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TOLERANCE REGIONS
A SURVEY OF ITS LITERATURE

III. Discussion of β-content Tolerance Regions

by

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$$C = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{\overline{X} - KV}^{\overline{X} + KV} \exp\{-(t-\mu)^2/2\sigma^2\}\, dt$$

$$= \Phi\left(\frac{\overline{X} - \mu}{\sigma} + K\frac{V}{\sigma}\right) - \Phi\left(\frac{\overline{X} - \mu}{\sigma} - K\frac{V}{\sigma}\right) \tag{4.3}$$

is exceedingly complicated, and the question of how then to choose K exactly so as to meet the requirement (4.1) for preassigned β and γ is quite hard to answer. (Although we do not know the distribution of the coverage, we remind the reader that we know the first moment, that is, the expectation of C. Indeed, using the results of Section 3 with K given, it is easy to see that

$$E(C) = \beta' \tag{4.4}$$

where β' is the root of the equation

Theorem 4.1. If $X_1, \ldots, X_n$ are n independent observations from the $N(\mu, \sigma^2)$ distribution, and if we wish to construct a β-content tolerance interval at confidence level γ of the form (4.2), then

$$S(X_1, \ldots, X_n) = (\bar{X} - k_1 k_2 V, \ \bar{X} + k_1 k_2 V], \qquad (4.6)$$

where k_1 and k_2 are defined by (4.5a) and (4.5b) respectively, has coverage C, where

$$C = \int_{\bar{X}-k_1k_2V}^{\bar{X}+k_1k_2V} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\{-(x-\mu)^2/2\sigma^2\} \, dx \qquad (4.7)$$

that satisfies the statement

$$\Pr_{\mu,\sigma^2}\{C \geq \beta\} = \gamma' \qquad (4.8)$$

with $|\gamma - \gamma'| = O(n^{-1})$, and further, n^{-1} is the exact rate of convergence except for $\gamma = \frac{1}{2}$.

The proof is rather lengthy and we do not give the details here -- the interested reader is referred to page 768 of Ellison (1964).

Bowker (1947) has given tables of K given by (4.5) for $\gamma = .75$, .90, .95 and .99, with $\beta = .75$, .90, .95, .99 and .999 for an extensive range of n up to 1,000. We reproduce this in Table 4.1. Weissberg and Beatty (1960) tabulate $k_1 = k_1(n;\beta)$ and $k_2 = k_2(f;\gamma)$ seperately,

Table 4.1 continued

n \ β	γ = 0.75: 0.75	0.90	0.95	0.99	0.999	γ = 0.90: 0.75	0.90	0.95	0.99	0.999
26	1.313	1.877	2.236	2.938	3.751	1.444	2.065	2.460	3.232	4.127
27	1.309	1.871	2.229	2.929	3.740	1.437	2.054	2.447	3.215	4.106
28	1.305	1.865	2.222	2.920	3.728	1.430	2.044	2.435	3.199	4.085
29	1.301	1.860	2.216	2.911	3.718	1.423	2.034	2.424	3.184	4.066
30	1.297	1.855	2.210	2.904	3.708	1.417	2.025	2.413	3.170	4.049
31	1.294	1.850	2.204	2.896	3.699	1.411	2.017	2.403	3.157	4.032
32	1.291	1.846	2.199	2.890	3.690	1.405	2.009	2.393	3.145	4.016
33	1.288	1.842	2.194	2.883	3.682	1.400	2.001	2.385	3.133	4.001
34	1.285	1.838	2.189	2.877	3.674	1.395	1.994	2.376	3.122	3.987
35	1.283	1.834	2.185	2.871	3.667	1.390	1.988	2.368	3.112	3.974
36	1.280	1.830	2.181	2.866	3.660	1.386	1.981	2.361	3.103	3.961
37	1.278	1.827	2.177	2.860	3.653	1.381	1.975	2.353	3.092	3.949
38	1.275	1.824	2.173	2.855	3.647	1.377	1.969	2.346	3.083	3.938
39	1.273	1.821	2.169	2.850	3.641	1.374	1.964	2.340	3.075	3.927
40	1.271	1.818	2.166	2.846	3.635	1.370	1.959	2.334	3.066	3.917
41	1.269	1.815	2.162	2.841	3.629	1.366	1.954	2.328	3.059	3.907
42	1.267	1.812	2.159	2.837	3.624	1.363	1.949	2.322	3.051	3.897
43	1.266	1.810	2.156	2.833	3.619	1.360	1.944	2.316	3.044	3.888
44	1.264	1.807	2.153	2.829	3.614	1.357	1.940	2.311	3.037	3.879
45	1.262	1.805	2.150	2.826	3.609	1.354	1.935	2.306	3.030	3.871
46	1.261	1.802	2.148	2.822	3.605	1.351	1.931	2.301	3.024	3.863
47	1.259	1.800	2.145	2.819	3.600	1.348	1.927	2.297	3.018	3.855
48	1.258	1.798	2.143	2.815	3.596	1.345	1.924	2.292	3.012	3.847
49	1.256	1.796	2.140	2.812	3.592	1.343	1.920	2.288	3.006	3.840
50	1.255	1.794	2.138	2.809	3.588	1.340	1.916	2.284	3.001	3.833
51	1.253	1.792	2.135	2.806	3.584	1.338	1.913	2.279	2.995	3.826
52	1.252	1.790	2.133	2.803	3.581	1.336	1.910	2.276	2.990	3.820
53	1.251	1.789	2.131	2.801	3.577	1.334	1.907	2.272	2.985	3.813
54	1.250	1.787	2.129	2.798	3.574	1.331	1.904	2.268	2.981	3.807
55	1.249	1.785	2.127	2.795	3.571	1.329	1.901	2.265	2.976	3.801
56	1.247	1.784	2.125	2.793	3.567	1.327	1.898	2.261	2.972	3.796
57	1.246	1.782	2.123	2.790	3.564	1.325	1.895	3.258	2.967	3.790
58	1.245	1.781	2.122	2.788	3.561	1.323	1.892	2.255	2.963	3.785
59	1.244	1.779	2.120	2.786	3.558	1.322	1.890	2.252	2.959	3.779
60	1.243	1.778	2.118	2.784	3.556	1.320	1.887	2.248	2.955	3.774
61	1.242	1.776	2.117	2.781	3.553	1.318	1.885	2.245	2.951	3.769
62	1.241	1.775	2.115	2.779	3.550	1.316	1.882	2.243	2.947	3.765
63	1.240	1.774	2.113	2.777	3.548	1.315	1.880	2.240	2.944	3.760
64	1.240	1.772	2.112	2.775	3.545	1.313	1.878	2.237	2.940	3.755

Table 4.1 continued

n \ β	γ = 0.75					γ = 0.90				
	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
65	1.239	1.771	2.110	2.773	3.543	1.312	1.875	2.235	2.937	3.751
66	1.238	1.770	2.109	2.771	3.540	1.310	1.873	2.232	2.933	3.747
67	1.237	1.769	2.108	2.770	3.538	1.309	1.871	2.229	2.930	3.742
68	1.236	1.768	2.106	2.768	3.536	1.307	1.869	2.227	2.927	3.738
69	1.235	1.766	2.105	2.766	3.533	1.306	1.867	2.225	2.923	3.734
70	1.235	1.765	2.104	2.764	3.531	1.304	1.865	2.222	2.920	3.730
71	1.234	1.764	2.102	2.763	3.529	1.303	1.863	2.220	2.917	3.727
72	1.233	1.763	2.101	2.761	3.527	1.302	1.861	2.218	2.915	3.723
73	1.233	1.762	2.100	2.760	3.525	1.300	1.859	2.216	2.912	3.719
74	1.232	1.761	2.099	2.758	3.523	1.299	1.858	2.214	2.909	3.716
75	1.231	1.760	2.098	2.757	3.521	1.298	1.856	2.211	2.906	3.712
76	1.230	1.759	2.096	2.755	3.519	1.297	1.854	2.209	2.904	3.709
77	1.230	1.758	2.095	2.754	3.517	1.295	1.853	2.207	2.901	3.706
78	1.229	1.758	2.094	2.752	3.516	1.295	1.851	2.206	2.898	3.702
79	1.229	1.757	2.093	2.751	3.514	1.293	1.849	2.204	2.896	3.699
80	1.228	1.756	2.092	2.749	3.512	1.292	1.848	2.202	2.894	3.696
81	1.227	1.755	2.091	2.748	3.510	1.291	1.846	2.200	2.891	3.693
82	1.227	1.754	2.090	2.747	3.509	1.290	1.845	2.198	2.889	3.690
83	1.226	1.753	2.089	2.746	3.507	1.289	1.843	2.196	2.887	3.687
84	1.226	1.753	2.088	2.744	3.506	1.288	1.842	2.195	2.884	3.684
85	1.225	1.752	2.087	2.743	3.504	1.287	1.841	2.193	2.882	3.682
86	1.225	1.751	2.086	2.742	3.503	1.286	1.839	2.191	2.880	3.679
87	1.224	1.750	2.086	2.741	3.501	1.285	1.838	2.190	2.878	3.676
88	1.224	1.749	2.085	2.740	3.500	1.284	1.837	2.188	2.876	3.674
89	1.223	1.749	2.084	2.738	3.498	1.284	1.835	2.187	2.874	3.671
90	1.223	1.748	2.083	2.737	3.497	1.283	1.834	2.185	2.872	3.669
91	1.222	1.747	2.082	2.736	3.495	1.282	1.833	2.184	2.870	3.666
92	1.222	1.747	2.081	2.735	3.494	1.281	1.832	2.182	2.868	3.664
93	1.221	1.746	2.081	2.734	3.493	1.280	1.830	2.181	2.866	3.661
94	1.221	1.745	2.080	2.733	3.491	1.279	1.829	2.180	2.864	3.659
95	1.220	1.745	2.079	2.732	3.490	1.278	1.828	2.178	2.863	3.657
96	1.220	1.744	2.078	2.731	3.489	1.278	1.827	2.177	2.861	3.654
97	1.219	1.743	2.077	2.730	3.488	1.277	1.826	2.175	2.859	3.652

Table 4.1 continued

	$\gamma = 0.75$					$\gamma = 0.90$				
n \ β	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
98	1.219	1.743	2.077	2.729	3.486	1.276	1.825	2.174	2.857	3.650
99	1.219	1.742	2.076	2.728	3.485	1.275	1.824	2.173	2.856	3.648
100	1.218	1.742	2.075	2.727	3.484	1.275	1.822	2.172	2.854	3.646
101	1.218	1.741	2.075	2.726	3.483	1.274	1.821	2.170	2.852	3.644
102	1.217	1.741	2.074	2.726	3.482	1.273	1.820	2.169	2.851	3.642
104	1.217	1.739	2.073	2.724	3.480	1.272	1.818	2.167	2.848	3.638
106	1.216	1.738	2.071	2.722	3.477	1.270	1.816	2.164	2.845	3.634
108	1.215	1.737	2.070	2.721	3.475	1.269	1.815	2.162	2.842	3.630
110	1.214	1.736	2.069	2.719	3.473	1.268	1.813	2.160	2.839	3.626
112	1.214	1.735	2.068	2.717	3.471	1.267	1.811	2.158	2.836	3.623
114	1.213	1.734	2.067	2.716	3.469	1.265	1.809	2.156	2.833	3.619
116	1.212	1.733	2.065	2.714	3.468	1.264	1.808	2.154	2.831	3.616
118	1.212	1.733	2.064	2.713	3.466	1.263	1.806	2.152	2.828	3.613
120	1.211	1.732	2.063	2.712	3.464	1.262	1.804	2.150	2.826	3.610
122	1.210	1.731	2.062	2.710	3.462	1.261	1.803	2.148	2.823	3.607
124	1.210	1.730	2.061	2.709	3.461	1.260	1.801	2.147	2.821	3.604
126	1.209	1.729	2.060	2.708	3.459	1.259	1.800	2.145	2.819	3.601
128	1.209	1.728	2.060	2.707	3.458	1.258	1.799	2.143	2.816	3.598
130	1.208	1.728	2.059	2.705	3.456	1.257	1.797	2.141	2.814	3.595
132	1.208	1.727	2.058	2.704	3.455	1.256	1.796	2.140	2.812	3.592
134	1.207	1.726	2.057	2.703	3.453	1.255	1.795	2.138	2.810	3.590
136	1.207	1.725	2.056	2.702	3.452	1.254	1.793	2.137	2.808	3.587
138	1.206	1.725	2.055	2.701	3.450	1.253	1.792	2.135	2.806	3.585
140	1.206	1.724	2.054	2.700	3.449	1.252	1.791	2.134	2.804	3.582
142	1.205	1.723	2.054	2.699	3.448	1.252	1.790	2.132	2.802	3.580
144	1.205	1.723	2.053	2.698	3.446	1.251	1.788	2.131	2.801	3.578
146	1.204	1.722	2.052	2.697	3.445	1.250	1.787	2.130	2.799	3.575
148	1.204	1.722	2.051	2.696	3.444	1.249	1.786	2.128	2.797	3.573
150	1.204	1.721	2.051	2.695	3.443	1.248	1.785	2.127	2.795	3.571
152	1.203	1.720	2.050	2.694	3.441	1.248	1.784	2.126	2.794	3.569
154	1.203	1.720	2.049	2.693	3.440	1.247	1.783	2.125	2.792	3.567
156	1.202	1.719	2.049	2.692	3.439	1.246	1.782	2.123	2.791	3.565
158	1.202	1.719	2.048	2.691	3.438	1.246	1.781	2.122	2.789	3.563

Table 4.1 continued

β / n	γ = 0.75					γ = 0.90				
	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
160	1.202	1.718	2.047	2.691	3.437	1.245	1.780	2.121	2.787	3.561
162	1.201	1.718	2.047	2.690	3.436	1.244	1.779	2.120	2.786	3.559
164	1.201	1.717	2.046	2.689	3.435	1.244	1.778	2.119	2.785	3.557
166	1.201	1.717	2.045	2.688	3.434	1.243	1.777	2.118	2.783	3.555
168	1.200	1.716	2.045	2.687	3.433	1.242	1.776	2.117	2.782	3.553
170	1.200	1.716	2.044	2.687	3.432	1.242	1.775	2.116	2.780	3.552
172	1.199	1.715	2.044	2.686	3.431	1.241	1.775	2.115	2.779	3.550
174	1.199	1.715	2.043	2.685	3.430	1.240	1.774	2.114	2.778	3.548
176	1.199	1.714	2.043	2.684	3.429	1.240	1.773	2.113	2.776	3.547
178	1.199	1.714	2.042	2.684	3.428	1.239	1.772	2.112	2.775	3.545
180	1.198	1.713	2.042	2.683	3.427	1.239	1.771	2.111	2.774	3.543
185	1.197	1.712	2.040	2.681	3.425	1.237	1.769	2.108	2.771	3.539
190	1.197	1.711	2.039	2.680	3.423	1.236	1.767	2.106	2.768	3.536
195	1.196	1.710	2.038	2.678	3.421	1.235	1.766	2.104	2.765	3.532
200	1.195	1.709	2.037	2.677	3.419	1.234	1.764	2.102	2.762	3.529
205	1.195	1.708	2.036	2.675	3.418	1.233	1.762	2.100	2.760	3.526
210	1.194	1.708	2.035	2.674	3.416	1.231	1.761	2.098	2.757	3.522
215	1.194	1.707	2.034	2.673	3.414	1.230	1.759	2.096	2.755	3.519
220	1.193	1.706	2.033	2.671	3.413	1.229	1.758	2.095	2.753	3.516
225	1.192	1.705	2.032	2.670	3.411	1.228	1.756	2.093	2.750	3.514
230	1.192	1.704	2.031	2.669	3.409	1.227	1.755	2.091	2.748	3.511
235	1.191	1.704	2.030	2.668	3.408	1.226	1.754	2.090	2.746	3.508
240	1.191	1.703	2.029	2.667	3.407	1.226	1.752	2.088	2.744	3.506
245	1.190	1.702	2.028	2.666	3.405	1.225	1.751	2.087	2.742	3.503
250	1.190	1.702	2.028	2.665	3.404	1.224	1.750	2.085	2.740	3.501
255	1.190	1.701	2.027	2.664	3.403	1.223	1.749	2.084	2.739	3.499
260	1.189	1.700	2.026	2.663	3.401	1.222	1.748	2.083	2.737	3.496
265	1.189	1.700	2.025	2.662	3.400	1.222	1.747	2.081	2.735	3.494
270	1.188	1.699	2.025	2.661	3.399	1.221	1.746	2.080	2.734	3.492
275	1.188	1.699	2.024	2.660	3.398	1.220	1.745	2.079	2.732	3.490
280	1.188	1.698	2.023	2.659	3.397	1.219	1.744	2.078	2.730	3.488
285	1.187	1.698	2.023	2.658	3.396	1.219	1.743	2.076	2.729	3.486

Table 4.1 continued

n \ β	γ = 0.75					γ = 0.90				
	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
290	1.187	1.697	2.022	2.658	3.395	1.218	1.742	2.075	2.727	3.484
295	1.186	1.697	2.022	2.657	3.394	1.217	1.741	2.074	2.726	3.482
300	1.186	1.696	2.021	2.656	3.393	1.217	1.740	2.073	2.725	3.481
310	1.185	1.695	2.020	2.655	3.391	1.216	1.738	2.071	2.722	3.477
320	1.185	1.694	2.019	2.653	3.389	1.215	1.737	2.069	2.719	3.474
330	1.184	1.693	2.018	2.652	3.388	1.213	1.735	2.067	2.717	3.471
340	1.184	1.693	2.017	2.651	3.386	1.212	1.734	2.066	2.715	3.468
350	1.183	1.692	2.016	2.649	3.384	1.211	1.732	2.064	2.713	3.465
360	1.183	1.691	2.015	2.648	3.383	1.210	1.731	2.062	2.710	3.463
370	1.182	1.690	2.014	2.647	3.382	1.210	1.730	2.061	2.708	3.460
380	1.182	1.690	2.013	2.646	3.380	1.209	1.728	2.059	2.707	3.458
390	1.181	1.689	2.013	2.645	3.379	1.208	1.727	2.058	2.705	3.455
400	1.181	1.688	2.012	2.644	3.378	1.207	1.726	2.057	2.703	3.453
425	1.180	1.687	2.010	2.642	3.375	1.205	1.723	2.054	2.699	3.448
450	1.179	1.686	2.009	2.640	3.372	1.204	1.721	2.051	2.695	3.443
475	1.178	1.685	2.007	2.638	3.370	1.202	1.719	2.048	2.692	3.438
500	1.177	1.683	2.006	2.636	3.368	1.201	1.717	2.046	2.689	3.434
525	1.177	1.682	2.005	2.635	3.366	1.199	1.715	2.043	2.686	3.431
550	1.176	1.681	2.004	2.633	3.364	1.198	1.713	2.041	2.683	3.427
575	1.175	1.681	2.003	2.632	3.362	1.197	1.712	2.039	2.680	3.424
600	1.175	1.680	2.002	2.631	3.360	1.196	1.710	2.038	2.678	3.421
625	1.174	1.679	2.001	2.629	3.359	1.195	1.709	2.036	2.676	3.418
650	1.174	1.678	2.000	2.628	3.357	1.194	1.707	2.034	2.674	3.416
675	1.173	1.678	1.999	2.627	3.356	1.193	1.706	2.033	2.672	3.413
700	1.173	1.677	1.998	2.626	3.355	1.192	1.705	2.032	2.670	3.411
725	1.172	1.676	1.998	2.625	3.354	1.192	1.704	2.030	2.668	3.408
750	1.172	1.676	1.997	2.624	3.352	1.191	1.703	2.029	2.667	3.406
800	1.171	1.675	1.996	2.623	3.350	1.189	1.701	2.027	2.663	3.402
850	1.171	1.674	1.994	2.621	3.348	1.188	1.699	2.025	2.661	3.399
900	1.170	1.673	1.993	2.620	3.347	1.187	1.697	2.023	2.658	3.396
950	1.169	1.672	1.992	2.619	3.345	1.186	1.696	2.021	2.656	3.393
1,000	1.169	1.671	1.992	2.617	3.344	1.185	1.695	2.019	2.654	3.390
∞	1.150	1.645	1.960	2.576	3.291	1.150	1.645	1.960	2.576	3.291

