	.2391	.5340	.3939	.3257	.2238
16	.7094	.5186	.4285	.2966	.6032	.4506	.3762	.2644	.5562	.4191	.3514	.2486
18	.7239	.5394	.4510	.3197	.6211	.4724	.3989	.2870	.5753	.4411	.3741	.2708
20	.7364	.5575	.4708	.3404	.6367	.4915	.4190	.3073	.5919	.4605	.3942	.2909
22	.7472	.5734	.4883	.3591	.6504	.5084	.4370	.3258	.6065	.4778	.4122	.3092
24	.7568	.5876	.5041	.3761	.6625	.5236	.4532	.3427	.6196	.4933	.4285	.3259
26	.7653	.6004	.5183	.3916	.6733	.5374	.4679	.3582	.6313	.5074	.4434	.3413
28	.7729	.6119	.5313	.4059	.6832	.5499	.4814	.3725	.6420	.5202	.4570	.3556
30	.7798	.6225	.5432	.4191	.6921	.5613	.4938	.3857	.6516	.5320	.4696	.3688
32	.7861	.6322	.5542	.4314	.7003	.5719	.5053	.3981	.6605	.5429	.4812	.3812
34	.7919	.6411	.5643	.4428	.7078	.5817	.5159	.4096	.6687	.5530	.4921	.3927
36	.7972	.6493	.5737	.4535	.7147	.5907	.5258	.4205	.6763	.5624	.5022	.4036
38	.8021	.6570	.5825	.4635	.7212	.5992	.5351	.4306	.6834	.5711	.5116	.4138
40	.8067	.6642	.5907	.4730	.7272	.6071	.5438	.4402	.6900	.5794	.5205	.4235
42	.8109	.6709	.5985	.4819	.7328	.6145	.5520	.4493	.6961	.5871	.5289	.4326
44	.8149	.6772	.6057	.4903	.7380	.6215	.5597	.4579	.7019	.5944	.5368	.4412
46	.8186	.6831	.6126	.4983	.7430	.6281	.5670	.4660	.7074	.6012	.5443	.4495
48	.8221	.6887	.6191	.5059	.7477	.6343	.5739	.4738	.7125	.6077	.5514	.4573
50	.8254	.6940	.6252	.5131	.7521	.6402	.5805	.4812	.7174	.6139	.5581	.4648
52	.8285	.6990	.6310	.5200	.7562	.6459	.5867	.4883	.7220	.6198	.5645	.4719
54	.8315	.7038	.6366	.5265	.7602	.6512	.5927	.4950	.7264	.6253	.5707	.4787
56	.8343	.7083	.6419	.5328	.7640	.6563	.5984	.5015	.7305	.6307	.5765	.4852
58	.8369	.7126	.6469	.5388	.7675	.6611	.6038	.5077	.7345	.6358	.5821	.4915
60	.8395	.7167	.6518	.5446	.7710	.6658	.6090	.5136	.7383	.6406	.5875	.4975
65	.8453	.7263	.6630	.5580	.7788	.6766	.6211	.5275	.7471	.6519	.5999	.5116
70	.8505	.7349	.6731	.5702	.7860	.6863	.6320	.5401	.7550	.6622	.6113	.5244
75	.8552	.7427	.6823	.5814	.7924	.6951	.6420	.5517	.7622	.7615	.6216	.5362
80	.8595	.7498	.6907	.5917	.7982	.7032	.6512	.5624	.7687	.6801	.6311	.5471
85	.8635	.7563	.6985	.6012	.8036	.7107	.6596	.5723	.7748	.6879	.6399	.5572
90	.8671	.7624	.7057	.6100	.8086	.7176	.6675	.5815	.7803	.6952	.6480	.5666
95	.8704	.7679	.7123	.6182	.8132	.7240	.6747	.5901	.7855	.7020	.6556	.5753
100	.8735	.7731	.7185	.6259	.8174	.7300	.6815	.5981	.7902	.7083	.6627	.5835

2. All μ_i known and less than a. For this case, we wish to select the population having least value of $\sigma_i/(a - \mu_i)$. We use, then, the following parameter-free procedure at level P*.

Procedure $N_{5,2}$: Retain population π_i in the subset if

$$Q_i^2 \leq d_{5,2} Q_{(1)}^2$$

where Q_i^2 are defined above and $d_{5,2} = d_{4,2}$.

3. All μ_i known, with $\mu_{[1]} \leq \mu_{[2]} \leq \cdots \leq \mu_{[k_1-1]} < \mu_{[k_1]} < a$, and $a < \mu_{[k_1+1]} \leq \cdots \leq \mu_{[k]}$, where $1 < k_1 < k$. Here, the properties of the normal distribution come into play and we note that as μ decreases, the coverage of the interval $(-\infty, a)$ increases. Hence we may eliminate from consideration the $k - k_1$ populations which have

means greater than a, and then apply procedure $N_{5,2}$ for $k = k_1$.

Case N_6: μ's unknown, $\mu_i = \mu$, $i = 1, \ldots, k$; σ^2's unknown and variable.

In this situation we are faced with the unpleasant fact that, not only do we not know the common value μ of the μ_i, but we do not know whether μ is greater or smaller than the known number a. Since we wish to find the population with least value of $(\mu - a)/\sigma_i$, $i = 1, \ldots, k$ (see (5.3)), this means that we do not know whether the best population is the one with the largest σ_i (as is the case if $\mu > a$) or the one with the smallest σ_i (as is the case if $\mu < a$). We do assume, however, that there exists an ordering of the σ_i such that

$$\sigma_{\{1\}} < \sigma_{\{2\}} \leq \cdots \leq \sigma_{\{k-1\}} < \sigma_{\{k\}} \tag{5.13}$$

Further, we may gain information as to whether μ is less or greater than a by using the combined estimator $\bar{\bar{X}}$ of μ, where

$$\bar{\bar{X}} = \frac{n\bar{X}_1 + \ldots + n\bar{X}_k}{kn} = \frac{1}{k}\sum_{i=1}^{k} \bar{X}_i \tag{5.14}$$

where $\bar{X}_i = n^{-1} \sum_{j=1}^{n} X_{ij}$. Denoting the usual unbiased estimator of σ_i^2 by V_i^2, that is $V_i^2 = (n-1)^{-1} \sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2$, we now state the following procedure which is parameter-free at level P*.

Procedure N_6: Compute the estimator $\overline{\overline{X}}$ of μ given by (5.14).

(1) If $\overline{\overline{X}} > a$, then retain π_i in the subset if

$$V_i^2 \geq d_{6,1} V_{(k)}^2$$

where $V_{(k)}^2 = \max_{i=1}^{k} V_i^2$, and where $d_{6,1}$ is given by

$$d_{6,1}(n) = d_{4,1}(n-1) \tag{5.15}$$

for $n \geq 2$.

(2) If $\overline{\overline{X}} < a$, then retain π_i in the subset if

$$V_i^2 \leq d_{6,2} V_{(1)}^2$$

where $V_{(1)}^2 = \min_{i=1}^{k} V_i^2$, and where $d_{6,2}$ is given by

$$d_{6,2}(n) = d_{4,2}(n-1) \tag{5.16}$$

for $n \geq 2$.

That this procedure is indeed parameter-free at level P^* is shown in Guttman (1968).

We now turn to the case where sampling is from exponential distributions; that is, we consider a collection Π of k populations which is such that the probability density function of the i^{th} population π_i is

$$f(x \mid \mu_i, \sigma_i) = \begin{cases} \sigma_i^{-1} \exp\{-(x-\mu_i)/\sigma_i\} & \text{if } x \geq \mu_i,\ \sigma_i > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{5.17}$$

$i = 1, \ldots, k$. Assuming that the set of interest is again of the form $A = (-\infty, a]$, where "a" is a known constant, and that there exists a 'best' population in the sense of Definition 5.1 with criterion function defined by (5.2) and (5.17), then the problem of finding selection procedures for making a "correct selection" splits itself into several cases. It is to be noted that the best population is that population having largest value among the k values of $(a - \mu_i)/\sigma_i$. Guttman (1961) has shown that the procedures listed below for the various cases are parameter-free at level P^*.