Table 4.1 continued

n \ β	γ = 0.95					γ = 0.99				
	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
2	22.858	32.019	37.674	48.430	60.573	114.363	160.193	188.491	242.300	303.054
3	5.922	8.380	9.916	12.861	16.208	13.378	18.930	22.401	29.055	36.616
4	3.779	5.369	6.370	8.299	10.502	6.614	9.398	11.150	14.527	18.383
5	3.002	4.275	5.079	6.634	8.415	4.643	6.612	7.855	10.260	13.015
6	2.604	3.712	4.414	5.775	7.337	3.743	5.337	6.345	8.301	10.548
7	2.361	3.369	4.007	5.248	6.676	3.233	4.613	5.488	7.187	9.142
8	2.197	3.136	3.732	4.891	6.226	2.905	4.147	4.936	6.468	8.234
9	2.078	2.967	3.532	4.631	5.899	2.677	3.822	4.550	5.966	7.600
10	1.987	2.839	3.379	4.433	5.649	2.508	3.582	4.265	5.594	7.129
11	1.916	2.737	3.259	4.277	5.452	2.378	3.397	4.045	5.308	6.766
12	1.858	2.655	3.162	4.150	5.291	2.274	3.250	3.870	5.079	6.477
13	1.810	2.587	3.081	4.044	5.158	2.190	3.130	3.727	4.893	6.240
14	1.770	2.529	3.012	3.955	5.045	2.120	3.029	3.608	4.737	6.043
15	1.735	2.480	2.954	3.878	4.949	2.060	2.945	3.507	4.605	5.876
16	1.705	2.437	2.903	3.812	4.865	2.009	2.872	3.421	4.492	5.732
17	1.679	2.400	2.858	3.754	4.791	1.965	2.808	3.345	4.393	5.607
18	1.655	2.366	2.819	3.702	4.725	1.926	2.753	3.279	4.307	5.497
19	1.635	2.337	2.784	3.656	4.667	1.891	2.703	3.221	4.230	5.399
20	1.616	2.310	2.752	3.615	4.614	1.860	2.659	3.168	4.161	5.312
21	1.599	2.286	2.723	3.577	4.567	1.833	2.620	3.121	4.100	5.234
22	1.584	2.264	2.697	3.543	4.523	1.808	2.584	3.078	4.044	5.163
23	1.570	2.244	2.673	3.512	4.484	1.785	2.551	3.040	3.993	5.098
24	1.557	2.225	2.651	3.483	4.447	1.764	2.522	3.004	3.947	5.039
25	1.545	2.208	2.631	3.457	4.413	1.745	2.494	2.972	3.904	4.985

Table 4.1 continued

n \ β	γ = 0.95					γ = 0.99				
	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
26	1.534	2.193	2.612	3.432	4.382	1.727	2.469	2.941	3.865	4.935
27	1.523	2.178	2.595	3.409	4.353	1.711	2.446	2.914	3.828	4.888
28	1.514	2.164	2.579	3.388	4.326	1.695	2.424	2.888	3.794	4.845
29	1.505	2.152	2.564	3.368	4.301	1.681	2.404	2.864	3.763	4.805
30	1.497	2.140	2.549	3.350	4.278	1.668	2.385	2.841	3.733	4.768
31	1.489	2.129	2.536	3.332	4.256	1.656	2.367	2.820	3.706	4.732
32	1.481	2.118	2.524	3.316	4.235	1.644	2.351	2.801	3.680	4.699
33	1.475	2.108	2.512	3.300	4.215	1.633	2.335	2.782	3.655	4.668
34	1.468	2.099	2.501	3.286	4.197	1.623	2.320	2.764	3.632	4.639
35	1.462	2.090	2.490	3.272	4.179	1.613	2.306	2.748	3.611	4.611
36	1.455	2.081	2.479	3.258	4.161	1.604	2.293	2.732	3.590	4.585
37	1.450	2.073	2.470	3.246	4.146	1.595	2.281	2.717	3.571	4.560
38	1.445	2.066	2.461	3.234	4.131	1.587	2.269	2.703	3.552	4.537
39	1.440	2.057	2.453	3.223	4.117	1.579	2.257	2.690	3.534	4.514
40	1.435	2.052	2.445	3.213	4.104	1.571	2.247	2.677	3.518	4.493
41	1.430	2.045	2.437	3.202	4.090	1.564	2.236	2.665	3.502	4.472
42	1.426	2.039	2.429	3.192	4.077	1.557	2.227	2.653	3.486	4.453
43	1.422	2.033	2.422	3.183	4.065	1.551	2.217	2.642	3.472	4.434
44	1.418	2.027	2.415	3.173	4.053	1.545	2.208	2.631	3.458	4.416
45	1.414	2.021	2.408	3.165	4.042	1.539	2.200	2.621	3.444	4.399
46	1.410	2.016	2.402	3.156	4.031	1.533	2.192	2.611	3.431	4.383
47	1.406	2.011	2.396	3.148	4.021	1.527	2.184	2.602	3.419	4.367
48	1.403	2.006	2.390	3.140	4.011	1.522	2.176	2.593	3.407	4.352
49	1.399	2.001	2.384	3.133	4.002	1.517	2.169	2.584	3.396	4.337
50	1.396	1.996	2.379	3.126	3.993	1.512	2.162	2.576	3.385	4.323
51	1.393	1.992	2.373	3.119	3.984	1.507	2.155	2.568	3.374	4.310
52	1.390	1.988	2.368	3.112	3.975	1.503	2.148	2.560	3.364	4.297
53	1.387	1.984	2.363	3.106	3.967	1.498	2.142	2.552	3.354	4.284
54	1.384	1.980	2.359	3.100	3.959	1.494	2.136	2.545	3.344	4.272
55	1.382	1.976	2.354	3.094	3.951	1.490	2.130	2.538	3.335	4.260
56	1.379	1.972	2.350	3.088	3.944	1.486	2.124	2.531	3.326	4.249
57	1.377	1.968	2.345	3.082	3.937	1.482	2.119	2.524	3.318	4.238
58	1.374	1.965	2.341	3.076	3.930	1.478	2.113	2.518	3.309	4.227
59	1.372	1.961	2.337	3.071	3.923	1.474	2.108	2.512	3.301	4.216
60	1.369	1.958	2.333	3.066	3.916	1.471	2.103	2.506	3.293	4.206
61	1.367	1.955	2.329	3.061	3.909	1.467	2.098	2.500	3.285	4.196
62	1.365	1.951	2.325	3.056	3.903	1.464	2.093	2.494	3.278	4.187
63	1.363	1.948	2.322	3.051	3.897	1.461	2.089	2.489	3.271	4.178
64	1.361	1.945	2.318	3.046	3.891	1.458	2.084	2.483	3.264	4.169

Table 4.1 continued

n \ β	γ = 0.95: 0.75	0.90	0.95	0.99	0.999	γ = 0.99: 0.75	0.90	0.95	0.99	0.999
65	1.359	1.943	2.315	3.042	3.886	1.455	2.080	2.478	3.257	4.160
66	1.357	1.940	2.311	3.037	3.880	1.452	2.076	2.473	3.250	4.152
67	1.355	1.937	2.308	3.033	3.874	1.449	2.071	2.468	3.244	4.143
68	1.353	1.934	2.305	3.029	3.869	1.446	2.067	2.463	3.237	4.135
69	1.351	1.932	2.302	3.025	3.864	1.443	2.063	2.459	3.231	4.127
70	1.349	1.929	2.299	3.021	3.859	1.440	2.060	2.454	3.225	4.120
71	1.347	1.927	2.296	3.017	3.854	1.438	2.056	2.450	3.219	4.112
72	1.346	1.924	2.293	3.013	3.849	1.435	2.052	2.445	3.214	4.105
73	1.344	1.922	2.290	3.009	3.844	1.433	2.049	2.441	3.208	4.098
74	1.343	1.920	2.287	3.006	3.840	1.430	2.045	2.437	3.203	4.091
75	1.341	1.917	2.285	3.002	3.835	1.428	2.042	2.433	3.197	4.084
76	1.839	1.915	2.282	2.999	3.831	1.426	2.039	2.429	3.192	4.078
77	1.338	1.913	2.279	2.996	3.826	1.423	2.035	2.425	3.187	4.071
78	1.336	1.911	2.277	2.992	3.822	1.421	2.032	2.421	3.182	4.065
79	1.335	1.909	2.274	2.989	3.818	1.419	2.029	2.418	3.177	4.059
80	1.334	1.907	2.272	2.986	3.814	1.417	2.026	2.414	3.173	4.053
81	1.332	1.905	2.270	2.983	3.810	1.415	2.023	2.411	3.168	4.047
82	1.331	1.903	2.267	2.980	3.806	1.413	2.020	2.407	3.163	4.041
83	1.329	1.901	2.265	2.977	3.803	1.411	2.017	2.404	3.159	4.035
84	1.328	1.899	2.263	2.974	3.799	1.409	2.014	2.400	3.155	4.030
85	1.327	1.897	2.261	2.971	3.795	1.407	2.012	2.397	3.150	4.024
86	1.326	1.896	2.259	2.968	3.792	1.405	2.009	2.394	3.146	4.019
87	1.324	1.894	2.257	2.966	3.788	1.403	2.007	2.391	3.142	4.014
88	1.323	1.892	2.255	2.963	3.785	1.402	2.004	2.388	3.138	4.009
89	1.322	1.890	2.253	2.960	3.781	1.400	2.001	2.385	3.134	4.004
90	1.321	1.889	2.251	2.958	3.778	1.398	1.999	2.382	3.130	3.999
91	1.320	1.887	2.249	2.955	3.775	1.396	1.997	2.379	3.127	3.994
92	1.319	1.886	2.247	2.953	3.772	1.395	1.994	2.376	3.123	3.989
93	1.318	1.884	2.245	2.950	3.769	1.393	1.992	2.373	3.119	3.985
94	1.317	1.882	2.243	2.948	3.766	1.392	1.990	2.371	3.116	3.980
95	1.315	1.881	2.241	2.945	3.763	1.390	1.987	2.368	3.112	3.976
96	1.314	1.880	2.240	2.943	3.760	1.388	1.985	2.366	3.109	3.971
97	1.313	1.878	2.238	2.941	3.757	1.387	1.983	2.363	3.105	3.967

Table 4.1 continued

n \ β	γ = 0.95 0.75	0.90	0.95	0.99	0.999	γ = 0.99 0.75	0.90	0.95	0.99	0.999
98	1.312	1.877	2.236	2.939	3.754	1.385	1.981	2.360	3.102	3.963
99	1.311	1.875	2.234	2.936	3.751	1.384	1.979	2.358	3.099	3.958
100	1.311	1.874	2.233	2.934	3.748	1.383	1.977	2.355	3.096	3.954
101	1.310	1.872	2.231	2.932	3.746	1.381	1.975	2.353	3.092	3.950
102	1.309	1.871	2.230	2.930	3.743	1.380	1.973	2.351	3.089	3.946
104	1.307	1.869	2.227	2.926	3.738	1.377	1.969	2.346	3.083	3.939
106	1.305	1.866	2.224	2.922	3.733	1.374	1.965	2.342	3.077	3.931
108	1.303	1.864	2.221	2.918	3.728	1.372	1.962	2.337	3.072	3.924
110	1.302	1.861	2.218	2.915	3.723	1.369	1.958	2.333	3.066	3.917
112	1.300	1.859	2.215	2.911	3.719	1.367	1.955	2.329	3.061	3.910
114	1.299	1.857	2.212	2.908	3.714	1.365	1.951	2.325	3.056	3.904
116	1.297	1.855	2.210	2.904	3.710	1.363	1.948	2.321	3.051	3.897
118	1.296	1.852	2.207	2.901	3.706	1.360	1.945	2.318	3.046	3.891
120	1.294	1.850	2.205	2.898	3.702	1.358	1.942	2.314	3.041	3.885
122	1.293	1.848	2.203	2.895	3.698	1.356	1.939	2.311	3.037	3.879
124	1.291	1.847	2.200	2.892	3.694	1.354	1.936	2.307	3.032	3.873
126	1.290	1.845	2.198	2.889	3.690	1.352	1.934	2.304	3.028	3.868
128	1.289	1.843	2.196	2.886	3.686	1.350	1.931	2.301	3.024	3.862
130	1.288	1.841	2.194	2.883	3.683	1.349	1.928	2.298	3.019	3.857
132	1.286	1.839	2.192	2.880	3.679	1.347	1.926	2.295	3.015	3.852
134	1.285	1.838	2.190	2.878	3.676	1.345	1.923	2.292	3.012	3.847
136	1.284	1.836	2.188	2.875	3.673	1.343	1.921	2.289	3.008	3.842
138	1.283	1.834	2.186	2.873	3.669	1.342	1.918	2.286	3.004	3.837
140	1.282	1.833	2.184	2.870	3.666	1.340	1.916	2.283	3.000	3.833
142	1.281	1.831	2.182	2.868	3.663	1.338	1.914	2.280	2.997	3.828
144	1.280	1.830	2.180	2.865	3.660	1.337	1.912	2.278	2.993	3.824
146	1.279	1.828	2.178	2.863	3.657	1.335	1.909	2.275	2.990	3.820
148	1.278	1.827	2.177	2.861	3.654	1.334	1.907	2.273	2.987	3.815
150	1.277	1.825	2.175	2.859	3.652	1.332	1.905	2.270	2.983	3.811
152	1.276	1.824	2.173	2.856	3.649	1.331	1.903	2.268	2.980	3.807
154	1.275	1.823	2.172	2.854	3.646	1.330	1.901	2.265	2.977	3.803
156	1.274	1.821	2.170	2.852	3.644	1.328	1.899	2.263	2.974	3.799
158	1.273	1.820	2.169	2.850	3.641	1.327	1.897	2.261	2.971	3.796

Table 4.1 continued

β / n	γ = 0.95					γ = 0.99				
	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
160	1.272	1.819	2.167	2.848	3.638	1.326	1.896	2.259	2.968	3.792
162	1.271	1.818	2.166	2.846	3.636	1.324	1.894	2.256	2.965	3.788
164	1.270	1.816	2.164	2.844	3.634	1.323	1.892	2.254	2.963	3.785
166	1.269	1.815	2.163	2.843	3.631	1.322	1.890	2.252	2.960	3.781
168	1.269	1.814	2.162	2.841	3.629	1.321	1.888	2.250	2.957	3.778
170	1.268	1.813	2.160	2.839	3.627	1.320	1.887	2.248	2.955	3.774
172	1.267	1.812	2.159	2.837	3.624	1.318	1.885	2.246	2.952	3.771
174	1.266	1.811	2.158	2.835	3.622	1.317	1.884	2.244	2.950	3.768
176	1.266	1.810	2.156	2.834	3.620	1.316	1.882	2.243	2.947	3.765
178	1.265	1.809	2.155	2.832	3.618	1.315	1.880	2.241	2.945	3.762
180	1.264	1.808	2.154	2.831	3.616	1.314	1.879	2.239	2.942	3.759
185	1.262	1.805	2.151	2.827	3.611	1.311	1.875	2.234	2.937	3.751
190	1.261	1.803	2.148	2.823	3.606	1.309	1.872	2.230	2.931	3.744
195	1.259	1.800	2.145	2.819	3.601	1.307	1.868	2.226	2.926	3.738
200	1.258	1.798	2.143	2.816	3.597	1.304	1.865	2.222	2.921	3.731
205	1.256	1.796	2.140	2.812	3.593	1.302	1.862	2.219	2.916	3.725
210	1.255	1.794	2.138	2.809	3.589	1.300	1.859	2.215	2.911	3.719
215	1.253	1.792	2.135	2.806	3.585	1.298	1.856	2.212	2.907	3.713
220	1.252	1.790	2.133	2.803	3.581	1.296	1.854	2.209	2.903	3.708
225	1.251	1.788	2.131	2.800	3.577	1.294	1.851	2.205	2.898	3.703
230	1.249	1.787	2.129	2.798	3.574	1.293	1.848	2.202	2.894	3.697
235	1.248	1.785	2.127	2.795	3.571	1.291	1.846	2.199	2.890	3.692
240	1.247	1.783	2.125	2.792	3.567	1.289	1.843	2.197	2.887	3.688
245	1.246	1.782	2.123	2.790	3.564	1.288	1.841	2.194	2.883	3.683
250	1.245	1.780	2.121	2.788	3.561	1.286	1.839	2.191	2.880	3.678
255	1.244	1.779	2.119	2.785	3.558	1.284	1.837	2.189	2.876	3.674
260	1.243	1.777	2.118	2.783	3.555	1.283	1.835	2.186	2.873	3.670
265	1.242	1.776	2.116	2.781	3.552	1.282	1.833	2.184	2.870	3.666
270	1.241	1.774	2.114	2.779	3.550	1.280	1.831	2.181	2.867	3.662
275	1.240	1.773	2.113	2.777	3.547	1.279	1.829	2.179	2.864	3.658
280	1.239	1.772	2.111	2.775	3.544	1.278	1.827	2.177	2.861	3.655
285	1.238	1.771	2.110	2.773	3.542	1.276	1.825	2.175	2.858	3.651

Table 4.1 continued

	γ = 0.95					γ = 0.99				
n \ β	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
290	1.237	1.769	2.108	2.771	3.540	1.275	1.823	2.173	2.855	3.647
295	1.237	1.768	2.107	2.769	3.537	1.274	1.822	2.170	2.853	3.644
300	1.236	1.767	2.106	2.767	3.535	1.273	1.820	2.169	2.850	3.641
310	1.234	1.765	2.103	2.764	3.531	1.271	1.817	2.165	2.845	3.634
320	1.233	1.763	2.100	2.760	3.526	1.268	1.814	2.161	2.840	3.628
330	1.231	1.761	2.098	2.757	3.522	1.266	1.811	2.158	2.836	3.623
340	1.230	1.759	2.096	2.754	3.519	1.265	1.808	2.154	2.831	3.617
350	1.229	1.757	2.094	2.752	3.515	1.263	1.805	2.151	2.827	3.612
360	1.228	1.755	2.092	2.749	3.512	1.261	1.803	2.148	2.823	3.607
370	1.227	1.754	2.090	2.746	3.508	1.259	1.801	2.146	2.820	3.602
380	1.225	1.752	2.088	2.744	3.505	1.258	1.798	2.143	2.816	3.598
390	1.224	1.751	2.086	2.742	3.502	1.256	1.796	2.140	2.813	3.593
400	1.223	1.749	2.084	2.739	3.499	1.255	1.794	2.138	2.809	3.589
425	1.221	1.746	2.080	2.734	3.492	1.251	1.789	2.132	2.802	3.579
450	1.219	1.743	2.077	2.729	3.486	1.248	1.785	2.127	2.795	3.570
475	1.217	1.740	2.073	2.725	3.481	1.245	1.781	2.122	2.788	3.562
500	1.215	1.737	2.070	2.721	3.475	1.243	1.777	2.117	2.783	3.555
525	1.213	1.735	2.067	2.717	3.471	1.240	1.773	2.113	2.777	3.548
550	1.212	1.733	2.065	2.713	3.466	1.238	1.770	2.109	2.772	3.541
575	1.210	1.731	2.062	2.710	3.462	1.236	1.767	2.106	2.767	3.535
600	1.209	1.729	2.060	2.707	3.458	1.234	1.764	2.102	2.763	3.530
625	1.208	1.727	2.058	2.704	3.455	1.232	1.762	2.099	2.759	3.525
650	1.207	1.725	2.056	2.702	3.451	1.230	1.759	2.096	2.755	3.520
675	1.205	1.724	2.054	2.699	3.448	1.229	1.757	2.094	2.752	3.515
700	1.204	1.722	2.052	2.697	3.445	1.227	1.755	2.091	2.748	3.511
725	1.203	1.721	2.050	2.694	3.442	1.226	1.753	2.089	2.745	3.507
750	1.202	1.719	2.049	2.692	3.439	1.225	1.751	2.086	2.742	3.503
800	1.201	1.717	2.046	2.688	3.434	1.222	1.747	2.082	2.736	3.495
850	1.199	1.714	2.043	2.685	3.430	1.220	1.744	2.078	2.731	3.489
900	1.198	1.712	2.040	2.682	3.426	1.218	1.741	2.075	2.726	3.483
950	1.196	1.710	2.038	2.679	3.422	1.216	1.738	2.071	2.722	3.477
1,000	1.195	1.709	2.036	2.676	3.418	1.214	1.736	2.068	2.718	3.472
∞	1.150	1.645	1.960	2.576	3.291	1.150	1.645	1.960	2.576	3.291

where f is the degrees of freedom associated with the estimator of σ^2, assumed independent of the estimator of $E(X) = \mu$. For the case discussed so far, $f = n-1$, but the Weissberg and Beatty tables do have the advantage that they may also be utilized for more complicated situations, such as linear regression in p independent variables, that is, $\mu = E(X) = \underset{\sim}{z}' \underset{\sim}{\theta}$ (in obvious notation), or the case where V^2 is an independent estimator of σ^2 from a different sample, and based on f degrees of freedom. In the regression case, we would have $f = n-p$. Wallis (1951) has discussed the case when $p = 2$. Weissberg and Beatty (1960) themselves give an application that makes use of an analysis of variance, so that f is the error degrees of freedom appropriate in their situation.