Case E_1: μ's known, $\mu_i = \mu$, $i=1,\ldots,k$, σ_i's unknown and variable.

If the known value of μ is such that $\mu > a$, the exponential density defined by (5.17) gives zero coverage to $[-\infty, a]$. This means that there would not be a best population in our collection Π, contrary to assumption, so that we assume that $\mu < a$. Under this assumption and the condition that defines case E_1, we see that the best population is that which has least value of the σ_i; that is, there is an ordering of the σ_i such that

$$\sigma_{[1]} < \sigma_{[2]} \leq \cdots \leq \sigma_{[k]}. \tag{5.18}$$

Now let k independent samples, each consisting of n independent observations, say X_{ij}, $j = 1, \ldots, n$, $i = 1, \ldots, k$, be taken. Let $Y_{ij} = X_{ij} - \mu$. The following procedure is parameter-free at level P^*.

<u>Procedure E_1:</u> Retain π_i in the subset if

$$\bar{Y}_i < f_1 \bar{Y}_{(1)}$$

where $\bar{Y}_i = n^{-1} \sum_{j=1}^{n} Y_{ij}$, $\bar{Y}_{(1)} = \min_{i=1}^{k} Y_i$ and f_1 is a constant chosen to make $Pr(CS) \geq P^*$, and satisfies the equation $f_1(n) = d_{4,2}(2n)$, where $d_{4,2}(2n)$ is defined in (5.12).

<u>Case E_2:</u> μ's unknown and variable; $\sigma_i = \sigma$, $i=1, \ldots, k$ and known.

We assume that there is one μ_i such that $\mu_i < a$. Let $U_i = \min_{j=1}^{n} X_{ij}$, and further let $U_{(1)}$ be the minimum of the U_i, $i = 1, \ldots, k$. The best population in our collection is the one with the least μ_i. The following procedure is parameter-free at level P^*.

<u>Procedure E_2:</u> Retain π_i in the subset if

$$U_i \leq U_{(1)} + f_2$$

where f_2 is a constant chosen to make the $P_r(CS) \geq P^*$ and satisfies

$$f_2 = (\sigma \log k P^*)/n(k-1) \qquad (5.19)$$

Case E_3: μ's unknown, variable; σ_i's known, variable.

If we let $\delta_i = (\mu_i - a)/\sigma_i$, then we may assume that there is an ordering of the δ_i into

$$\delta_{[1]} < \delta_{[2]} \leq \delta_{[3]} \leq \cdots \leq \delta_{[k]}. \qquad (5.20)$$

and the best population is now the one having as its value of δ, the value $\delta_{[1]}$. Denote $\min_{j=1}^{n} X_{ij}$ by U_i and let $Z_i = (U_i - a)/\sigma_i$. The following procedure is parameter-free at level P*.

Procedure E_3: Retain π_i if

$$Z_i \leq Z_{(1)} + f_3$$

where $Z_{(1)} = \min_{i=1}^{k} Z_i$, and f_3 is a constant chosen to make $\Pr(CS) \geq P^*$ and satisfies the equation

$$f_3 = f_3(n;\, k,\, P^*) = f_2(n;\, k,\, P^*;\, \sigma = 1) \qquad (5.21)$$

where f_2 is defined in (5.19).

Case E_4: μ's known and variable; σ's unknown and variable.

Since we wish to find the population with least value of $(\mu_i - a)/\sigma_i$, the condition defining case E_4 leads us into the following cases.

1. <u>All μ_i known and less than a.</u> Suppose we order the observations X_{ij} taken from the population π_i into $(X_{i(1)}, \ldots, X_{i(n)})$, where

$$X_{i(1)} < X_{i(2)} < \ldots < X_{i(n)} \qquad (5.22)$$

where $i = 1, \ldots, k$. Define $S_i = (n-1)^{-1} \sum_{j=2}^{n} (X_{i(j)} - X_{i(1)})$, and $Z_i = S_i/(a - \mu_i)$. It is easy to see that we seek the population with least value of $\tau_i = \sigma_i/(a - \mu_i)$, and we assume that there is a best population, that is there is an ordering of the τ_i such that

$$\tau_{[1]} < \tau_{[2]} \leq \cdots \leq \tau_{[k]} \qquad (5.23)$$

The following procedure is parameter-free at level P*.