Now if $n > 1,000$, that is, if n lies outside the range of Table 4.1, then we may make use of an approximation due to Bowker (1946) and set the k of (4.2) equal to

$$z_{\frac{1-\beta}{2}} \left[1 + z_{1-\gamma}/\sqrt{2n} + (5z^2_{1-\gamma} + 10)/12n \right] \qquad (4.7)$$

where z_τ is the point exceeded with probability τ when using the $N(0,1)$ distribution, and (4.2) will, to good approximation when n is large, then satisfy (4.1). We remind the reader that for the appropriate K, the resulting β-content region at confidence level γ enjoys the property of being of β-expectation, where β' is the solution of equation (4.4a).

We are now in the position of being able to construct a tolerance interval with ability to pick up $A_C = (\mu - z_{\frac{1-\beta}{2}}\sigma,\ \mu + z_{\frac{1-\beta}{2}}\sigma]$, and, of course, the interval is of the form (4.2). Indeed, by selecting the correct value of K in (4.2), the interval can be constructed so that it is a β-expectation tolerance interval (see Section 3) or a β-content interval at confidence level γ (by the results of this section). It is interesting at this point, then, to seek an approximation to the (very complicated) distribution of the coverage C of such an interval (4.2), where, as we have seen, C is given by (4.3).

Now since C does lie in the interval (0,1), it is natural to try to approximate the distribution of C by $I(p,q)$, the Incomplete Beta distribution of order, say (p,q), whose distribution function has the form given by (2.3). Such a distribution has mean and variance given by, respectively

$$\mu_1' = p/(p+q) \qquad \text{and} \qquad \mu_2 = pq/(p+q)^2(p+q+1) \tag{4.8}$$

Now to approximate the distribution of C by the $I(p,q)$ distribution, we proceed by fitting by moments, that is, we equate the mean and variance of C, ($\mu_C = E(C)$, $\sigma_C^2 = \mathrm{Var}(C)$), to μ_1' and μ_2 of (4.8) respectively. Solving for p and q, we find

$$\begin{aligned} p &= [\mu_C^2(1-\mu_C) - \mu_C\sigma_C^2]/\sigma_C^2 \\ q &= [\mu_C(1-\mu_C)^2 - (1-\mu_C)\sigma_C^2]/\sigma_C^2 \end{aligned} \tag{4.9}$$

We need, then, the following theorem.

Theorem 4.2. If the coverage of a tolerance interval S of the form (4.2) has mean and variance μ_C and σ_C^2, respectively, then, to first order approximation,

$$\mu_C = [2\,\Phi(K) - 1] + 2\,K\,\phi(K)\;E(V-1)$$
$$\sigma_C^2 = 4K^2\,\phi^2(K)\;\mathrm{Var}\;(V). \tag{4.10}$$

Proof. From (4.3), it is easy to see that C has distribution which does not depend on μ and σ^2, and indeed, that we may write (4.3) as

$$C = \Phi(\overline{X} + KV) - \Phi(\overline{X} - KV) \tag{4.11}$$

where, for this proof only, $\overline{X} = N(0,\ 1/n)$, and $V^2 = \chi^2_{n-1}/n-1$, with χ^2_{n-1} denoting a Chi-Squared variable with (n-1) degrees of freedom. Now if we expand $C = C(\overline{X}, V)$ about the point (0,1) by a first order Taylor Series, we obtain

$$\begin{aligned} C = C(\overline{X}, V) &= C(0,1) + 2K\,\phi(K)\;(V-1) \\ &= [\,2\Phi(K) - 1] + 2\,K\,\phi(K)\;(V-1) \end{aligned} \tag{4.11a}$$

since $\frac{\partial}{\partial \overline{X}} C(\overline{X}, V)$ vanishes when evaluated at (0,1). Hence we have that (to first order approximation)

$$E(C) = [2\Phi(K)-1] + 2K\phi(K)E(V-1). \tag{4.12}$$

Now squaring both sides of (4.11a) and taking expectations yields

$$E(C^2) = [2\Phi(K)-1]^2 + 4K\phi(K)\,[2\Phi(K)-1]\,E(V-1) + 4K^2\phi^2(K)\,E(V-1)^2 \tag{4.13}$$

so that the variance of C is

$$\begin{aligned}\sigma_C^2 &= E(C^2) - [E(C)]^2 \\ &= 4K^2\phi^2(K)\,\mathrm{Var}\,(V)\end{aligned}$$

and the theorem is proved.

Before applying the result of Theorem 4.2, it is useful to state and prove the following corollary.

Corollary 4.2. If X_i, $i = 1, \ldots, n$ are n independent observations from $N(\mu, \sigma^2)$, and if a tolerance interval S of the form (4.2) is constructed, then its coverage C has mean and variance, to terms of order $1/n$, given by

$$\begin{aligned}\mu_C &= [2\Phi(K)-1] - K\phi(K)(2n)^{-1} \\ \sigma_C^2 &= [2K^2\phi^2(K)]/n = [K^2\exp\{-K^2\}]/\pi n\end{aligned} \tag{4.14}$$

Proof. Using the results of Romanovsky (1925), it may easily be verified that

$$E(V-1) = -1/4n - 7/32n^2 - \dots \quad (4.15)$$

$$\text{Var}(V) = 1/2n + 3/4n^2 + \dots \quad (4.16)$$

Substituting (4.15) and (4.16) in (4.10), yields (4.14).

We can now "compare" distribution-free tolerance intervals which are β-content at confidence level γ, with tolerance intervals of the form (4.2), constructed so that they too are β-content at confidence level γ.

Suppose first that we are sampling from a population whose distribution function is continuous but otherwise unknown, and that we wish to construct a tolerance interval with ability to give information about the center $100\beta\%$ of the population. A reasonable candidate is, clearly, the interval

$$S_1(X_1, \dots, X_n) = (X_{(r)}, X_{(n-r+1)}] \quad (4.17)$$

where we have encluded $m = 2r$ blocks and use the $n-m+1$ blocks $(X_{(i-1)}, X_{(i)}]$, $i = r+1, \dots, n-r+1$. The reader will recall that the distribution of the coverage, say C, of the interval (4.17) is the Incomplete Beta distribution, with parameters $(n-m+1, m)$. Hence, we have

$$\text{Var}(C_1) = \frac{(n-m+1)(m)}{(n+1)^2(n+2)} = \tau(1-\tau)/(n+2) \quad (4.18)$$

where $\tau = (1 - \frac{m}{n+1})$. If S_1 is constructed so that it is of β-content at confidence level γ, we may determine m so that (4.1) holds by the methods of Section 2, or, if we may, consult Table 2.1 or Figure 2.3 etc., to find the required m for fixed β, γ and n for (4.1) to hold.

Now suppose we are told that the distribution function is not only continuous, but is a normal distribution function, so that we should construct the β-content at confidence γ interval to be of the form (4.2). Denoting this tolerance interval by S_2, we know from Corollary 4.2 that its coverage C_2 has variance given by

$$\mathrm{Var}(C_2) = [2K^2\phi^2(K)]/n = [K^2 \exp\{-K^2\}]/\pi n \tag{4.19}$$

For a large sample, then, say $n \geq 100$, we may compare the performance of S_1 with respect to S_2 by computing the relative efficiency (Rel. Eff.) of S_1 to S_2 as

$$\begin{aligned} \text{Rel. Eff.} &= \mathrm{Var}(C_2)/\mathrm{Var}(C_1) \\ &= \frac{n+2}{\pi n}[K^2 \exp\{-K^2\}/\tau(1-\tau)] \end{aligned} \tag{4.20}$$

We give some results for selected $n \geq 100$ in Table 4.2. For example, if $n = 1{,}000$ with $\beta = \gamma = .90$, Table 2.1 gives the values $m = 88$, so that $\tau = \frac{913}{1{,}001}$. Table 4.1 gives the value $K = 1.695$. Substitution of τ, n and K in (4.20) yields, for this case, the value of Rel. Eff. = .65. Remarkably, then, knowledge of the functional form of the distribution function effects a reduction in variance of over 35% in the variance of the coverage of the correctly constructed interval (4.2).

Table 4.2

Relative efficiencies of the β-content at confidence level γ tolerance intervals S_1(see (4.17)) with respect to S_2(see (4.2))

n \ β	γ = .75				γ = .90				γ = .95				γ = .99			
	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99
100	.641	.650	.654	-	.654	.698	.705	-	.683	.723	.570	-	.725	.670	.717	-
110	.646	.644	.552	-	.657	.674	.541	-	.683	.688	.658	-	.719	.774	.855	-
120	.632	.638	.611	-	.659	.658	.608	-	.683	.663	.749	-	.713	.709	.999	-
130	.636	.633	.538	-	.661	.646	.676	-	.661	.644	.564	-	.708	.668	.578	-
140	.639	.631	.586	.228	.663	.636	.561	-	.662	.704	.627	-	.705	.740	.658	-
150	.629	.675	.530	.250	.649	.687	.614	-	.664	.686	.691	-	.681	.703	.735	-
170	.624	.664	.525	.305	.653	.666	.577	-	.666	.657	.619	-	.679	.654	.603	-
200	.624	.654	.551	.359	.646	.645	.591	-	.656	.678	.616	-	.679	.667	.584	-
300	.619	.658	.576	.296	.637	.666	.583	.428	.642	.651	.581	.350	.660	.654	.601	-
400	.616	.646	.557	.277	.629	.660	.546	.316	.636	.664	.576	.532	.653	.656	.567	-
500	.615	.652	.548	.358	.624	.657	.558	.421	.633	.657	.575	.362	.645	.660	.597	.538
600	.611	.645	.562	.330	.624	.646	.566	.354	.628	.654	.577	.464	.640	.652	.585	.710
700	.611	.641	.557	.315	.622	.648	.553	.322	.628	.653	.558	.378	.637	.658	.580	.445
800	.609	.646	.551	.304	.620	.566	.562	.304	.624	.652	.563	.338	.635	.654	.577	.538
900	.610	.643	.549	.297	.619	.645	.554	.349	.622	.654	.554	.390	.633	.651	.576	.423
1000	.608	.641	.557	.292	.629	.646	.562	.329	.621	.646	.559	.357	.630	.649	.561	.488

As another application of the results of Theorem 4.2 and its Corollary 4.2a, we may inquire into the following question. Suppose sampling is from the $N(\mu, \sigma^2)$ distribution, and we construct a tolerance interval of the form (4.2) to be of β-expectation, that is, we choose K to be such that $K = \sqrt{1+n^{-1}}\ t_{n-1;\ \frac{1-\beta}{2}}$. We may now ask with what confidence, say γ, we have that the coverage of this β-expectation region exceeds β, or in general, any specified value, say δ, that is, we seek the value of γ given by

$$\gamma = P_r\,(C \geq \delta\,) \tag{4.21}$$

As we do not know the distribution of C, and as has been remarked on in this section before we stated Theorem 4.2, we must approximate the distribution of C by $I(p,q)$ with (p,q) given by (4.9). Now when $K = \sqrt{1+n^{-1}}\ t_{n-1;\ \frac{1-\beta}{2}}$, then, of course, $E(C) = \beta$ exactly; the use of (4.14) gives rise, then to a percentage error of

$$100 \left| \frac{\beta - [2\Phi(K) - 1] + K\phi(K)(2n)^{-1}}{\beta} \right| \ . \tag{4.22}$$

This percentage error, for $n \geq 100$ is less than 0.60% for $\beta = .75$, and less than 0.45% for $\beta \geq .90$. It is for this reason that we use $\mu_C = \beta$ in (4.9) when approximating the distribution of C by $I(p,q)$, and we have, for $n \geq 100$,

$$\gamma \simeq \int_{\delta}^{1} \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} t^{p-1} (1-t)^{q-1}\, dt \tag{4.23}$$

with

$$\begin{aligned} p &= [\beta^2 (1-\beta) - \beta\sigma_C^2] / \sigma_C^2 \\ q &= [\beta (1-\beta)^2 - (1-\beta)\sigma_C^2] / \sigma_C^2 \end{aligned} \tag{4.24}$$

and, of course σ_C^2 given by 4.14. Table 4.3 gives values of γ for selected $n \geq 100$ and $\beta = \delta = .75, .90, .95$ and $.99$. For example, the tolerance interval (4.2) constructed on the basis of a sample of size $n = 100$ to be of expectation .75, is also of .75 content with confidence $\gamma = .5119$.

Up to now we have been discussing the case of sampling from the univariate normal, and we now turn to the situation where sampling is from the k-variate normal. The reader will recall from section 3 that the minimax and most stringent β-expectation tolerance region with ability to pick up the "center" $100\beta\%$ set $A_C^{(k)}$ defined by (3.44) is given by the set S in R_k, where S is defined in (3.52) and (3.53). Intuitively, then, if we wish to construct a tolerance region S which is of β-content at confidence level γ, with ability to pick up $A_C^{(k)}$, it seems natural to use S given by

$$S(\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n) = \{\underset{\sim}{Y} \mid (\underset{\sim}{Y} - \overline{\underset{\sim}{X}})' V^{-1} (\underset{\sim}{Y} - \overline{\underset{\sim}{X}}) \leq K^{(k)}\} \tag{4.25}$$

where $K^{(k)} = K^{(k)}(n; \gamma, \beta)$ is chosen to give this region the property

Table 4.3

Approximate values of the confidence level γ that the β-expectation tolerance region (4.2), with $K = \sqrt{1 + n^{-1}}\ t_{n-1;\ \frac{1-\beta}{2}}$ has coverage that exceeds β.

n \ β	.75	.90	.95	.99
100	.5119	.5278	.5390	.5603
101	.5118	.5276	.5388	.5601
102	.5118	.5275	.5386	.5599
103	.5117	.5274	.5384	.5597
104	.5117	.5273	.5383	.5595
105	.5116	.5271	.5381	.5593
106	.5115	.5270	.5379	.5591
107	.5115	.5269	.5378	.5589
108	.5114	.5268	.5376	.5587
109	.5114	.5267	.5375	.5585
110	.5113	.5265	.5373	.5583
111	.5113	.5264	.5372	.5581
112	.5112	.5263	.5370	.5579
113	.5112	.5262	.5369	.5577
114	.5111	.5261	.5367	.5575
115	.5111	.5260	.5366	.5573
116	.5110	.5259	.5364	.5571
117	.5110	.5258	.5363	.5569
118	.5109	.5257	.5361	.5568
119	.5109	.5256	.5360	.5566
120	.5108	.5254	.5359	.5564
121	.5108	.5253	.5357	.5562
122	.5108	.5252	.5356	.5560
123	.5107	.5251	.5355	.5559
124	.5107	.5250	.5353	.5557
125	.5106	.5250	.5352	.5555
126	.5106	.5249	.5351	.5554
127	.5105	.5248	.5349	.5552
128	.5105	.5247	.5348	.5550
129	.5105	.5246	.5347	.5549
130	.5104	.5245	.5346	.5547
131	.5104	.5244	.5344	.5545
132	.5103	.5243	.5343	.5544
133	.5103	.5242	.5342	.5542
134	.5103	.5241	.5341	.5540
135	.5102	.5240	.5340	.5539

Table 4.3 (Continued)

n \ β	.75	.90	.95	.99
136	.5102	.5240	.5338	.5537
137	.5102	.5239	.5337	.5536
138	.5101	.5238	.5336	.5534
139	.5101	.5237	.5335	.5533
140	.5100	.5236	.5334	.5531
141	.5100	.5235	.5333	.5530
142	.5100	.5235	.5332	.5528
143	.5099	.5234	.5331	.5527
144	.5099	.5233	.5330	.5525
145	.5099	.5232	.5328	.5524
146	.5098	.5231	.5327	.5522
147	.5098	.5231	.5326	.5521
148	.5098	.5230	.5325	.5519
149	.5097	.5229	.5324	.5518
150	.5097	.5228	.5323	.5517
151	.5097	.5228	.5322	.5515
152	.5096	.5227	.5321	.5514
153	.5096	.5226	.5320	.5513
154	.5096	.5226	.5319	.5511
155	.5095	.5225	.5318	.5510
156	.5095	.5224	.5317	.5508
157	.5095	.5223	.5316	.5507
158	.5095	.5223	.5315	.5506
159	.5094	.5222	.5315	.5505
160	.5094	.5221	.5314	.5503
161	.5094	.5221	.5313	.5502
162	.5093	.5220	.5312	.5501
163	.5093	.5219	.5311	.5499
164	.5093	.5219	.5310	.5498
165	.5093	.5218	.5309	.5497
166	.5092	.5217	.5308	.5496
167	.5092	.5217	.5307	.5494
168	.5092	.5216	.5307	.5493
169	.5091	.5216	.5306	.5492
170	.5091	.5215	.5305	.5491
171	.5091	.5214	.5304	.5490
172	.5091	.5214	.5303	.5488
173	.5090	.5213	.5302	.5487
174	.5090	.5213	.5301	.5486
175	.5090	.5212	.5301	.5485

Table 4.3 (Continued)

n \ β	.75	.90	.95	.99
176	.5090	.5211	.5300	.5484
177	.5089	.5211	.5299	.5483
178	.5089	.5210	.5298	.5481
179	.5089	.5210	.5297	.5480
180	.5089	.5209	.5297	.5479
181	.5088	.5208	.5296	.5478
182	.5088	.5208	.5295	.5477
183	.5088	.5207	.5294	.5476
184	.5088	.5207	.5294	.5475
185	.5087	.5206	.5293	.5474
186	.5087	.5206	.5292	.5473
187	.5087	.5205	.5291	.5471
188	.5087	.5205	.5291	.5470
189	.5086	.5204	.5290	.5469
190	.5086	.5204	.5289	.5468
191	.5086	.5203	.5288	.5467
192	.5086	.5203	.5288	.5466
193	.5086	.5202	.5287	.5465
194	.5085	.5202	.5286	.5464
195	.5085	.5201	.5286	.5463
196	.5085	.5201	.5285	.5462
197	.5085	.5200	.5284	.5461
198	.5084	.5200	.5284	.5460
199	.5084	.5199	.5283	.5459
200	.5084	.5199	.5282	.5458
210	.5082	.5194	.5276	.5449
220	.5080	.5189	.5270	.5440
230	.5078	.5185	.5264	.5431
240	.5077	.5182	.5259	.5423
250	.5075	.5178	.5254	.5415
260	.5074	.5175	.5249	.5408
270	.5072	.5171	.5244	.5401
280	.5071	.5168	.5240	.5395
290	.5070	.5165	.5236	.5389
300	.5069	.5163	.5232	.5383
320	.5066	.5158	.5225	.5372
340	.5064	.5153	.5218	.5361
360	.5063	.5149	.5213	.5352
380	.5061	.5145	.5207	.5343
400	.5059	.5141	.5202	.5335

Table 4. 3 (Continued)

β / n	.75	.90	.95	.99
420	.5058	.5138	.5197	.5328
440	.5057	.5135	.5193	.5321
460	.5055	.5132	.5189	.5314
480	.5054	.5129	.5185	.5308
500	.5053	.5126	.5181	.5302
550	.5051	.5121	.5173	.5289
600	.5049	.5115	.5165	.5277
650	.5047	.5111	.5159	.5266
700	.5045	.5107	.5153	.5257
750	.5043	.5103	.5148	.5249
800	.5042	.5100	.5144	.5241
850	.5041	.5097	.5139	.5234
900	.5040	.5095	.5136	.5228
950	.5039	.5092	.5132	.5222
1000	.5038	.5090	.5129	.5217

that it is of β-content at confidence level γ. We point out immediately that if we knew $K^{(k)}$, then the β-content, γ confidence region S given by (4.25) would be of β'-expectation, where β' is the solution of the equation (see (3.53)),

$$F_{k,n-k;\,1-\beta'} = K^{(k)}\,(n-k)/[k(n-1)(1+n^{-1})]; \tag{4.26}$$

this follows immediately from the well known fact that $U^2 = (\underset{\sim}{Y} - \overline{\underset{\sim}{X}})'\,V^{-1}(\underset{\sim}{Y} - \overline{\underset{\sim}{X}}) = (1 + n^{-1})\,T^2_{n-1}$, where here T^2_{n-1} is Hotelling's T^2-statistic with $n-1$ degrees of freedom, which is itself distributed as an $(n-1)k\,F_{k,n-k}/(n-k)$ variable. The trouble at this point with equation (4.26), which we shall return to later in this section, is that we do not know the constants $K^{(k)}$, even for large n. To help us find $K^{(k)}$, we need the following theorem, due to Guttman (1968).

<u>Theorem 4.3.</u> If sampling from the k-variate normal distribution, the tolerance region S given by (4.25) has coverage C whose mean and variance are, to terms of order $1/n$,

$$\mu_C = \Psi_k(K^{(k)}) - [K^{(k)}]^{k/2} e^{-K^{(k)}/2}\left[2^{\frac{k}{2}+1}\Gamma\left(\tfrac{k}{2}\right)n\right]^{-1} \tag{4.27a}$$

and

$$\sigma^2_C = [K^{(k)}]^k\, e^{-K^{(k)}}\left[k2^{k-1}\Gamma^2\left(\tfrac{k}{2}\right)n\right]^{-1} \tag{4.27b}$$

respectively, where

$$\Psi_k(K^{(k)}) = P_r\,(\chi_k^2 \leq K^{(k)}) \quad . \tag{4.27c}$$

The reader should note that for $k = 1$, Theorem 4.3 corresponds with Corollary 4.2a if the constant K of Corollary 4.2a is put equal to $[K^{(1)}]^{\frac{1}{2}}$. This is not surprising, for after all, if $k = 1$, the variance-covariance matrix $\underline{V}$ is equal to the scaler V^2, and the region (4.25) takes the form

$$S(X_1, \ldots, X_n) = \{Y \,|\, (Y - \bar{X})^2/V^2 \leq K^{(1)}\} \tag{4.28}$$

which is equivalent to (4.2) with $K^{(1)} = K^2$. For $k = 2$, the result (4.27b) of Theorem 4.3 was obtained by Paulson (1943) when the bivariate region (4.25) was constructed to be of β-expectation, so that $K^{(2)}$ is put equal to $(1 + n^{-1})\ T^2_{n-1;\,1-\beta} = (1 + n^{-1})(n-1)2F_{2,\,n-2;1-\beta}/n-2$.

The proof of Theorem 4.3 is straightforward but tediously long and is omitted here -- the interested reader is referred to Guttman (1968).

Now that we have the mean and variance of the coverage C of the region S to terms of order $1/n$, we may use this information to approximate the distribution of C by the Incomplete Beta with parameters p and q given by (4.9), with μ_C and σ^2_C given by (4.27a) and (4.27b) respectively. Having done this, we may approximate the $K^{(k)}$ needed to make

(4.25) β-content at confidence level γ, where

$$\gamma = \int_{\beta}^{1} \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} t^{p-1} (1-t)^{q-1} dt \qquad (4.29)$$

Tables 4.4(a, b and c) give approximate values of $K^{(k)}$, $k = 2, 3$ and 4 for selected $n \geq 100$. Starting guesses were made by setting $K^{(k)} = K_0^{(k)} = \chi^2_{k;1-\beta}$ and iterating until the desired γ is achieved. As a check, constants $K^{(1)}$ were computed for selected $n \geq 100$ by this method and compared with K^2, where of course, K may be found in Table 4.1. For $n = 100$, the largest percentage error occurs for the $\gamma = .99$, $\beta = .99$ entry, and equals 8.6%. The percentage error for increasing n then drops off quite quickly--for example, if $n = 1,000$, and $\beta = \gamma = .99$, the percentage error is only 0.8%

Having obtained the $K^{(k)}$, we may now inquire about the content (coverage) of β-expectation regions (4.25), that is, regions (4.25) constructed with $K^{(k)} = (1+n^{-1})(n-1)kF_{k;n-k;1-\beta}/(n-k)$. Use of this value of $K^{(k)}$ makes $E(C) = \beta$ so that the approximation (4.27a) is in error. For $k = 2, 3$ and 4, with $\beta = .75$ and $n \geq 100$, the percentage error is less than 0.70%, and for $\beta \geq .90$ less than .60% (the largest errors occur for $k = 4$). Thus we may now approximate the distribution of the coverage C of the β-expectation regions (4.25) by the Incomplete Beta with (p, q) given by (4.9), with $\mu_C = \beta$ and σ^2_C given by (4.27b). We can now compute the confidence γ that we have in the coverage of the β-expectation regions of the form (4.25), exceeding β, or in general, any specified value, say δ, by using this approximation. Table 4.5(a, b and c) gives the values of $\gamma = \Pr(C \geq \delta)$ so obtained for $n \geq 100$, $\delta = \beta = .75$, .90, .95 and .99.

Further, we may compare, for large n, distribution-free tolerance regions which are β-content at confidence level γ, with tolerance intervals of the form (4.25), constructed so that they too are β-content at

Table 4.4a

Approximate values of $K^{(2)}$ needed to make the region (4.25), with $k = 2$, β-content at confidence level γ, when sampling from the bivariate normal

	$\gamma = .75$				$\gamma = .90$			
n \ β	.75	.90	.95	.99	.75	.90	.95	.99
100	2.9724	4.9146	6.3737	9.7329	3.1612	5.2181	6.7582	10.2901
120	2.9540	4.8880	6.3426	9.6958	3.1248	5.1636	6.6929	10.2063
140	2.9398	4.8673	6.3181	9.6659	3.0968	5.1213	6.6417	10.1396
160	2.9285	4.8505	6.2982	9.6410	3.0744	5.0873	6.6003	10.0849
180	2.9191	4.8366	6.2815	9.6199	3.0560	5.0591	6.5658	10.0389
200	2.9113	4.8248	6.2673	9.6017	3.0406	5.0353	6.5366	9.9996
220	2.9045	4.8146	6.2550	9.5858	3.0274	5.0149	6.5114	9.9654
240	2.8987	4.8058	6.2442	9.5717	3.0159	4.9971	6.4894	9.9354
260	2.8935	4.7979	6.2347	9.5592	3.0058	4.9814	6.4700	9.9087
280	2.8890	4.7909	6.2261	9.5478	2.9969	4.9674	6.4527	9.8848
300	2.8849	4.7847	6.2185	9.5376	2.9889	4.9549	6.4371	9.8632
320	2.8812	4.7790	6.2115	9.5282	2.9817	4.9436	6.4230	9.8436
340	2.8778	4.7738	6.2051	9.5197	2.9751	4.9332	6.4101	9.8256
360	2.8748	4.7691	6.1993	9.5118	2.9692	4.9238	6.3983	9.8091
380	2.8719	4.7647	6.1939	9.5044	2.9637	4.9152	6.3875	9.7939
400	2.8693	4.7607	6.1889	9.4977	2.9586	4.9072	6.3775	9.7798
420	2.8669	4.7569	6.1843	9.4913	2.9539	4.8997	6.3682	9.7667
440	2.8647	4.7534	6.1800	9.4854	2.9496	4.8929	6.3595	9.7544
460	2.8626	4.7502	6.1759	9.4798	2.9455	4.8864	6.3514	9.7430
480	2.8607	4.7471	6.1722	9.4746	2.9417	4.8804	6.3438	9.7322
500	2.8588	4.7443	6.1686	9.4697	2.9382	4.8747	6.3367	9.7221
520	2.8571	4.7416	6.1652	9.4650	2.9348	4.8694	6.3300	9.7125
540	2.8555	4.7390	6.1621	9.4606	2.9317	4.8644	6.3237	9.7035
560	2.8539	4.7366	6.1591	9.4564	2.9287	4.8596	6.3177	9.6949
580	2.8525	4.7343	6.1562	9.4524	2.9259	4.8551	6.3120	9.6868
600	2.8511	4.7322	6.1535	9.4486	2.9232	4.8508	6.3066	9.6790
620	2.8498	4.7301	6.1509	9.4450	2.9207	4.8468	6.3015	9.6717
640	2.8485	4.7281	6.1485	9.4416	2.9183	4.8429	6.2966	9.6646
660	2.8474	4.7263	6.1461	9.4383	2.9160	4.8392	6.2919	9.6579
680	2.8462	4.7245	6.1439	9.4351	2.9138	4.8357	6.2875	9.6515
700	2.8451	4.7228	6.1417	9.4321	2.9117	4.8323	6.2832	9.6453
720	2.8441	4.7211	6.1396	9.4292	2.9097	4.8291	6.2791	9.6394
740	2.8431	4.7195	6.1377	9.4264	2.9077	4.8260	6.2752	9.6338
760	2.8421	4.7180	6.1358	9.4237	2.9059	4.8230	6.2714	9.6283
780	2.8412	4.7166	6.1339	9.4211	2.9041	4.8202	6.2678	9.6231
800	2.8403	4.7152	6.1322	9.4186	2.9024	4.8174	6.2643	9.6181
820	2.8395	4.7138	6.1305	9.4162	2.9008	4.8148	6.2610	9.6132
840	2.8387	4.7125	6.1288	9.4139	2.8992	4.8123	6.2577	9.6085
860	2.8379	4.7113	6.1273	9.4116	2.8976	4.8098	6.2546	9.6040
880	2.8371	4.7100	6.1257	9.4094	2.8962	4.8074	6.2516	9.5996
900	2.8364	4.7089	6.1243	9.4073	2.8947	4.8051	6.2487	9.5953
920	2.8357	4.7077	6.1228	9.4053	2.8934	4.8029	6.2459	9.5912
940	2.8350	4.7066	6.1215	9.4033	2.8920	4.8008	6.2431	9.5873
960	2.8343	4.7056	6.1201	9.4014	2.8908	4.7987	6.2405	9.5834
980	2.8337	4.7045	6.1188	9.3996	2.8895	4.7967	6.2379	9.5797
1000	2.8330	4.7035	6.1176	9.3977	2.8883	4.7947	6.2354	9.5761
∞	2.7726	4.6052	5.9915	9.2103	2.7726	4.6052	5.9915	9.2103

Table 4.4a (Continued)

Approximate values of $K^{(2)}$ needed to make the region (4.25), with $k = 2$, β-content at confidence level γ, when sampling from the trivariate normal

n \ β	γ = .95				γ = .99			
	.75	.90	.95	.99	.75	.90	.95	.99
100	3.2807	5.4072	6.9948	10.6230	3.5182	5.7757	7.4486	11.2396
120	3.2323	5.3349	6.9081	10.5118	3.4453	5.6683	7.3211	11.0795
140	3.1953	5.2789	6.8404	10.4235	3.3896	5.5852	7.2216	10.9521
160	3.1657	5.2339	6.7856	10.3511	3.3454	5.5184	7.1411	10.8477
180	3.1415	5.1967	6.7401	10.2903	3.3092	5.4634	7.0743	10.7601
200	3.1211	5.1653	6.7016	10.2384	3.2789	5.4170	7.0177	10.6852
220	3.1038	5.1384	6.6684	10.1934	3.2531	5.3772	6.9691	10.6201
240	3.0887	5.1150	6.6394	10.1538	3.2308	5.3427	6.9266	10.5630
260	3.0755	5.0943	6.6138	10.1187	3.2112	5.3123	6.8892	10.5123
280	3.0638	5.0759	6.5910	10.0872	3.1939	5.2853	6.8558	10.4669
300	3.0533	5.0595	6.5705	10.0588	3.1784	5.2611	6.8259	10.4259
320	3.0438	5.0446	6.5520	10.0330	3.1645	5.2393	6.7988	10.3887
340	3.0352	5.0311	6.5351	10.0095	3.1518	5.2194	6.7742	10.3547
360	3.0274	5.0187	6.5197	9.9879	3.1403	5.2013	6.7516	10.3235
380	3.0202	5.0074	6.5054	9.9679	3.1298	5.1847	6.7309	10.2947
400	3.0136	4.9969	6.4923	9.9494	3.1201	5.1694	6.7117	10.2681
420	3.0075	4.9872	6.4801	9.9322	3.1111	5.1552	6.6940	10.2433
440	3.0018	4.9782	6.4688	9.9162	3.1028	5.1420	6.6775	10.2201
460	2.9965	4.9697	6.4582	9.9011	3.0950	5.1297	6.6621	10.1985
480	2.9916	4.9619	6.4483	9.8871	3.0878	5.1182	6.6476	10.1782
500	2.9870	4.9545	6.4390	9.8738	3.0810	5.1074	6.6341	10.1591
520	2.9826	4.9475	6.4302	9.8613	3.0746	5.0972	6.6213	10.1411
540	2.9785	4.9409	6.4219	9.8495	3.0686	5.0876	6.6093	10.1240
560	2.9746	4.9347	6.4141	9.8383	3.0630	5.0786	6.5979	10.1079
580	2.9709	4.9289	6.4067	9.8276	3.0576	5.0700	6.5871	10.0926
600	2.9674	4.9233	6.3996	9.8175	3.0525	5.0619	6.5769	10.0780
620	2.9641	4.9180	6.3929	9.8079	3.0477	5.0542	6.5672	10.0642
640	2.9610	4.9129	6.3865	9.7987	3.0431	5.0469	6.5579	10.0510
660	2.9580	4.9081	6.3804	9.7900	3.0387	5.0399	6.5491	10.0384
680	2.9551	4.9035	6.3746	9.7816	3.0346	5.0332	6.5407	10.0263
700	2.9524	4.8991	6.3690	9.7735	3.0306	5.0268	6.5326	10.0147
720	2.9498	4.8949	6.3637	9.7658	3.0268	5.0207	6.5249	10.0037
740	2.9473	4.8909	6.3586	9.7585	3.0231	5.0148	6.5175	9.9930
760	2.9449	4.8870	6.3537	9.7514	3.0196	5.0092	6.5103	9.9828
780	2.9425	4.8833	6.3490	9.7445	3.0163	5.0038	6.5035	9.9730
800	2.9403	4.8797	6.3444	9.7379	3.0131	4.9986	6.4969	9.9635
820	2.9382	4.8763	6.3401	9.7316	3.0100	4.9936	6.4906	9.9544
840	2.9361	4.8730	6.3358	9.7255	3.0070	4.9888	6.4845	9.9456
860	2.9341	4.8698	6.3318	9.7196	3.0041	4.9842	6.4786	9.9371
880	2.9322	4.8667	6.3278	9.7139	3.0013	4.9797	6.4730	9.9289
900	2.9304	4.8637	6.3241	9.7083	2.9986	4.9754	6.4675	9.9210
920	2.9286	4.8608	6.3204	9.7030	2.9960	4.9712	6.4622	9.9133
940	2.9269	4.8580	6.3168	9.6978	2.9935	4.9672	6.4570	9.9059
960	2.9252	4.8553	6.3134	9.6928	2.9911	4.9633	6.4521	9.8987
980	2.9236	4.8527	6.3101	9.6880	2.9888	4.9595	6.4472	9.8917
1000	2.9220	4.8501	6.3068	9.6832	2.9865	4.9558	6.4426	9.8849
∞	2.7726	4.6052	5.9915	9.2103	2.7726	4.6052	5.9915	9.2103

Table 4.4b

Approximate values of $K^{(3)}$ needed to make the region (4.25), with $k = 3$, β-content at confidence level γ, when sampling from the trivariate normal

	$\gamma = .75$				$\gamma = .90$			
n \ β	.75	.90	.95	.99	.75	.90	.95	.99
100	4.3518	6.5985	8.2290	11.8881	4.5746	6.9271	8.6300	12.4404
120	4.3293	6.5683	8.1946	11.8477	4.5312	6.8673	8.5605	12.3543
140	4.3119	6.5448	8.1676	11.8155	4.4978	6.8209	8.5061	12.2859
160	4.2981	6.5259	8.1457	11.7888	4.4712	6.7835	8.4621	12.2299
180	4.2866	6.5101	8.1274	11.7664	4.4492	6.7525	8.4255	12.1830
200	4.2770	6.4968	8.1119	11.7471	4.4307	6.7263	8.3944	12.1429
220	4.2688	6.4854	8.0985	11.7303	4.4149	6.7038	8.3677	12.1080
240	4.2617	6.4754	8.0868	11.7155	4.4012	6.6842	8.3443	12.0775
260	4.2554	6.4666	8.0764	11.7023	4.3892	6.6669	8.3236	12.0503
280	4.2498	6.4587	8.0671	11.6904	4.3784	6.6516	8.3052	12.0260
300	4.2448	6.4517	8.0587	11.6797	4.3689	6.6378	8.2886	12.0041
320	4.2403	6.4453	8.0512	11.6699	4.3602	6.6253	8.2736	11.9841
340	4.2362	6.4395	8.0443	11.6610	4.3524	6.6139	8.2600	11.9659
360	4.2324	6.4342	8.0379	11.6527	4.3452	6.6036	8.2475	11.9492
380	4.2290	6.4293	8.0321	11.6451	4.3386	6.5940	8.2359	11.9338
400	4.2258	6.4247	8.0267	11.6381	4.3325	6.5852	8.2253	11.9195
420	4.2229	6.4205	8.0217	11.6315	4.3269	6.5770	8.2154	11.9062
440	4.2202	6.4166	8.0170	11.6253	4.3217	6.5694	8.2062	11.8938
460	4.2176	6.4130	8.0127	11.6196	4.3168	6.5623	8.1976	11.8821
480	4.2153	6.4096	8.0086	11.6142	4.3123	6.5557	8.1896	11.8712
500	4.2130	6.4064	8.0047	11.6091	4.3080	6.5494	8.1820	11.8610
520	4.2109	6.4033	8.0011	11.6042	4.3040	6.5435	8.1748	11.8513
540	4.2090	6.4005	7.9977	11.5997	4.3002	6.5380	8.1681	11.8421
560	4.2071	6.3978	7.9944	11.5954	4.2966	6.5328	8.1617	11.8335
580	4.2053	6.3952	7.9913	11.5912	4.2932	6.5278	8.1557	11.8252
600	4.2036	6.3928	7.9884	11.5873	4.2900	6.5231	8.1500	11.8174
620	4.2020	6.3905	7.9856	11.5836	4.2870	6.5186	8.1445	11.8100
640	4.2005	6.3883	7.9830	11.5801	4.2841	6.5143	8.1393	11.8029
660	4.1991	6.3862	7.9804	11.5766	4.2813	6.5103	8.1344	11.7961
680	4.1977	6.3842	7.9780	11.5734	4.2787	6.5064	8.1296	11.7896
700	4.1964	6.3823	7.9757	11.5703	4.2761	6.5026	8.1251	11.7834
720	4.1951	6.3804	7.9735	11.5673	4.2737	6.4991	8.1207	11.7774
740	4.1939	6.3787	7.9713	11.5644	4.2714	6.4957	8.1166	11.7717
760	4.1927	6.3770	7.9693	11.5616	4.2692	6.4924	8.1126	11.7662
780	4.1916	6.3754	7.9673	11.5590	4.2670	6.4892	8.1087	11.7609
800	4.1905	6.3738	7.9654	11.5564	4.2650	6.4862	8.1050	11.7558
820	4.1895	6.3723	7.9636	11.5539	4.2630	6.4833	8.1015	11.7509
840	4.1885	6.3708	7.9618	11.5515	4.2611	6.4805	8.0980	11.7461
860	4.1875	6.3694	7.9601	11.5492	4.2593	6.4778	8.0947	11.7415
880	4.1866	6.3681	7.9585	11.5470	4.2575	6.4751	8.0915	11.7371
900	4.1857	6.3668	7.9569	11.5448	4.2558	6.4726	8.0884	11.7328
920	4.1848	6.3655	7.9554	11.5428	4.2541	6.4702	8.0854	11.7287
940	4.1840	6.3643	7.9539	11.5407	4.2525	6.4678	8.0825	11.7247
960	4.1832	6.3631	7.9524	11.5388	4.2510	6.4655	8.0797	11.7208
980	4.1824	6.3619	7.9510	11.5369	4.2495	6.4633	8.0770	11.7170
1000	4.1817	6.3608	7.9497	11.5350	4.2480	6.4611	8.0743	11.7134
∞	4.1084	6.2514	7.8147	11.3449	4.1084	6.2514	7.8147	11.3449

Table 4.4b (Continued)

Approximate values of $K^{(3)}$ needed to make the region (4.7.5) with k = 3, β-content at confidence level γ when sampling from the trivariate normal