<u>Procedure $E_{4,1}$</u>: Retain π_i if

$$Z_i \leq f_{4,1} \; Z_{(1)}$$

where f_4 is such that $P_r(CS) \geq P^*$ and satisfies

$$f_{4,1}(n) = d_{4,2}(2n-2) \qquad (5.24)$$

for $n > 2, 3, \ldots$.

2. All μ_i known with $\mu_{[1]} \leq \cdots \leq \mu_{[k_1-1]} < \mu_{[k_1]} < a$, <u>and</u> $a < \mu_{[k_1+1]} \leq \cdots \leq \mu_{[k]}$, where $1 < k_1 < k$. Because an exponential

density of the type defined by (5.17) gives zero coverage to the interval $(-\infty, a]$ if $\mu_i > a$, we may disregard the $k - k_1$ populations for which $\mu_i > a$ and apply procedure $E_{4,1}$ with the k of that procedure put equal to k_1.

3. All μ_i known with $\mu_i > a$, $i = 1, \ldots, k$. For reasons just cited we may disregard this case entirely.

Case E_5: μ's unknown and variable; σ_i unknown, $\sigma_i = \sigma$, $i=1, \ldots, k$.

This case requires the use of a pooled estimator of σ, the common value of σ_i. Accordingly, define

$$S = \frac{(n-1)\,S_1 + \ldots + (n-1)\,S_k}{k\,(n-1)} = \frac{1}{k} \sum_{i=1}^{k} S_i \tag{5.25}$$

where the S_i are defined immediately after (5.22). Let $X_{i(1)} = \min_{j=1}^{n} X_{ij}$, $i = 1, \ldots, k$ and $V_i = n\,X_{i(1)}$, with $V_{(1)} = \min_{i=1}^{k} V_i$. We again wish to find the population with least value of $(\mu_i - a)/\sigma_i = (\mu_i - a)/\sigma$, that is, the population with the least value of μ_i, where we tacitly assume that there is at least one $\mu_i < a$. The following procedure is parameter-free at level P^*.

Procedure E_5. Retain π_i if

$$V_i < V_{(1)} + f_5\, S$$

where f_5 is such that the $\Pr(CS) \geq P^*$ and satisfies

$$P^* = \int_0^\infty \{[1 + (u-f_5)/k(n-1)]^{-k(k-1)(n-1)}\} \{[1 + u/k(n-1)]^{-k(n-1)-1}\} du. \tag{5.26}$$

Table 5.4 gives values of f_5 for selected $n \geq 2$, $P^* = .75, .90, .95$ and $.99$ and $k = 2, 3$ and 4. Details of proofs that procedures E_1-E_5 are parameter-free at level P^* may be found in Guttman (1961).

We will return to best population problems in the next part of this monograph, namely, when we discuss the Bayesian approach to best population problems.

Table 5.4

Values of the constant f_5 needed to make the Procedure E_5 parameter-free at level P^*