	$\gamma = .95$				$\gamma = .99$			
n \ β	.75	.90	.95	.99	.75	.90	.95	.99
100	4.7138	7.1297	8.8744	12.7686	4.9866	7.5198	9.3390	13.3740
120	4.6570	7.0513	8.7834	12.6557	4.9027	7.4055	9.2073	13.2138
140	4.6133	6.9905	8.7122	12.5663	4.8384	7.3168	9.1043	13.0865
160	4.5784	6.9416	8.6546	12.4931	4.7872	7.2454	9.0209	12.9822
180	4.5498	6.9011	8.6067	12.4317	4.7451	7.1864	8.9517	12.8946
200	4.5257	6.8669	8.5662	12.3793	4.7098	7.1366	8.8930	12.8198
220	4.5051	6.8375	8.5312	12.3338	4.6796	7.0939	8.8424	12.7549
240	4.4872	6.8119	8.5007	12.2939	4.6535	7.0567	8.7982	12.6979
260	4.4715	6.7894	8.4737	12.2585	4.6306	7.0239	8.7593	12.6473
280	4.4576	6.7694	8.4497	12.2268	4.6103	6.9948	8.7245	12.6020
300	4.4451	6.7514	8.4281	12.1982	4.5921	6.9687	8.6933	12.5611
320	4.4338	6.7351	8.4086	12.1722	4.5757	6.9451	8.6651	12.5240
340	4.4236	6.7203	8.3908	12.1485	4.5609	6.9237	8.6394	12.4901
360	4.4143	6.7068	8.3745	12.1267	4.5473	6.9041	8.6158	12.4589
380	4.4058	6.6944	8.3595	12.1066	4.5349	6.8861	8.5942	12.4362
400	4.3979	6.6829	8.3456	12.0880	4.5235	6.8695	8.5742	12.4056
420	4.3906	6.6723	8.3327	12.0707	4.5129	6.8541	8.5556	12.3788
440	4.3838	6.6624	8.3208	12.0545	4.5030	6.8398	8.5384	12.3557
460	4.3775	6.6532	8.3096	12.0394	4.4939	6.8265	8.5223	12.3341
480	4.3716	6.6445	8.2991	12.0252	4.4853	6.8140	8.5072	12.3138
500	4.3660	6.6364	8.2893	12.0119	4.4773	6.8022	8.4930	12.2948
520	4.3608	6.6288	8.2800	11.9993	4.4697	6.7912	8.4796	12.2768
540	4.3559	6.6216	8.2713	11.9874	4.4626	6.7808	8.4670	12.2598
560	4.3513	6.6148	8.2630	11.9762	4.4559	6.7710	8.4551	12.2437
580	4.3469	6.6083	8.2551	11.9655	4.4496	6.7617	8.4438	12.2284
600	4.3427	6.6022	8.2477	11.9553	4.4436	6.7529	8.4331	12.2138
620	4.3388	6.5964	8.2406	11.9456	4.4378	6.7445	8.4229	12.2000
640	4.3350	6.5908	8.2339	11.9364	4.4324	6.7365	8.4132	12.1868
660	4.3314	6.5855	8.2274	11.9276	4.4272	6.7289	8.4039	12.1742
680	4.3280	6.5805	8.2213	11.9191	4.4223	6.7216	8.3951	12.1622
700	4.3247	6.5757	8.2154	11.9111	4.4176	6.7146	8.3866	12.1506
720	4.3216	6.5711	8.2097	11.9033	4.4130	6.7080	8.3785	12.1396
740	4.3186	6.5666	8.2043	11.8959	4.4087	6.7016	8.3707	12.1289
760	4.3157	6.5624	8.1991	11.8887	4.4045	6.6955	8.3633	12.1187
780	4.3129	6.5583	8.1941	11.8819	4.4006	6.6896	8.3561	12.1089
800	4.3103	6.5544	8.1893	11.8573	4.3967	6.6839	8.3492	12.0995
820	4.3077	6.5506	8.1847	11.8689	4.3930	6.6785	8.3425	12.0904
840	4.3052	6.5469	8.1803	11.8627	4.3895	6.6732	8.3361	12.0816
860	4.3029	6.5434	8.1759	11.8568	4.3861	6.6682	8.3299	12.0731
880	4.3006	6.5400	8.1718	11.8511	4.3827	6.6633	8.3240	12.0649
900	4.2983	6.5367	8.1678	11.8455	4.3796	6.6586	8.3182	12.0570
920	4.2962	6.5336	8.1639	11.8401	4.3765	6.6540	8.3126	12.0493
940	4.2941	6.5305	8.1601	11.8349	4.3735	6.6496	8.3072	12.0419
960	4.2921	6.5275	8.1565	11.8299	4.3706	6.6453	8.3020	12.0347
980	4.2902	6.5246	8.1530	11.8250	4.3678	6.6412	8.2970	12.0277
1000	4.2883	6.5218	8.1495	11.8203	4.3651	6.6371	8.2920	12.0209
∞	4.1084	6.2514	7.8147	11.3449	4.1084	6.2514	7.8147	11.3449

Table 4.4c

Approximate values of $K^{(4)}$ needed to make the region (4.25), with $k = 4$, β-content at confidence level γ, when sampling from the quadrinormal

n \ β	γ = .75 .75	γ = .75 .90	γ = .75 .95	γ = .75 .99	γ = .90 .75	γ = .90 .90	γ = .90 .95	γ = .90 .99
100	5.6639	8.1577	9.9296	13.8403	5.9131	8.5070	10.3460	14.3949
120	5.6380	8.1245	9.8924	13.7972	5.8641	8.4426	10.2726	14.3062
140	5.6180	8.0987	9.8632	13.7630	5.8264	8.3927	10.2153	14.2359
160	5.6021	8.0778	9.8396	13.7348	5.7962	8.3524	10.1689	14.1784
180	5.5889	8.0606	9.8200	13.7111	5.7714	8.3191	10.1303	14.1303
200	5.5779	8.0460	9.8033	13.6908	5.7506	8.2909	10.0976	14.0892
220	5.5685	8.0335	9.7889	13.6732	5.7327	8.2667	10.0695	14.0535
240	5.5603	8.0226	9.7763	13.6577	5.7172	8.2456	10.0448	14.0222
260	5.5531	8.0130	9.7652	13.6439	5.7035	8.2270	10.0231	13.9945
280	5.5467	8.0044	9.7553	13.6315	5.6914	8.2105	10.0037	13.9696
300	5.5409	7.9967	9.7464	13.6203	5.6806	8.1956	9.9862	13.9472
320	5.5358	7.9897	9.7383	13.6102	5.6708	8.1822	9.9705	13.9269
340	5.5311	7.9834	9.7309	13.6009	5.6619	8.1699	9.9561	13.9083
360	5.5268	7.9776	9.7241	13.5923	5.6538	8.1588	9.9429	13.8912
380	5.5229	7.9722	9.7179	13.5844	5.6463	8.1485	9.9308	13.8754
400	5.5192	7.9673	9.7122	13.5771	5.6395	8.1390	9.9196	13.8608
420	5.5159	7.9627	9.7068	13.5703	5.6331	8.1302	9.9092	13.8473
440	5.5128	7.9585	9.7018	13.5639	5.6272	8.1220	9.8995	13.8346
460	5.5098	7.9545	9.6972	13.5579	5.6217	8.1144	9.8904	13.8228
480	5.5071	7.9508	9.6928	13.5523	5.6165	8.1072	9.8820	13.8116
500	5.5046	7.9473	9.6887	13.5470	5.6117	8.1005	9.8740	13.8012
520	5.5022	7.9440	9.6849	13.5420	5.6071	8.0942	9.8665	13.7913
540	5.4999	7.9409	9.6812	13.5373	5.6028	8.0882	9.8594	13.7820
560	5.4978	7.9380	9.6777	13.5329	5.5988	8.0825	9.8527	13.7732
580	5.4957	7.9352	9.6745	13.5286	5.5950	8.0772	9.8464	13.7648
600	5.4938	7.9325	9.6714	13.5246	5.5913	8.0721	9.8403	13.7568
620	5.4920	7.9300	9.6684	13.5207	5.5879	8.0673	9.8346	13.7492
640	5.4903	7.9276	9.6656	13.5171	5.5846	8.0627	9.8291	13.7420
660	5.4886	7.9254	9.6629	13.5136	5.5814	8.0583	9.8239	13.7351
680	5.4870	7.9232	9.6603	13.5102	5.5784	8.0541	9.8189	13.7284
700	5.4855	7.9211	9.6578	13.5070	5.5756	8.0501	9.8141	13.7221
720	5.4841	7.9191	9.6555	13.5039	5.5728	8.0463	9.8096	13.7160
740	5.4827	7.9172	9.6532	13.5009	5.5702	8.0426	9.8052	13.7102
760	5.4814	7.9153	9.6510	13.4981	5.5677	8.0391	9.8010	13.7046
780	5.4801	7.9136	9.6490	13.4953	5.5653	8.0357	9.7969	13.6992
800	5.4789	7.9119	9.6469	13.4927	5.5629	8.0324	9.7930	13.6940
820	5.4777	7.9102	9.6450	13.4902	5.5607	8.0293	9.7893	13.6890
840	5.4765	7.9087	9.6431	13.4877	5.5585	8.0262	9.7857	13.6842
860	5.4754	7.9071	9.6413	13.4853	5.5564	8.0233	9.7822	13.6795
880	5.4744	7.9057	9.6396	13.4830	5.5544	8.0205	9.7788	13.6750
900	5.4734	7.9042	9.6379	13.4808	5.5525	8.0178	9.7755	13.6707
920	5.4724	7.9029	9.6363	13.4787	5.5506	8.0151	9.7724	13.6664
940	5.4714	7.9015	9.6347	13.4766	5.5488	8.0126	9.7693	13.6624
960	5.4705	7.9002	9.6332	13.4746	5.5470	8.0101	9.7664	13.6584
980	5.4696	7.8990	9.6317	13.4726	5.5453	8.0077	9.7635	13.6546
1000	5.4687	7.8978	9.6302	13.4707	5.5437	8.0054	9.7607	13.6509
∞	5.3853	7.7794	9.4877	13.2767	5.3853	7.7794	9.4877	13.2767

Table 4.4c (Continued)

Approximate values of $K^{(4)}$ needed to make the region (4.25), with $k = 4$, β-content at confidence level γ, when sampling from the quadrinormal

n \ β	γ = .95 .75	.90	.95	.99	γ = .99 .75	.90	.95	.99
100	6.0675	8.7208	10.5983	14.7233	6.3679	9.1300	11.0753	15.3278
120	6.0039	8.6372	10.5030	14.6081	6.2753	9.0094	10.9391	15.1658
140	5.9550	8.5723	10.4286	14.5169	6.2041	8.9158	10.8325	15.0371
160	5.9159	8.5200	10.3683	14.4423	6.1472	8.8403	10.7461	14.9317
180	5.8837	8.4767	10.3183	14.3798	6.1005	8.7779	10.6743	14.8433
200	5.8567	8.4402	10.2758	14.3265	6.0612	8.7252	10.6134	14.7677
220	5.8335	8.4088	10.2392	14.2802	6.0277	8.6799	10.5609	14.7021
240	5.8134	8.3814	10.2073	14.2397	5.9985	8.6404	10.5150	14.6445
260	5.7957	8.3573	10.1790	14.2036	5.9730	8.6057	10.4745	14.5934
280	5.7800	8.3358	10.1539	14.1714	5.9503	8.5747	10.4384	14.5477
300	5.7660	8.3165	10.1313	14.1423	5.9300	8.5470	10.4060	14.5064
320	5.7533	8.2991	10.1108	14.1160	5.9117	8.5219	10.3766	14.4689
340	5.7418	8.2833	10.0922	14.0919	5.8951	8.4991	10.3499	14.4346
360	5.7313	8.2688	10.0751	14.0697	5.8800	8.4783	10.3254	14.4032
380	5.7217	8.2555	10.0594	14.0493	5.8661	8.4591	10.3029	14.3741
400	5.7128	8.2432	10.0449	14.0304	5.8533	8.4415	10.2821	14.3473
420	5.7046	8.2318	10.0314	14.0128	5.8414	8.4251	10.2628	14.3223
440	5.6970	8.2212	10.0188	13.9964	5.8304	8.4098	10.2448	14.2990
460	5.6898	8.2113	10.0071	13.9811	5.8201	8.3956	10.2280	14.2771
480	5.6832	8.2020	9.9961	13.9667	5.8105	8.3823	10.2123	14.2567
500	5.6769	8.1933	9.9858	13.9531	5.8015	8.3698	10.1975	14.2374
520	5.6710	8.1852	9.9761	13.9404	5.7931	8.3581	10.1836	14.2192
540	5.6655	8.1774	9.9669	13.9283	5.7851	8.3470	10.1705	14.2020
560	5.6603	8.1701	9.9583	13.9169	5.7776	8.3365	10.1580	14.1858
580	5.6553	8.1632	9.9500	13.9060	5.7704	8.3266	10.1463	14.1703
600	5.6506	8.1566	9.9422	13.8957	5.7637	8.3171	10.1351	14.1557
620	5.6461	8.1504	9.9348	13.8859	5.7573	8.3082	10.1245	14.1417
640	5.6419	8.1445	9.9277	13.8765	5.7512	8.2997	10.1143	14.1284
660	5.6378	8.1388	9.9210	13.8675	5.7453	8.2915	10.1047	14.1156
680	5.6340	8.1334	9.9145	13.8590	5.7398	8.2838	10.0954	14.1035
700	5.6303	8.1282	9.9084	13.8508	5.7345	8.2763	10.0866	14.0918
720	5.6267	8.1232	9.9025	13.8429	5.7294	8.2692	10.0781	14.0806
740	5.6233	8.1185	9.8968	13.8354	5.7245	8.2624	10.0700	14.0699
760	5.6201	8.1139	9.8913	13.8281	5.7199	8.2559	10.0622	14.0596
780	5.6169	8.1095	9.8861	13.8212	5.7154	8.2496	10.0548	14.0497
800	5.6139	8.1053	9.8811	13.8145	5.7111	8.2435	10.0475	14.0401
820	5.6111	8.1013	9.8762	13.8080	5.7069	8.2377	10.0406	14.0309
840	5.6083	8.0973	9.8715	13.8018	5.7029	8.2321	10.0339	14.0220
860	5.6056	8.0936	9.8670	13.7957	5.6991	8.2267	10.0275	14.0135
880	5.6030	8.0899	9.8627	13.7899	5.6953	8.2215	10.0212	14.0052
900	5.6005	8.0864	9.8585	13.7843	5.6918	8.2164	10.0152	13.9972
920	5.5981	8.0830	9.8544	13.7788	5.6883	8.2116	10.0094	13.9894
940	5.5957	8.0797	9.8504	13.7736	5.6849	8.2068	10.0038	13.9819
960	5.5934	8.0765	9.8466	13.7684	5.6817	8.2023	9.9983	13.9747
980	5.5913	8.0734	9.8429	13.7635	5.6785	8.1978	9.9930	13.9676
1000	5.5891	8.0704	9.8393	13.7587	5.6755	8.1935	9.9879	13.9608
∞	5.3853	7.7794	9.4877	13.2767	5.3853	7.7794	9.4877	13.2767

Table 4.5a

Approximate values of the confidence level γ that the β-expectation tolerance region (4.25), k = 2, has coverage that exceeds β

n \ β	.75	.90	.95	.99
100	.5122	.5258	.5341	.5471
101	.5121	.5256	.5339	.5470
102	.5121	.5255	.5338	.5469
103	.5120	.5254	.5337	.5468
104	.5119	.5253	.5335	.5467
105	.5119	.5252	.5334	.5466
106	.5118	.5251	.5333	.5464
107	.5118	.5250	.5332	.5463
108	.5117	.5249	.5330	.5462
109	.5117	.5248	.5329	.5461
110	.5116	.5247	.5328	.5460
111	.5116	.5246	.5327	.5459
112	.5115	.5245	.5326	.5458
113	.5115	.5244	.5324	.5357
114	.5114	.5243	.5323	.5356
115	.5114	.5242	.5322	.5455
116	.5113	.5241	.5321	.5454
117	.5113	.5240	.5320	.5453
118	.5112	.5239	.5319	.5452
119	.5112	.5238	.5318	.5451
120	.5111	.5237	.5317	.5449
121	.5111	.5237	.5316	.5448
122	.5110	.5236	.5314	.5447
123	.5110	.5235	.5313	.5446
124	.5110	.5234	.5312	.5445
125	.5109	.5233	.5311	.5444
126	.5109	.5232	.5310	.5443
127	.5108	.5231	.5309	.5442
128	.5108	.5231	.5308	.5441
129	.5107	.5230	.5307	.5440
130	.5107	.5229	.5306	.5439
131	.5107	.5228	.5305	.5438
132	.5106	.5227	.5304	.5437
133	.5106	.5227	.5303	.5436
134	.5105	.5226	.5302	.5435
135	.5105	.5225	.5301	.5434
136	.5105	.5224	.5301	.5433
137	.5104	.5224	.5300	.5432
138	.5104	.5223	.5299	.5431
139	.5104	.5222	.5298	.5430
140	.5103	.5221	.5297	.5429

Table 4.5a (Continued)

n \ β	.75	.90	.95	.99
141	.5103	.5221	.5296	.5428
142	.5102	.5220	.5295	.5427
143	.5102	.5219	.5294	.5426
144	.5102	.5218	.5293	.5426
145	.5101	.5218	.5292	.5425
146	.5101	.5217	.5292	.5424
147	.5101	.5216	.5291	.5423
148	.5100	.5216	.5290	.5422
149	.5100	.5215	.5289	.5421
150	.5100	.5214	.5288	.5420
151	.5099	.5214	.5287	.5419
152	.5099	.5213	.5287	.5418
153	.5099	.5212	.5286	.5417
154	.5098	.5212	.5285	.5416
155	.5098	.5211	.5284	.5415
156	.5098	.5211	.5283	.5414
157	.5098	.5210	.5283	.5414
158	.5097	.5209	.5282	.5413
159	.5097	.5209	.5281	.5412
160	.5097	.5208	.5280	.5411
161	.5096	.5207	.5280	.5410
162	.5096	.5207	.5279	.5409
163	.5096	.5206	.5278	.5408
164	.5095	.5206	.5277	.5408
165	.5095	.5205	.5277	.5407
166	.5095	.5205	.5276	.5406
167	.5095	.5204	.5275	.5405
168	.5094	.5203	.5274	.5404
169	.5094	.5203	.5274	.5403
170	.5094	.5202	.5273	.5402
171	.5094	.5202	.5272	.5402
172	.5093	.5201	.5272	.5401
173	.5093	.5201	.5271	.5400
174	.5093	.5200	.5270	.5399
175	.5092	.5200	.5270	.5398
176	.5092	.5199	.5269	.5398
177	.5092	.5198	.5268	.5397
178	.5092	.5198	.5268	.5396
179	.5091	.5197	.5267	.5395
180	.5091	.5197	.5266	.5394
181	.5091	.5196	.5266	.5394
182	.5091	.5196	.5265	.5393
183	.5090	.5195	.5264	.5392

Table 4.5a (Continued)

β / n	.75	.90	.95	.99
184	.5090	.5195	.5264	.5391
185	.5090	.5194	.5263	.5391
186	.5090	.5194	.5262	.5390
187	.5089	.5193	.5262	.5389
188	.5089	.5193	.5261	.5388
189	.5089	.5192	.5260	.5387
190	.5089	.5192	.5260	.5387
191	.5089	.5192	.5259	.5486
192	.5088	.5191	.5259	.5485
193	.5088	.5191	.5258	.5385
194	.5088	.5190	.5257	.5384
195	.5088	.5190	.5257	.5383
196	.5087	.5189	.5256	.5382
197	.5087	.5189	.5256	.5382
198	.5087	.5188	.5255	.5381
199	.5087	.5188	.5255	.5380
200	.5087	.5187	.5254	.5379
210	.5084	.5183	.5248	.5372
220	.5083	.5179	.5243	.5366
230	.5081	.5175	.5238	.5360
240	.5079	.5172	.5234	.5354
250	.5077	.5169	.5230	.5348
260	.5076	.5165	.5225	.5343
270	.5075	.5162	.5222	.5337
280	.5073	.5160	.5218	.5332
290	.5072	.5157	.5214	.5328
300	.5071	.5154	.5211	.5323
320	.5069	.5150	.5205	.5314
340	.5067	.5145	.5199	.5306
360	.5065	.5141	.5194	.5299
380	.5063	.5138	.5189	.5292
400	.5061	.5134	.5184	.5286
420	.5060	.5131	.5180	.5279
440	.5059	.5128	.5176	.5274
460	.5057	.5126	.5172	.5268
480	.5056	.5123	.5169	.5263
500	.5055	.5120	.5166	.5259
550	.5052	.5115	.5158	.5248
600	.5050	.5110	.5152	.5238
650	.5048	.5106	.5146	.5229
700	.5046	.5102	.5141	.5222
750	.5045	.5099	.5136	.5214
800	.5043	.5096	.5132	.5208
850	.5042	.5093	.5128	.5202
900	.5041	.5090	.5125	.5197
950	.5040	.5088	.5121	.5192
1000	.5039	.5086	.5118	.5187

Table 4. 5b

Approximate values of the confidence level γ that the β-expectation tolerance region (4.25), k = 3, has coverage that exceeds β

n \ β	.75	.90	.95	.99
100	.5120	.5240	.5306	.5389
101	.5120	.5239	.5305	.5389
102	.5119	.5238	.5304	.5388
103	.5119	.5237	.5303	.5388
104	.5118	.5236	.5302	.5388
105	.5118	.5236	.5301	.5387
106	.5117	.5235	.5300	.5387
107	.5117	.5234	.5299	.5386
108	.5116	.5233	.5298	.5386
109	.5116	.5232	.5297	.5385
110	.5115	.5231	.5296	.5385
111	.5115	.5230	.5295	.5384
112	.5114	.5229	.5294	.5384
113	.5114	.5229	.5294	.5383
114	.5113	.5228	.5293	.5383
115	.5113	.5227	.5292	.5382
116	.5112	.5226	.5291	.5382
117	.5112	.5225	.5290	.5381
118	.5111	.5225	.5289	.5381
119	.5111	.5224	.5288	.5380
120	.5110	.5223	.5287	.5380
121	.5110	.5222	.5287	.5379
122	.5110	.5222	.5286	.5379
123	.5109	.5221	.5285	.5378
124	.5109	.5220	.5284	.5377
125	.5108	.5219	.5283	.5377
126	.5108	.5219	.5282	.5376
127	.5107	.5218	.5282	.5376
128	.5107	.5217	.5281	.5375
129	.5107	.5216	.5280	.5375
130	.5106	.5216	.5279	.5374
131	.5106	.5215	.5278	.5373
132	.5106	.5214	.5278	.5373
133	.5105	.5214	.5277	.5372
134	.5105	.5213	.5276	.5372
135	.5104	.5212	.5275	.5371
136	.5104	.5212	.5275	.5370
137	.5104	.5211	.5274	.5370
138	.5103	.5210	.5273	.5369
139	.5103	.5210	.5272	.5369
140	.5103	.5209	.5272	.5368
141	.5102	.5208	.5271	.5367
142	.5102	.5208	.5270	.5367
143	.5102	.5207	.5269	.5366
144	.5101	.5206	.5269	.5366

Table 4.5b (Continued)

n \ β	.75	.90	.95	.99
145	.5101	.5206	.5268	.5365
146	.5101	.5205	.5267	.5365
147	.5100	.5205	.5267	.5364
148	.5100	.5204	.5266	.5363
149	.5100	.5203	.5265	.5363
150	.5099	.5203	.5264	.5362
151	.5099	.5202	.5264	.5362
152	.5099	.5202	.5263	.5361
153	.5098	.5201	.5262	.5360
154	.5098	.5201	.5262	.5360
155	.5098	.5200	.5261	.5359
156	.5097	.5199	.5260	.5359
157	.5097	.5199	.5260	.5358
158	.5097	.5198	.5259	.5357
159	.5096	.5198	.5259	.5357
160	.5096	.5197	.5258	.5356
161	.5096	.5197	.5257	.5356
162	.5096	.5196	.5257	.5355
163	.5095	.5196	.5256	.5354
164	.5095	.5195	.5255	.5354
165	.5095	.5195	.5255	.5353
166	.5094	.5194	.5254	.5353
167	.5094	.5194	.5254	.5352
168	.5094	.5193	.5253	.5352
169	.5094	.5193	.5252	.5351
170	.5093	.5192	.5252	.5350
171	.5093	.5192	.5251	.5350
172	.5093	.5191	.5251	.5349
173	.5093	.5191	.5250	.5349
174	.5092	.5190	.5249	.5348
175	.5092	.5190	.5249	.5348
176	.5092	.5189	.5248	.5347
177	.5092	.5189	.5248	.5346
178	.5091	.5188	.5247	.5346
179	.5091	.5188	.5247	.5345
180	.5091	.5187	.5246	.5345
181	.5091	.5187	.5245	.5344
182	.5090	.5186	.5245	.5344
183	.5090	.5186	.5244	.5343
184	.5090	.5185	.5244	.5343
185	.5090	.5185	.5243	.5342
186	.5089	.5184	.5243	.5341
187	.5089	.5184	.5242	.5341
188	.5089	.5184	.5242	.5340

Table 4.5b (Continued)

n \ β	.75	.90	.95	.99
189	.5089	.5183	.5241	.5340
190	.5088	.5183	.5241	.5339
191	.5088	.5182	.5240	.5339
192	.5088	.5182	.5240	.5338
193	.5088	.5181	.5239	.5338
194	.5088	.5181	.5239	.5337
195	.5087	.5181	.5238	.5337
196	.5087	.5180	.5237	.5336
197	.5087	.5180	.5237	.5335
198	.5087	.5179	.5236	.5335
199	.5086	.5179	.5236	.5334
200	.5086	.5179	.5236	.5334
210	.5084	.5175	.5231	.5329
220	.5082	.5171	.5226	.5324
230	.5081	.5168	.5222	.5319
240	.5079	.5164	.5218	.5314
250	.5077	.5161	.5214	.5310
260	.5076	.5158	.5210	.5305
270	.5074	.5156	.5207	.5301
280	.5073	.5153	.5204	.5297
290	.5072	.5150	.5201	.5293
300	.5071	.5148	.5198	.5290
320	.5069	.5144	.5192	.5283
340	.5066	.5140	.5187	.5276
360	.5065	.5136	.5182	.5270
380	.5063	.5132	.5178	.5264
400	.5061	.5129	.5173	.5258
420	.5060	.5126	.5170	.5253
440	.5059	.5123	.5166	.5248
460	.5057	.5121	.5163	.5244
480	.5056	.5118	.5159	.5239
500	.5055	.5116	.5156	.5235
550	.5052	.5111	.5149	.5226
600	.5050	.5106	.5143	.5217
650	.5048	.5102	.5138	.5210
700	.5047	.5099	.5133	.5203
750	.5045	.5095	.5129	.5197
800	.5044	.5092	.5125	.5191
850	.5042	.5090	.5121	.5186
900	.5041	.5087	.5118	.5181
950	.5040	.5085	.5115	.5176
1000	.5039	.5083	.5112	.5172

Table 4.5c

Approximate values of the confidence level γ that the β-expectation tolerance region (4.25), k = 4, has coverage that exceeds β

n \ β	.75	.90	.95	.99
100	.5118	.5225	.5277	.5327
101	.5118	.5224	.5276	.5327
102	.5117	.5223	.5275	.5327
103	.5117	.5222	.5275	.5327
104	.5116	.5222	.5274	.5327
105	.5116	.5221	.5273	.5328
106	.5115	.5220	.5273	.5328
107	.5115	.5219	.5272	.5328
108	.5114	.5219	.5271	.5328
109	.5114	.5218	.5271	.5328
110	.5113	.5217	.5270	.5328
111	.5113	.5217	.5269	.5328
112	.5112	.5216	.5269	.5328
113	.5112	.5215	.5268	.5328
114	.5111	.5214	.5267	.5327
115	.5111	.5214	.5267	.5327
116	.5110	.5213	.5266	.5327
117	.5110	.5212	.5265	.5327
118	.5110	.5212	.5265	.5327
119	.5109	.5211	.5264	.5327
120	.5109	.5210	.5263	.5327
121	.5108	.5210	.5263	.5327
122	.5108	.5209	.5262	.5326
123	.5107	.5208	.5261	.5326
124	.5107	.5208	.5261	.5326
125	.5107	.5207	.5260	.5326
126	.5106	.5207	.5260	.5326
127	.5106	.5206	.5259	.5325
128	.5106	.5205	.5258	.5325
129	.5105	.5205	.5258	.5325
130	.5105	.5204	.5257	.5325
131	.5104	.5203	.5256	.5324
132	.5104	.5203	.5256	.5324
133	.5104	.5202	.5255	.5324
134	.5103	.5202	.5255	.5324
135	.5103	.5201	.5254	.5323
136	.5103	.5201	.5253	.5323
137	.5102	.5200	.5253	.5323
138	.5102	.5199	.5252	.5323
139	.5102	.5199	.5252	.5322
140	.5101	.5198	.5251	.5322
141	.5101	.5198	.5251	.5322
142	.5101	.5197	.5250	.5321
143	.5100	.5197	.5249	.5321
144	.5100	.5196	.5249	.5321

n \ β	.75	.90	.95	.99
145	.5100	.5196	.5248	.5320
146	.5099	.5195	.5248	.5320
147	.5099	.5194	.5247	.5320
148	.5099	.5194	.5247	.5319
149	.5098	.5193	.5246	.5319
150	.5098	.5193	.5245	.5319
151	.5098	.5192	.5245	.5318
152	.5097	.5192	.5244	.5318
153	.5097	.5191	.5244	.5318
154	.5097	.5191	.5243	.5317
155	.5096	.5190	.5243	.5317
156	.5096	.5190	.5242	.5317
157	.5096	.5189	.5242	.5316
158	.5096	.5189	.5241	.5316
159	.5095	.5188	.5241	.5316
160	.5095	.5188	.5240	.5315
161	.5095	.5187	.5239	.5315
162	.5094	.5187	.5239	.5315
163	.5094	.5186	.5238	.5314
164	.5094	.5186	.5238	.5314
165	.5094	.5186	.5237	.5313
166	.5093	.5185	.5237	.5313
167	.5093	.5185	.5236	.5313
168	.5093	.5184	.5236	.5312
169	.5093	.5184	.5235	.5312
170	.5092	.5183	.5235	.5312
171	.5092	.5183	.5234	.5311
172	.5092	.5182	.5234	.5311
173	.5092	.5182	.5233	.5311
174	.5091	.5182	.5233	.5310
175	.5091	.5181	.5232	.5310
176	.5091	.5181	.5232	.5309
177	.5091	.5180	.5231	.5309
178	.5090	.5180	.5231	.5309
179	.5090	.5179	.5230	.5308
180	.5090	.5179	.5230	.5308
181	.5090	.5179	.5230	.5308
182	.5089	.5178	.5229	.5307
183	.5089	.5178	.5229	.5307
184	.5089	.5177	.5228	.5306
185	.5089	.5177	.5228	.5306
186	.5088	.5176	.5227	.5306
187	.5088	.5176	.5227	.5305
188	.5088	.5176	.5226	.5305
189	.5088	.5175	.5226	.5304
190	.5088	.5175	.5225	.5304

Table 4. 5c (Continued)

n \ β	.75	.90	.95	.99
191	.5087	.5175	.5225	.5304
192	.5087	.5174	.5225	.5303
193	.5087	.5174	.5224	.5303
194	.5087	.5173	.5224	.5303
195	.5086	.5173	.5223	.5302
196	.5086	.5173	.5223	.5302
197	.5086	.5172	.5222	.5301
198	.5086	.5172	.5222	.5301
199	.5086	.5171	.5221	.5301
200	.5085	.5171	.5221	.5300
210	.5083	.5168	.5217	.5297
220	.5082	.5164	.5213	.5293
230	.5080	.5161	.5209	.5289
240	.5078	.5158	.5206	.5286
250	.5077	.5155	.5202	.5282
260	.5075	.5153	.5199	.5279
270	.5074	.5150	.5196	.5275
280	.5073	.5148	.5193	.5272
290	.5071	.5145	.5190	.5269
300	.5070	.5143	.5188	.5266
320	.5068	.5139	.5182	.5260
340	.5066	.5135	.5178	.5254
360	.5064	.5132	.5173	.5249
380	.5063	.5128	.5169	.5244
400	.5061	.5125	.5165	.5240
420	.5060	.5122	.5162	.5235
440	.5058	.5120	.5159	.5231
460	.5057	.5117	.5155	.5227
480	.5056	.5115	.5152	.5223
500	.5055	.5113	.5150	.5219
550	.5052	.5108	.5143	.5211
600	.5050	.5103	.5138	.5203
650	.5048	.5100	.5133	.5197
700	.5046	.5096	.5128	.5190
750	.5045	.5093	.5124	.5185
800	.5043	.5090	.5120	.5180
850	.5042	.5087	.5117	.5175
900	.5041	.5085	.5114	.5170
950	.5040	.5083	.5111	.5166
1000	.5039	.5081	.5108	.5162

Table 4.6

Approximate relative efficiencies of the β-content, confidence level γ tolerance regions (4.25) with respect to the corresponding distribution-free regions.

k=2

n \ β	$\gamma = .75$				$\gamma = .90$			
	.75	.90	.95	.99	.75	.90	.95	.99
100	.6768	.6197	.6131		.6800	.6732	.6968	
120	.6570	.6037	.5629		.6859	.6279	.5837	
140	.6756	.5936	.5337	.2133	.6906	.6024	.5295	
200	.6597	.6095	.4924	.3179	.6752	.6030	.5400	
300	.6556	.6101	.5065	.2500	.6685	.6171	.5198	.3849
400	.6540	.5965	.4864	.2290	.6609	.6091	.4823	.2740
500	.6532	.6013	.4764	.2918	.6570	.6062	.4890	.3577
600	.6497	.5950	.4879	.2669	.6582	.5945	.4945	.2961
700	.6500	.5910	.4813	.2523	.6562	.5960	.4820	.2661
800	.6480	.5953	.4768	.2428	.6549	.5976	.4880	.2486
900	.6485	.5924	.4736	.2362	.6541	.5921	.4802	.2845
1000	.6471	.5903	.4812	.2314	.6514	.5942	.4857	.2674

n \ β	$\gamma = .95$				$\gamma = .99$			
	.75	.90	.95	.99	.75	.90	.95	.99
100	.7047	.7105	.5892		.7401	.6938		
120	.7045	.6398	.7458		.7270	.7120		
140	.6848	.6731	.6094		.7197	.7286		
200	.6806	.6360	.5745		.6955	.6360	.5712	
300	.6692	.6037	.5248	.3292	.6807	.6113	.5603	
400	.6652	.6126	.5115	.4775	.6762	.6081	.5161	
500	.6635	.6052	.5073	.3170	.6698	.6098	.5347	.5036
600	.6593	.6019	.5063	.3972	.6663	.5999	.5199	.6440
700	.6595	.6004	.4880	.3200	.6643	.6051	.5122	.3954
800	.6573	.5998	.4915	.2822	.6631	.6006	.5080	.4704
900	.6557	.5997	.4811	.3247	.6623	.5978	.5057	.3651
1000	.6545	.5936	.4853	.2945	.6597	.5960	.4906	.4173

500	.6508	.5414	.4081	.2322	.6510	.5455	.4209	.2904
600	.6477	.5359	.4180	.2123	.6529	.5350	.4254	.2398
700	.6484	.5324	.4124	.2005	.6515	.5365	.4145	.2150
800	.6465	.5364	.4085	.1929	.6507	.5380	.4195	.2005
900	.6473	.5339	.4057	.1875	.6501	.5332	.4127	.2291
1000	.6460	.5320	.4122	.1836	.6478	.5351	.4173	.2152

	$\gamma = .95$				$\gamma = .99$			
n \ β	.75	.90	.95	.99	.75	.90	.95	.99
100	.6834	.6393	.5171		.7111	.6310		
120	.6852	.5754	.6529		.7006	.6460		
140	.6675	.6053	.5324		.6953	.6600		
200	.6664	.5719	.4998		.6756	.5744	.5048	
300	.6581	.5430	.4549	.2741	.6648	.5511	.4912	
400	.6559	.5511	.4424	.3946	.6626	.5478	.4505	
500	.6555	.5446	.4382	.2606	.6579	.5492	.4654	.4244
600	.6521	.5417	.4370	.3253	.6556	.5402	.4516	.5394
700	.6530	.5404	.4208	.2613	.6545	.5448	.4443	.3295
800	.6513	.5400	.4235	.2299	.6540	.5407	.4401	.3905
900	.6502	.5400	.4144	.2639	.6539	.5382	.4377	.3021
1000	.6494	.5345	.4178	.2390	.6518	.5367	.4244	.3443

Suppose then, we are sampling on a k-dimensional random variable and that we wish to construct a tolerance region with ability to pick up the center $100\beta\%$ of the population being sampled and which is of β-expectation. If the functional form of the distribution of the population is unknown, we may construct a distribution-free tolerance region composed of $n-m+1$ "inner" blocks, that is, formed by discarding m outer blocks in the manner of Section 2, where now we impose the restriction

$$m = (n+1)(1-\beta)$$

or (4.31)

$$n-m+1 = (n+1)\beta$$

since we wish this region to be of β-expectation. (For example, if $k = 1$ we would use the tolerance interval

$$S(X_1, \ldots, X_n) = (X_{(r)}, X_{(n-r+1)}] \tag{4.32}$$

where $2r = m$ and m satisfying (4.31).)

Now the variance of the coverage C_{DF} of such regions is given by

$$\mathrm{Var}(C_{DF}) = \frac{(n-m+1)\, m}{(n+1)^2(n+2)} = \beta\,(1-\beta)\,/\,(n+2) \tag{4.33}$$

Now if the distribution function has known functional form, and if this is that of the k-variate normal, then we would construct a tolerance region of the form (4.25), with $K^{(k)}$ set equal to c_k, where c_k is given by

$$c_k = [(n-1)\,k/(n-k)]\,[1+n^{-1}]\,F_{k,\ n-k;\ 1-\beta} \tag{4.34}$$

Using Theorem 4.3, we have that the variance of the coverage C_N of (4.25) with $K^{(k)} = c_k$ is

$$\mathrm{Var}\,(C_N) = [c_k]^k\, e^{-c_k}\, [k\, 2^{k-1}\, \Gamma^2(\tfrac{k}{2})\, n]^{-1} \tag{4.35}$$

to terms of order $1/n$. Hence, the relative efficiency (large sample) of the β-expectation distribution-free regions with respect to the β-expectation region (4.25) with $K^{(k)} = c_k$ is the ratio of (4.33) to (4.35). We note then that the limiting relative efficiency is

$$\begin{aligned} \text{Lim. Rel. Eff.} &= \lim_{n\to\infty} \mathrm{Var}\,(C_N)/\mathrm{Var}\,(C_{DF}) \\ &= [\chi^2_{k;\ 1-\beta}]^k\, e^{-\chi^2_{k;\ 1-\beta}}\, [\beta(1-\beta)\, k2^{k-1}\, \Gamma^2(\tfrac{k}{2})]^{-1} \end{aligned} \tag{4.36}$$

Table 4.7 gives values of the limiting relative efficiency (4.36) for $\beta = .75,\ .90,\ .95,\ .975,\ .99$ and $.995$ for $k = 1,\ 2,\ 3$ and 4. The result (4.36) for $k = 1$ was obtained by Wilks (1941).

We now return to β-content tolerance regions and we suppose again that sampling is from an $N(\mu, \sigma^2)$ population, and that it is desired to construct a β-content (at confidence level γ) tolerance interval S which has ability to pick up the left hand sets A_L (see (3.13)) of the $N(\mu, \sigma^2)$

Table 4.7

Values of the limiting relative efficiency (4.36) of the β-expectation distribution-free tolerance regions with respect to the β-expectation tolerance regions (4.25)

k \ β	.75	.90	.95	.99
1	.5981	.6395	.5525	.2803
2	.6406	.5891	.4723	.2142
3	.6449	.5552	.4304	.1851
4	.6425	.5319	.4040	.1681

population being sampled. Intuitively, the results of Section 3 would lead us to consider intervals of the form

$$S(X_1, \ldots, X_n) = (-\infty,\ \bar{X} + K'\, V\,] \tag{4.37}$$

with $(\bar{X}, V^2) = (n^{-1} \sum_{i=1}^{n} X_i,\ (n-1) \sum_{i=1}^{n} (X_i - \bar{X})^2)$, and where we wish to select $K' = K'(n, \gamma, \beta)$ so that the interval S of form (4.37) is β-content at confidence level γ. Now although the distribution of the coverage C of S is very complicated -- C is of course given by

$$\begin{aligned} C &= (2\,\pi\,\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\bar{X}+K'V} \exp\{-(t-\mu)^2/2\sigma^2\}\ dt \\ &= \Phi\left(\frac{\bar{X}-\mu}{\sigma} + K'\,\frac{V}{\sigma}\right), \end{aligned} \tag{4.38}$$

we may determine the constant $K' = K'(n;\gamma, \beta)$ by the following theorem.

<u>Theorem 4.4.</u> If sampling from the $N(\mu, \sigma^2)$ population, and using the above notation, the coverage C (defined by (4.38))of the interval S given by (4.37) is such that

$$\Pr(C \geq \beta) = \Pr[\,T^*_{n-1}(\sqrt{n}\, z_{1-\beta}) \leq \sqrt{n}\, K'\,] \tag{4.39}$$

where $T^*_f(\delta)$ is the Non-Central Student-t variable with f degrees of freedom and non-centrality parameter δ, and where $z_{1-\beta}$ is the point exceeded with probability $1-\beta$ when using the $N(0,1)$ distribution.