P^* / n	K = 2				K = 3				K = 4			
	.75	.90	.95	.99	.75	.90	.95	.99	.75	.90	.95	.99
2	.4529	.6246	.6722	.7075	.4194	.5056	.5306	.5495	.3710	.4293	.4463	.4593
4	.4241	.6053	.6580	.6978	.4109	.5007	.5272	.5473	.3681	.4281	.4458	.4593
6	.4170	.5989	.6522	.6927	.4088	.4992	.5259	.5463	.3674	.4277	.4455	.4591
8	.4138	.5960	.6495	.6902	.4079	.4985	.5253	.5457	.3671	.4275	.4454	.4591
10	.4120	.5942	.6479	.6887	.4074	.4981	.5250	.5454	.3669	.4274	.4453	.4590
12	.4108	.5931	.6469	.6878	.4070	.4979	.5247	.5452	.3667	.4273	.4453	.4590
14	.4100	.5923	.6461	.6871	.4068	.4977	.5246	.5451	.3667	.4273	.4452	.4589
16	.4094	.5917	.6456	.6866	.4066	.4975	.5245	.5450	.3666	.4272	.4452	.4589
18	.4090	.5913	.6451	.6862	.4065	.4974	.5244	.5449	.3666	.4272	.4452	.4589
20	.4086	.5909	.6448	.6859	.4064	.4974	.5243	.5449	.3665	.4272	.4452	.4589
22	.4083	.5906	.6445	.6856	.4063	.4973	.5242	.5448	.3665	.4272	.4451	.4589
24	.4081	.5904	.6443	.6854	.4062	.4972	.5242	.5448	.3665	.4271	.4451	.4588
26	.4079	.5902	.6441	.6852	.4062	.4972	.5242	.5447	.3664	.4271	.4451	.4588
28	.4077	.5900	.6440	.6851	.4061	.4971	.5241	.5447	.3664	.4271	.4451	.4588
30	.4075	.5899	.6438	.6849	.4061	.4971	.5241	.5447	.3664	.4271	.4451	.4588
32	.4074	.5897	.6437	.6848	.4060	.4971	.5241	.5446	.3664	.4271	.4451	.4588
34	.4073	.5896	.6436	.6847	.4060	.4971	.5240	.5446	.3664	.4271	.4451	.4588
36	.4072	.5895	.6435	.6846	.4060	.4970	.5240	.5446	.3664	.4271	.4451	.4588
38	.4071	.5894	.6434	.6845	.4059	.4970	.5240	.5446	.3664	.4271	.4451	.4588
40	.4070	.5893	.6433	.6845	.4059	.4970	.5240	.5446	.3664	.4271	.4451	.4588
42	.4069	.5893	.6432	.6844	.4059	.4970	.5240	.5446	.3664	.4271	.4451	.4588
44	.4069	.5892	.6432	.6843	.4059	.4970	.5240	.5445	.3663	.4271	.4451	.4588
46	.4068	.5891	.6431	.6843	.4059	.4969	.5239	.5445	.3663	.4271	.4451	.4588
48	.4067	.5891	.6431	.6842	.4058	.4969	.5239	.5445	.3663	.4271	.4451	.4588
50	.4067	.5890	.6430	.6842	.4058	.4969	.5239	.5445	.3663	.4271	.4451	.4588
52	.4066	.5890	.6430	.6842	.4058	.4969	.5239	.5445	.3663	.4271	.4451	.4588
54	.4066	.5889	.6429	.6841	.4058	.4969	.5239	.5445	.3663	.4271	.4451	.4588
56	.4066	.5889	.6429	.6841	.4058	.4969	.5239	.5445	.3663	.4271	.4451	.4588
58	.4065	.5888	.6429	.6840	.4058	.4969	.5239	.5445	.3663	.4270	.4451	.4588
60	.4065	.5888	.6428	.6840	.4058	.4969	.5239	.5445	.3663	.4270	.4451	.4588
65	.4064	.5887	.6428	.6839	.4057	.4968	.5239	.5445	.3663	.4270	.4450	.4588
70	.4063	.5887	.6427	.6839	.4057	.4968	.5238	.5444	.3663	.4270	.4450	.4588
75	.4063	.5886	.6426	.6838	.4057	.4968	.5238	.5444	.3663	.4270	.4450	.4588
80	.4062	.5886	.6426	.6838	.4057	.4968	.5238	.5444	.3663	.4270	.4450	.4588
85	.4062	.5885	.6425	.6837	.4057	.4968	.5238	.5444	.3663	.4270	.4450	.4588
90	.4061	.5885	.6425	.6837	.4057	.4968	.5238	.5444	.3663	.4270	.4450	.4588
95	.4061	.5884	.6425	.6837	.4057	.4968	.5238	.5444	.3663	.4270	.4450	.4588
100	.4061	.5884	.6424	.6836	.4056	.4968	.5238	.5444	.3663	.4270	.4450	.4588
∞	.4055	.5878	.6419	.6331	.4055	.4966	.5237	.5443	.3662	.4270	.4450	.4587

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