Proof: It is evident from (4.38) that the distribution of C does not depend on μ and σ, and we write, for this proof only

$$C = \Phi(\bar{X} + K' V)$$

where $\bar{X} = N(0, n^{-1})$ and $V^2 = \chi^2_{n-1}/n-1$, $\bar{X}$ and V independent. Now the following events are easily seen to be equivalent:

$$C \geq \beta \quad ; \qquad \bar{X} + K' V \geq z_{1-\beta} \quad ;$$

$$\frac{\sqrt{n}\,(U + Z_{1-\beta})}{V} \leq \sqrt{n}\,K', \text{ where } U = N(0, n^{-1}), \tag{4.40}$$

$$T^*_{n-1}(\sqrt{n}\; z_{1-\beta}) \leq \sqrt{n}\; K'$$

that is, we have that

$$Pr(C \geq \beta) = Pr(T^*_{n-1}(\sqrt{n}\; z_{1-\beta}) \leq \sqrt{n}\; K') \tag{4.40a}$$

and the theorem is proved.

Owen (1963) has used the result of Theorem 4.4 to tabulate values of K' so that $\gamma = Pr(C \geq \beta)$, for preassigned γ, β and n. We reproduce his Table in 4.8. He also noted that if V^2 is any estimator of σ^2, independent of $\bar{X}$, and based on f degrees of freedom, then

$$Pr(C \geq \beta) = Pr(T^*_f(\sqrt{n}\; z_{1-\beta}) \leq \sqrt{n}\; K') = \gamma \tag{4.40b}$$

Table 4.8 *

Values of $K' = K'(n;\gamma,\beta)$ necessary to make $S = (-\infty, \overline{X}+K'V]$ β-content at confidence level γ

γ=.75				γ=.90				γ=.95				γ=.99			
β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99
2.225	3.992	5.122	7.267	5.842	10.253	13.090	18.500	11.763	20.581	26.260	37.094	58.939	103.029	131.426	185.617
1.464	2.501	3.152	4.396	2.603	4.258	5.311	7.340	3.806	6.155	7.656	10.553	8.728	13.995	17.370	23.896
1.255	2.134	2.681	3.726	1.972	3.188	3.957	5.438	2.618	4.162	5.144	7.042	4.715	7.380	9.083	12.387
1.152	1.962	2.463	3.421	1.698	2.742	3.400	4.666	2.150	3.407	4.203	5.741	3.454	5.362	6.578	8.939
1.088	1.859	2.336	3.244	1.540	2.494	3.092	4.243	1.895	3.006	3.708	5.062	2.848	4.411	5.406	7.335
1.043	1.790	2.250	3.126	1.435	2.333	2.894	3.972	1.732	2.755	3.399	4.642	2.491	3.859	4.723	6.412
1.010	1.740	2.188	3.042	1.360	2.219	2.754	3.783	1.618	2.582	3.187	4.354	2.253	3.497	4.285	5.812
.985	1.701	2.141	2.977	1.302	2.133	2.650	3.641	1.532	2.454	3.031	4.143	2.083	3.240	3.972	5.389
.964	1.671	2.104	2.927	1.257	2.066	2.568	3.532	1.465	2.355	2.911	3.981	1.954	3.048	3.738	5.074
.947	1.645	2.073	2.885	1.219	2.011	2.503	3.443	1.411	2.275	2.815	3.852	1.853	2.898	3.556	4.829
.932	1.624	2.048	2.851	1.188	1.966	2.448	3.371	1.366	2.210	2.736	3.747	1.771	2.777	3.410	4.633
.920	1.606	2.026	2.822	1.162	1.928	2.402	3.309	1.328	2.155	2.671	3.659	1.703	2.677	3.290	4.472
.909	1.591	2.007	2.797	1.139	1.895	2.363	3.257	1.296	2.109	2.614	3.585	1.645	2.593	3.189	4.337
.899	1.577	1.991	2.775	1.119	1.867	2.329	3.212	1.268	2.068	2.566	3.520	1.595	2.521	3.102	4.222
.891	1.565	1.976	2.756	1.101	1.842	2.299	3.172	1.243	2.033	2.524	3.464	1.552	2.459	3.028	4.123
.883	1.554	1.963	2.739	1.085	1.819	2.272	3.137	1.220	2.002	2.486	3.414	1.514	2.405	2.963	4.037
.876	1.545	1.952	2.723	1.071	1.800	2.249	3.105	1.201	1.974	2.453	3.370	1.481	2.357	2.905	3.960
.870	1.536	1.941	2.710	1.058	1.782	2.227	3.077	1.183	1.949	2.423	3.331	1.450	2.314	2.854	3.892
.864	1.528	1.932	2.697	1.046	1.765	2.208	3.052	1.166	1.926	2.396	3.295	1.423	2.276	2.808	3.832
.859	1.521	1.923	2.685	1.035	1.750	2.190	3.028	1.152	1.905	2.371	3.263	1.399	2.241	2.766	3.777
.854	1.514	1.915	2.675	1.025	1.737	2.174	3.007	1.138	1.886	2.349	3.233	1.376	2.209	2.729	3.727
.849	1.508	1.908	2.665	1.016	1.724	2.159	2.987	1.125	1.869	2.328	3.206	1.355	2.180	2.694	3.681
.845	1.502	1.901	2.656	1.007	1.712	2.145	2.969	1.114	1.853	2.309	3.181	1.336	2.154	2.662	3.640
.841	1.497	1.895	2.648	1.000	1.702	2.132	2.952	1.103	1.838	2.292	3.158	1.319	2.129	2.633	3.601
.838	1.492	1.889	2.640	.992	1.691	2.120	2.937	1.093	1.824	2.275	3.136	1.303	2.106	2.606	3.566
.834	1.487	1.883	2.633	.985	1.682	2.109	2.922	1.083	1.811	2.260	3.116	1.287	2.085	2.581	3.533
.831	1.483	1.878	2.626	.979	1.673	2.099	2.909	1.075	1.799	2.246	3.098	1.273	2.065	2.558	3.502
.828	1.478	1.873	2.620	.973	1.665	2.089	2.896	1.066	1.788	2.232	3.080	1.260	2.047	2.536	3.473
.825	1.475	1.869	2.614	.967	1.657	2.080	2.884	1.058	1.777	2.220	3.064	1.247	2.030	2.515	3.447
.822	1.471	1.864	2.608	.961	1.650	2.071	2.872	1.051	1.767	2.208	3.048	1.236	2.014	2.496	3.421
.820	1.467	1.860	2.602	.956	1.643	2.063	2.862	1.044	1.758	2.197	3.034	1.225	1.998	2.478	3.398
.817	1.464	1.856	2.597	.951	1.636	2.055	2.852	1.037	1.749	2.186	3.020	1.214	1.984	2.461	3.375
.815	1.461	1.353	2.593	.947	1.630	2.049	2.842	1.031	1.740	2.176	3.007	1.204	1.970	2.445	3.354
.813	1.458	1.849	2.588	.942	1.624	2.041	2.833	1.025	1.732	2.167	2.995	1.195	1.957	2.430	3.334

Table 4.3 (Continued)

	γ=.75			γ=.90				γ=.95				γ=.99			
=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99
.810	1.455	1.846	2.584	.938	1.618	2.034	2.824	1.019	1.725	2.159	2.983	1.186	1.945	2.415	3.315
.808	1.452	1.842	2.579	.934	1.613	2.028	2.816	1.014	1.717	2.149	2.972	1.177	1.934	2.402	3.297
.806	1.450	1.839	2.575	.930	1.608	2.022	2.808	1.009	1.710	2.141	2.961	1.169	1.922	2.389	3.280
.805	1.447	1.836	2.572	.926	1.603	2.016	2.800	1.004	1.704	2.133	2.951	1.161	1.912	2.376	3.264
.803	1.445	1.834	2.568	.923	1.598	2.010	2.793	.999	1.697	2.125	2.941	1.154	1.902	2.364	3.249
.801	1.443	1.831	2.564	.919	1.593	2.005	2.786	.994	1.691	2.118	2.932	1.147	1.892	2.353	3.234
.799	1.440	1.828	2.561	.916	1.589	2.000	2.780	.990	1.685	2.111	2.923	1.140	1.883	2.342	3.220
.798	1.438	1.826	2.558	.913	1.585	1.995	2.773	.986	1.680	2.105	2.914	1.133	1.874	2.331	3.206
.796	1.436	1.824	2.555	.910	1.581	1.990	2.767	.982	1.674	2.098	2.906	1.127	1.865	2.321	3.193
.795	1.434	1.821	2.552	.907	1.577	1.986	2.761	.978	1.669	2.092	2.898	1.121	1.857	2.312	3.180
.793	1.432	1.819	2.549	.904	1.573	1.981	2.756	.974	1.664	2.086	2.890	1.115	1.849	2.303	3.168
.792	1.431	1.817	2.546	.901	1.570	1.977	2.750	.971	1.659	2.081	2.883	1.110	1.842	2.294	3.157
.791	1.429	1.815	2.543	.899	1.566	1.973	2.745	.967	1.654	2.075	2.876	1.104	1.835	2.285	3.146
.789	1.427	1.813	2.541	.896	1.563	1.969	2.740	.964	1.650	2.070	2.869	1.099	1.828	2.277	3.135
.788	1.425	1.811	2.538	.894	1.559	1.965	2.735	.960	1.646	2.065	2.862	1.094	1.821	2.269	3.125
.787	1.424	1.809	2.536	.891	1.556	1.962	2.730	.957	1.641	2.060	2.856	1.089	1.814	2.261	3.115
.786	1.422	1.807	2.534	.889	1.553	1.958	2.726	.954	1.637	2.055	2.850	1.084	1.808	2.254	3.105
.785	1.421	1.805	2.531	.887	1.550	1.955	2.721	.951	1.633	2.051	2.844	1.080	1.802	2.247	3.096
.783	1.419	1.804	2.52	.884	1.547	1.951	2.717	.948	1.630	2.046	2.838	1.075	1.796	2.240	3.087
.782	1.418	1.802	2.527	.882	1.545	1.948	2.713	.945	1.626	2.042	2.833	1.071	1.790	2.233	3.078
.781	1.417	1.801	2.525	.880	1.542	1.945	2.709	.943	1.622	2.038	2.827	1.067	1.785	2.226	3.070
.780	1.415	1.799	2.523	.878	1.539	1.942	2.705	.940	1.619	2.034	2.822	1.063	1.779	2.220	3.061
.779	1.414	1.797	2.521	.876	1.537	1.939	2.701	.938	1.615	2.030	2.817	1.059	1.774	2.214	3.053
.778	1.413	1.796	2.519	.875	1.534	1.936	2.697	.935	1.612	2.026	2.812	1.055	1.769	2.208	3.046
.777	1.411	1.795	2.517	.873	1.532	1.933	2.694	.933	1.609	2.022	2.807	1.052	1.764	2.202	3.038
.777	1.410	1.793	2.515	.871	1.530	1.931	2.690	.930	1.606	2.019	2.802	1.048	1.759	2.197	3.031
.776	1.409	1.792	2.514	.869	1.527	1.928	2.687	.928	1.603	2.015	2.798	1.045	1.755	2.191	3.024
.775	1.408	1.791	2.512	.868	1.525	1.925	2.683	.926	1.600	2.012	2.793	1.041	1.750	2.186	3.017
.774	1.407	1.789	2.510	.866	1.523	1.923	2.680	.924	1.597	2.008	2.789	1.038	1.746	2.181	3.010
.773	1.406	1.788	2.509	.864	1.521	1.920	2.677	.921	1.594	2.005	2.785	1.035	1.741	2.176	3.004
.772	1.405	1.787	2.507	.863	1.519	1.918	2.674	.919	1.591	2.002	2.781	1.032	1.737	2.171	2.998
.772	1.404	1.786	2.506	.861	1.517	1.916	2.671	.917	1.589	1.999	2.777	1.028	1.733	2.166	2.991
.771	1.403	1.784	2.504	.860	1.515	1.913	2.668	.915	1.586	1.996	2.773	1.025	1.729	2.162	2.985
.770	1.402	1.783	2.503	.858	1.513	1.911	2.665	.913	1.584	1.993	2.769	1.023	1.725	2.157	2.980
.769	1.401	1.782	2.501	.857	1.511	1.909	2.662	.911	1.581	1.990	2.765	1.020	1.722	2.153	2.974

4.8 (Continued)

γ=.75				γ=.90				γ=.95				γ=.99			
β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99
.769	1.400	1.781	2.500	.855	1.509	1.907	2.660	.910	1.579	1.987	2.762	1.017	1.718	2.148	2.968
.768	1.399	1.780	2.498	.854	1.508	1.905	2.657	.908	1.576	1.984	2.758	1.014	1.714	2.144	2.963
.767	1.398	1.779	2.497	.853	1.506	1.903	2.654	.906	1.574	1.982	2.755	1.012	1.711	2.140	2.958
.767	1.397	1.778	2.496	.851	1.504	1.901	2.652	.904	1.572	1.979	2.751	1.009	1.707	2.136	2.952
.766	1.396	1.777	2.495	.850	1.503	1.899	2.649	.903	1.570	1.976	2.748	1.006	1.704	2.132	2.947
.765	1.396	1.776	2.493	.849	1.501	1.897	2.647	.901	1.568	1.974	2.745	1.004	1.701	2.128	2.942
.765	1.395	1.775	2.492	.848	1.499	1.895	2.644	.899	1.565	1.971	2.742	1.002	1.698	2.125	2.938
.764	1.394	1.774	2.491	.846	1.498	1.893	2.642	.898	1.563	1.969	2.739	.999	1.694	2.121	2.933
.763	1.393	1.773	2.490	.845	1.496	1.892	2.640	.896	1.561	1.967	2.736	.997	1.691	2.117	2.928
.763	1.392	1.772	2.489	.844	1.495	1.890	2.638	.895	1.599	1.964	2.733	.995	1.688	2.114	2.924
.762	1.392	1.771	2.487	.843	1.493	1.888	2.635	.893	1.557	1.962	2.730	.992	1.685	2.110	2.919
.762	1.391	1.771	2.486	.842	1.492	1.887	2.633	.892	1.556	1.960	2.727	.990	1.682	2.107	2.915
.761	1.390	1.770	2.485	.841	1.490	1.885	2.631	.890	1.554	1.958	2.724	.988	1.680	2.104	2.911
.761	1.390	1.769	2.484	.840	1.489	1.883	2.629	.889	1.552	1.956	2.721	.986	1.677	2.100	2.907
.760	1.389	1.768	2.483	.839	1.488	1.882	2.627	.888	1.550	1.954	2.719	.984	1.674	2.097	2.902
.759	1.388	1.767	2.482	.838	1.486	1.880	2.625	.886	1.548	1.952	2.716	.982	1.672	2.094	2.898
.759	1.387	1.767	2.481	.837	1.485	1.879	2.623	.885	1.547	1.950	2.714	.980	1.669	2.091	2.895
.758	1.387	1.766	2.480	.836	1.484	1.877	2.621	.884	1.545	1.948	2.711	.978	1.666	2.088	2.891
.758	1.386	1.765	2.479	.835	1.483	1.876	2.619	.882	1.543	1.946	2.709	.976	1.664	2.085	2.887
.757	1.386	1.764	2.478	.834	1.481	1.874	2.618	.881	1.542	1.944	2.706	.974	1.661	2.082	2.883
.757	1.385	1.764	2.477	.833	1.480	1.873	2.616	.880	1.540	1.942	2.704	.972	1.659	2.080	2.880
.756	1.384	1.763	2.476	.832	1.479	1.871	2.614	.879	1.538	1.940	2.701	.970	1.657	2.077	2.876
.756	1.384	1.762	2.475	.831	1.478	1.870	2.612	.877	1.537	1.938	2.699	.969	1.654	2.074	2.872
.756	1.383	1.762	2.475	.830	1.477	1.869	2.611	.876	1.535	1.937	2.697	.967	1.652	2.071	2.869
.755	1.383	1.761	2.474	.829	1.475	1.867	2.609	.875	1.534	1.935	2.695	.965	1.650	2.069	2.866
.755	1.382	1.760	2.473	.828	1.474	1.866	2.607	.874	1.532	1.933	2.692	.963	1.647	2.066	2.862
.754	1.381	1.760	2.472	.827	1.473	1.865	2.606	.873	1.531	1.931	2.690	.962	1.645	2.064	2.859
.754	1.381	1.759	2.471	.827	1.472	1.864	2.604	.872	1.530	1.930	2.688	.960	1.643	2.061	2.856
.753	1.380	1.758	2.470	.826	1.471	1.862	2.602	.871	1.528	1.928	2.686	.959	1.641	2.059	2.853
.753	1.380	1.758	2.470	.825	1.470	1.861	2.601	.870	1.527	1.927	2.684	.957	1.639	2.056	2.850
.749	1.375	1.752	2.462	.818	1.460	1.850	2.587	.860	1.514	1.912	2.665	.942	1.620	2.034	2.821
.746	1.371	1.747	2.456	.811	1.452	1.841	2.574	.851	1.503	1.899	2.649	.930	1.604	2.015	2.797
.743	1.367	1.743	2.450	.805	1.445	1.832	2.564	.844	1.494	1.888	2.635	.919	1.589	1.999	2.776
.740	1.364	1.739	2.445	.800	1.439	1.825	2.554	.837	1.485	1.879	2.622	.909	1.577	1.984	2.757
.738	1.361	1.735	2.441	.796	1.433	1.818	2.546	.832	1.478	1.870	2.611	.901	1.566	1.971	2.740
.736	1.358	1.732	2.437	.792	1.428	1.812	2.538	.826	1.471	1.862	2.601	.893	1.556	1.960	2.726
.734	1.355	1.730	2.434	.788	1.423	1.807	2.531	.822	1.465	1.855	2.592	.886	1.547	1.949	2.712
.732	1.353	1.727	2.430	.785	1.419	1.802	2.525	.817	1.459	1.849	2.584	.880	1.538	1.940	2.700

4.8 (Continued)

γ=.75				γ=.90				γ=.95				γ=.99			
β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99	β=.75	β=.90	β=.95	β=.99
.731	1.351	1.725	2.427	.782	1.415	1.797	2.519	.813	1.454	1.843	2.577	.874	1.531	1.931	2.689
.729	1.349	1.723	2.425	.779	1.411	1.793	2.514	.809	1.450	1.837	2.570	.868	1.524	1.923	2.679
.728	1.348	1.721	2.422	.776	1.408	1.789	2.509	.806	1.445	1.832	2.563	.863	1.518	1.916	2.669
.726	1.346	1.719	2.420	.774	1.405	1.786	2.505	.803	1.441	1.828	2.557	.859	1.512	1.909	2.660
.725	1.345	1.717	2.418	.772	1.402	1.783	2.501	.800	1.437	1.823	2.552	.854	1.506	1.902	2.652
.724	1.343	1.715	2.415	.769	1.399	1.780	2.497	.797	1.434	1.819	2.547	.850	1.501	1.896	2.645
.723	1.342	1.714	2.414	.767	1.397	1.777	2.493	.795	1.431	1.815	2.542	.847	1.496	1.891	2.638
.722	1.341	1.712	2.412	.766	1.394	1.774	2.489	.792	1.428	1.812	2.537	.843	1.492	1.886	2.631
.721	1.339	1.711	2.410	.764	1.392	1.771	2.486	.790	1.425	1.809	2.533	.840	1.487	1.881	2.625
.720	1.338	1.710	2.408	.762	1.390	1.769	2.483	.788	1.422	1.805	2.529	.837	1.483	1.876	2.619
.720	1.337	1.709	2.407	.761	1.388	1.767	2.480	.786	1.419	1.802	2.525	.834	1.480	1.872	2.613
.719	1.336	1.708	2.406	.759	1.386	1.765	2.477	.784	1.417	1.800	2.522	.831	1.476	1.868	2.608
.717	1.334	1.705	2.402	.756	1.382	1.760	2.471	.779	1.411	1.793	2.514	.824	1.468	1.858	2.596
.715	1.332	1.703	2.399	.753	1.378	1.755	2.466	.775	1.406	1.787	2.506	.819	1.461	1.850	2.585
.714	1.330	1.701	2.397	.750	1.375	1.751	2.461	.772	1.402	1.782	2.500	.813	1.454	1.842	2.576
.713	1.329	1.699	2.394	.747	1.372	1.748	2.456	.769	1.398	1.778	2.494	.809	1.448	1.836	2.567
.712	1.327	1.697	2.392	.745	1.369	1.745	2.452	.766	1.394	1.774	2.489	.805	1.443	1.830	2.560
.710	1.326	1.696	2.390	.743	1.366	1.742	2.448	.763	1.391	1.770	2.484	.801	1.438	1.824	2.553
.710	1.325	1.694	2.389	.741	1.364	1.739	2.445	.761	1.388	1.766	2.480	.797	1.434	1.819	2.546
.709	1.324	1.693	2.387	.740	1.362	1.736	2.442	.758	1.385	1.763	2.475	.794	1.430	1.814	2.540
.707	1.322	1.691	2.384	.736	1.358	1.732	2.436	.754	1.380	1.757	2.468	.788	1.422	1.806	2.530
.706	1.320	1.689	2.381	.734	1.355	1.728	2.431	.751	1.376	1.752	2.462	.783	1.416	1.799	2.520
.704	1.318	1.687	2.379	.731	1.352	1.725	2.427	.748	1.372	1.748	2.456	.779	1.411	1.792	2.512
.703	1.317	1.685	2.377	.729	1.349	1.722	2.423	.745	1.368	1.744	2.451	.775	1.406	1.787	2.505
.701	1.315	1.683	2.374	.726	1.344	1.717	2.417	.740	1.363	1.737	2.443	.768	1.397	1.777	2.493
.700	1.313	1.680	2.371	.723	1.341	1.712	2.411	.736	1.358	1.732	2.436	.763	1.390	1.769	2.483
.698	1.311	1.678	2.369	.720	1.338	1.709	2.407	.733	1.354	1.727	2.430	.758	1.385	1.762	2.475
.674	1.282	1.645	2.326	.674	1.282	1.645	2.326	.674	1.282	1.645	2.326	.674	1.282	1.645	2.326

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and has given tables of K' for preassigned γ, β and f for low n in Owen (1958) -- see also Owen (1962, 1963). Using these tables, then, we are now at the happy point of knowing how to construct tolerance intervals of the form (4.37), which have the ability to give us information about the left-hand $100\beta\%$ sets of the $N(\mu, \sigma^2)$ distribution being sampled, and are of β-content at confidence level γ.

Now the result of (4.39) of Theorem 4.4 may be utilized to answer the following question. Suppose a (one-sided) tolerance interval S of the form (4.37), is to be constructed so that it has the property of being β-expectation, that is, we set $K' = \sqrt{1+n^{-1}}\; t_{n-1;\,1-\beta}$. If we now fix a constant δ so that $0 < \delta < 1$, we have from the arguments given in Theorem 4.3 that the interval $S =$ $(-\infty,\ \bar{X} + \sqrt{1 + n^{-1}}\; t_{n-1;\,1-\beta}]$ has coverage C which is such that

$$\gamma = \Pr(C \geq \delta) = \Pr[T^*_{n-1}(\sqrt{n}\; z_{1-\delta}) \leq \sqrt{n+1}\; t_{n-1;\,1-\beta}] \tag{4.41}$$

that is, the β-expectation interval S is δ-content at confidence level γ, with γ given by (4.41). Table 4.9 gives values of γ for $\beta = \delta = .75$, .90, .95 and .99 for various values of n.

The "converse" question, namely, if S of the form (4.31) is β-content at confidence level γ, then what is the value of β', where $\beta' = E(C)$, is easily answered and in the same manner as (4.4a) we find that β' is given by the solution to the equation

Table 4.9

Values of the confidence level γ that the β-expectation tolerance region (4.37) has coverage that exceeds β (see(4.41))

n \ β	.75	.90	.95	.99
2	.5543	.7341	.8318	.9524
3	.5642	.6911	.7762	.9158
4	.5635	.6651	.7400	.8807
5	.5602	.6475	.7149	.8509
6	.5565	.6346	.6962	.8264
7	.5531	.6246	.6817	.8061
8	.5500	.6165	.6701	.7889
9	.5474	.6099	.6604	.7743
10	.5451	.6042	.6522	.7616
11	.5431	.5994	.6451	.7505
12	.5413	.5952	.6390	.7407
13	.5397	.5915	.6335	.7319
14	.5383	.5881	.6287	.7240
15	.5371	.5852	.6243	.7169
16	.5359	.5825	.6204	.7104
17	.5349	.5800	.6168	.7044
18	.5339	.5778	.6135	.6990
19	.5330	.5757	.6105	.6939
20	.5322	.5738	.6077	.6892
21	.5314	.5720	.6052	.6848
22	.5307	.5704	.6027	.6808
23	.5301	.5688	.6005	.6769
24	.5294	.5674	.5984	.6733
25	.5289	.5660	.5964	.6700
26	.5283	.5648	.5945	.6668
27	.5278	.5635	.5928	.6638
28	.5273	.5624	.5911	.6609
29	.5268	.5613	.5895	.6582
30	.5264	.5603	.5880	.6556
35	.5244	.5558	.5815	.6443
40	.5229	.5522	.5763	.6352
45	.5216	.5493	.5719	.6276
50	.5205	.5467	.5682	.6212
55	.5195	.5446	.5650	.6156
60	.5187	.5427	.5623	.6108
65	.5180	.5410	.5598	.6065
70	.5173	.5395	.5577	.6027
75	.5167	.5382	.5557	.5992
80	.5162	.5370	.5539	.5961
85	.5157	.5359	.5523	.5933
90	.5153	.5348	.5509	.5907
95	.5149	.5339	.5495	.5883

$$t_{n-1;\ 1-\beta'} = K'/(1 + n^{-1})^{\frac{1}{2}} \tag{4.42}$$

Another interesting point about the tolerance intervals S of the form (4.37) is contained in the following theorem, due to Wilks (1941).

Theorem 4.5. Suppose sampling from $N(\mu, \sigma^2)$, and that a tolerance interval S is constructed of the form (4.37). If C denotes the coverage of S, where C is given by (4.38), then, to terms of order $1/n$,

$$\mathrm{Var}(C) = \sigma_C^2 = \{[(K')^2 + 2]\exp[-(K')^2]\}/4\pi n \tag{4.43}$$

Proof. The proof is straightforward and proceeds as in Theorem 4.2 by expanding C in a Taylor Series and taking the required expectations to arrive at (4.43).

The result of Theorem 4.5 may be utilized in the same manner that the result (4.19) was to produce Table 4.2. Suppose then, that we are sampling from a population with continuous, but otherwise unknown, distribution function and that we wish to construct a tolerance interval with ability to give information about the left-hand $100\beta\%$ of the population. A reasonable interval is clearly

$$S_3(X_1, \ldots, X_n) = (-\infty,\ X_{(n-m+1)}] \tag{4.44}$$

that is, the interval composed of the $(n-m+1)$ left-hand most blocks $(X_{(i-1)}, X_{(i)}]$, $i = 1, \ldots, n-m+1$ so that the m blocks $(X_{(j-1)}, X_j]$, $j = n-m+2, \ldots, n+1$ have been excluded. Again, the variance of the coverage of (4.44), say C_3 is $\tau(1-\tau)/(n+2)$, where $\tau = (1-m/(n+1))$ -- see (4.18). If we wish S_3 to be of β-content at confidence level γ, then we must choose m suitably, perhaps with the aid of Table 2.1 etc. Now if we are told that we are sampling from a normal distribution, then we would construct the interval of form (4.37), say S_4, with K' chosen to make S_4 a β-content at confidence level γ interval. The coverage C_4 of this latter interval has variance for large n given by (4.43) of Theorem 4.5 so that the relative efficiency of S_3 to S_4 is given by

$$\text{Rel. Eff.} = \frac{n+2}{4\pi n} \{[(K')^2 + 2] \exp[-(K')^2]\} / \tau(1-\tau) \tag{4.45}$$

We give some values of (4.45) in Table 4.10 for selected n, β and γ.

To end this section we turn to the case which is of interest in life-testing applications and suppose that sampling is from the single exponential distribution whose density function is given by

$$\begin{cases} \sigma^{-1} \exp\{-x/\sigma\} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.46}$$

Suppose we wish to construct a tolerance interval of β-content at confidence level γ which has ability to pick up the right hand $100\beta\%$ set that is, the interval composed of the $(n-m+1)$ left-hand most blocks $(X_{(i-1)}, X_{(i)}]$, $i = 1, \ldots, n-m+1$ so that the m blocks $(X_{(j-1)}, X_j]$, $j = n-m+2, \ldots, n+1$ have been excluded. Again, the variance of the coverage of (4.44), say C_3 is $\tau(1-\tau)/(n+2)$, where $\tau = (1-m/(n+1))$ -- see (4.18). If we wish S_3 to be of β-content at confidence level γ, then we must choose m suitably, perhaps with the aid of Table 2.1 etc. Now if we are told that we are sampling from a normal distribution, then we would construct the interval of form (4.37), say S_4, with K' chosen to make S_4 a β-content at confidence level γ interval. The coverage C_4 of this latter interval has variance for large n given by (4.43) of Theorem 4.5 so that the relative efficiency of S_3 to S_4 is given by

$$\text{Rel. Eff.} = \frac{n+2}{4\pi n}\{[(K')^2 + 2]\exp[-(K')^2]\} / \tau(1-\tau) \tag{4.45}$$

We give some values of (4.45) in Table 4.10 for selected n, β and γ.

To end this section we turn to the case which is of interest in life-testing applications and suppose that sampling is from the single exponential distribution whose density function is given by

$$\begin{array}{ll} \sigma^{-1}\exp\{-x/\sigma\} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{array} \tag{4.46}$$

Suppose we wish to construct a tolerance interval of β-content at confidence level γ which has ability to pick up the right hand $100\beta\%$ set basis of this sample, of the form (4.48) which has the property that it is of β-content at confidence level γ. Then the constant K appearing in (4.48) is such that

$$K = K(n;\gamma, \beta) = (2n \log 1/\beta)/\chi^2_{2n;\ 1-\gamma} \tag{4.51}$$

and the interval (4.42) so obtained is of β'-expectation, where

$$\beta' = [n(n + K)^{-1}]^n \tag{4.52}$$

<u>Proof:</u> It is easy to see that the distribution of C does not depend on σ and that

$$\Pr(C \geq \beta) = \int_0^{\frac{1}{K}\log 1/\beta} \frac{n^n}{\Gamma(n)} t^{n-1} e^{-nt}\, dt \tag{4.53}$$

Setting $2nt = u$ inside the integral yields

$$\Pr(C \geq \beta) = \int_0^{\frac{2n}{K}\log 1/\beta} \frac{1}{2^n \Gamma(n)} u^{n-1} e^{-u/2}\, du$$

$$= \Pr(\chi^2_{2n} \leq \frac{2n}{K} \log 1/\beta) \tag{4.54}$$

Setting $\Pr(C \geq \beta) = \gamma$ implies that

$$\frac{2n}{K} \log 1/\beta = \chi^2_{2n;\ 1-\gamma} \tag{4.55}$$

of the above single exponential distribution, which is given by

$$A_R = \{x \mid x \geq \sigma \log \frac{1}{\beta}\} \tag{4.47}$$

Recalling the results of section 3, a reasonable candidate is the interval given by

$$S(X_1, \ldots, X_n) = [K\overline{X}, \infty) \tag{4.48}$$

where $K = K(n;\gamma, \beta)$ is chosen so that S is a β-content interval at confidence level γ. Now the coverage C of the interval (4.42) is easily seen to be given by

$$C = \exp\{-(K\overline{X})/\sigma\} \tag{4.49}$$

where $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$ is distributed with density function

$$\begin{cases} \dfrac{n^n}{\sigma^n \Gamma(n)} \overline{x}^{n-1} e^{-(n\overline{x})/\sigma} & \text{if } \overline{x} \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.50}$$

Thus we are able to state the following theorem.

<u>Theorem 4.6.</u> Suppose $(X_1, \ldots, X_n)$ is a random sample of n independent observations from the single exponential distribution whose density function is given by (4.40). Suppose further that we wish to construct a tolerance interval, on the

and the result (4.51) follows immediately.

To prove the result (4.52), we have from (4.48) and (4.50) that

$$\beta' = E(C) = \int_0^\infty e^{-K(\frac{\bar{x}}{\sigma})} \frac{n^n}{\sigma^n \Gamma(n)} \bar{x}^{n-1} e^{-(n\bar{x})/\sigma} \, d\bar{x}$$

$$= \int_0^\infty \frac{n^n}{\Gamma(n)} t^{n-1} e^{-t(n+K)} \, dt$$

$$= \frac{n^n}{(n+K)^n}$$

and the Theorem is proved.

We tabulate values of $K(n;\gamma,\beta)$ for γ, β = .75, .90, .95 and .99 for selected n in Table 4.11.

Table 4.11

Values of K which makes the tolerance interval (4.48) β-content at confidence level γ, when sampling from the single exponential density given by (4.46)

n \ β	γ = .75				γ = .90			
	.75	.90	.95	.99	.75	.90	.95	.99
1	.2075	.0760	.0370	[illegible]	.1249	.0458	.0223	.0044
2	.2137	.0783	.0381	.0075	.1479	.0542	.0264	.0052
3	.2201	.0806	.0393	.0077	.1622	.0594	.0289	.0057
4	.2252	.0825	.0402	.0079	.1722	.0631	.0307	.0060
5	.2293	.0840	.0409	.0080	.1800	.0659	.0321	.0063
6	.2326	.0852	.0415	.0081	.1861	.0682	.0332	.0065
7	.2353	.0862	.0420	.0082	.1912	.0700	.0341	.0067
8	.2376	.0870	.0424	.0083	.1955	.0716	.0349	.0068
9	.2397	.0878	.0427	.0084	.1993	.0730	.0355	.0070
10	.2415	.0884	.0431	.0084	.2025	.0742	.0361	.0071
11	.2431	.0890	.0433	.0085	.2054	.0752	.0366	.0072
12	.2445	.0895	.0436	.0085	.2080	.0762	.0371	.0073
13	.2458	.0900	.0438	.0086	.2103	.0770	.0376	.0074
14	.2469	.0904	.0440	.0086	.2125	.0778	.0379	.0074
15	.2480	.0908	.0442	.0087	.2144	.0785	.0382	.0075
16	.2490	.0912	.0444	.0087	.2162	.0792	.0385	.0076
17	.2499	.0915	.0446	.0087	.2178	.0798	.0388	.0076
18	.2507	.0918	.0447	.0088	.2194	.0803	.0391	.0077
19	.2515	.0921	.0449	.0088	.2208	.0809	.0394	.0077
20	.2523	.0924	.0450	.0088	.2221	.0814	.0396	.0078
21	.2530	.0926	.0451	.0088	.2234	.0818	.0398	.0078
22	.2536	.0929	.0452	.0089	.2245	.0822	.0400	.0078
23	.2542	.0931	.0453	.0089	.2257	.0827	.0402	.0079
24	.2548	.0933	.0454	.0089	.2267	.0830	.0404	.0079
25	.2553	.0935	.0455	.0089	.2277	.0834	.0406	.0080
26	.2559	.0937	.0456	.0089	.2287	.0837	.0408	.0080
27	.2564	.0939	.0457	.0090	.2296	.0841	.0409	.0080
28	.2568	.0941	.0458	.0090	.2304	.0844	.0411	.0081
29	.2573	.0942	.0459	.0090	.2312	.0847	.0412	.0081
30	.2577	.0944	.0460	.0090	.2320	.0850	.0414	.0081
31	.2581	.0945	.0460	.0090	.2328	.0853	.0415	.0081
32	.2585	.0947	.0461	.0090	.2335	.0855	.0416	.0082
33	.2589	.0948	.0462	.0090	.2342	.0858	.0418	.0082
34	.2592	.0949	.0462	.0091	.2348	.0860	.0419	.0082
35	.2596	.0951	.0463	.0091	.2355	.0862	.0420	.0082
36	.2599	.0952	.0463	.0091	.2361	.0865	.0421	.0083
37	.2602	.0953	.0464	.0091	.2367	.0867	.0422	.0083
38	.2606	.0954	.0465	.0091	.2372	.0869	.0423	.0083
39	.2609	.0955	.0465	.0091	.2378	.0871	.0424	.0083
40	.2611	.0956	.0466	.0091	.2383	.0873	.0425	.0083
41	.2614	.0957	.0466	.0091	.2388	.0875	.0426	.0083
42	.2617	.0958	.0467	.0091	.2393	.0876	.0427	.0084
43	.2620	.0959	.0467	.0092	.2399	.0879	.0428	.0084
44	.2622	.0960	.0468	.0092	.2403	.0880	.0428	.0084
45	.2625	.0961	.0468	.0092	.2407	.0882	.0429	.0084
46	.2627	.0962	.0468	.0092	.2411	.0883	.0430	.0084
47	[illegible]	[illegible]	[illegible]	[illegible]	.2416	.0885	.0431	.0084

Table 4.11 (Continued)

Values of K which makes the tolerance interval (4.48) β-content at confidence level γ, when sampling from the single exponential density given by (4.46)

	$\gamma = .95$				$\gamma = .99$			
n \ β	.75	.90	.95	.99	.75	.90	.95	.99
1	.0960	.0352	.0171	.0034	.0625	.0229	.0111	.0022
2	.1213	.0444	.0216	.0042	.0867	.0317	.0155	.0030
3	.1371	.0502	.0244	.0048	.1027	.0376	.0183	.0036
4	.1484	.0544	.0265	.0052	.1146	.0420	.0204	.0040
5	.1571	.0576	.0280	.0055	.1240	.0454	.0221	.0043
6	.1642	.0601	.0293	.0057	.1317	.0482	.0235	.0046
7	.1701	.0623	.0303	.0059	.1382	.0506	.0246	.0048
8	.1750	.0641	.0312	.0061	.1438	.0527	.0257	.0050
9	.1794	.0657	.0320	.0063	.1488	.0545	.0265	.0052
10	.1832	.0671	.0327	.0064	.1532	.0561	.0273	.0054
11	.1866	.0683	.0333	.0065	.1571	.0575	.0280	.0055
12	.1896	.0694	.0338	.0066	.1606	.0588	.0286	.0056
13	.1924	.0705	.0343	.0067	.1639	.0600	.0292	.0057
14	.1949	.0714	.0347	.0068	.1669	.0611	.0298	.0058
15	.1972	.0722	.0352	.0069	.1696	.0621	.0302	.0059
16	.1993	.0730	.0355	.0070	.1721	.0630	.0307	.0060
17	.2013	.0737	.0359	.0070	.1745	.0639	.0311	.0061
18	.2031	.0744	.0362	.0071	.1767	.0647	.0315	.0062
19	.2048	.0750	.0365	.0072	.1787	.0655	.0319	.006[illegible]
20	.2064	.0756	.0368	.0072	.1807	.0662	.0322	.006[illegible]
21	.2079	.0761	.0371	.0073	.1825	.0668	.0325	.0064
22	.2093	.0767	.0373	.0073	.1842	.0675	.0329	.0064
23	.2106	.0771	.0376	.0074	.1859	.0681	.0331	.0065
24	.2119	.0776	.0378	.0074	.1874	.0686	.0334	.0066
25	.2131	.0780	.0380	.0074	.1889	.0692	.0337	.0066
26	.2142	.0785	.0382	.0075	.1903	.0697	.0339	.0067
27	.2153	.0789	.0384	.0075	.1916	.0702	.0342	.0067
28	.2163	.0792	.0386	.0076	.1929	.0707	.0344	.0067
29	.2173	.0796	.0388	.0076	.1941	.0711	.0346	.0068
30	.2183	.0799	.0389	.0076	.1953	.0715	.0348	.0068
31	.2192	.0803	.0391	.0077	.1964	.0719	.0350	.0069
32	.2200	.0806	.0392	.0077	.1975	.0723	.0352	.0069
33	.2209	.0809	.0394	.0077	.1986	.0727	.0354	.0069
34	.2217	.0812	.0395	.0077	.1996	.0731	.0356	.0070
35	.2224	.0815	.0397	.0078	.2005	.0734	.0358	.0070
36	.2232	.0817	.0398	.0078	.2015	.0738	.0359	.0070
37	.2239	.0820	.0399	.0078	.2024	.0741	.0361	.0071
38	.2246	.0823	.0400	.0079	.2032	.0744	.0362	.0071
39	.2253	.0825	.0402	.0079	.2041	.0747	.0364	.0071
40	.2259	.0827	.0403	.0079	.2049	.0750	.0365	.0072
41	.2265	.0830	.0404	.0079	.2057	.0753	.0367	.0072
42	.2271	.0832	.0405	.0079	.2064	.0756	.0368	.007[illegible]
43	.2277	.0834	.0406	.0080	.2072	.0759	.0369	.0072
44	.2283	.0836	.0407	.0080	.2079	.0761	.0371	.0073
45	.2288	.0838	.0408	.0080	.2086	.0764	.0372	.0073
46	.2294	.0840	.0409	.0080	.2093	.0767	.0373	.0073
47	[illegible]	[illegible]	.0410	.0080	.2100	.0769	.0374	.0073

Table 4.11 (Continued)

Values of K which makes the tolerance interval (4.48) β-content at confidence level γ, when sampling from the single exponential density given by (4.46)

	$\gamma = .95$				$\gamma = .99$			
n \ β	.75	.90	.95	.99	.75	.90	.95	.99
1	.0960	.0352	.0171	.0034	.0625	.0229	.0111	.0022
2	.1213	.0444	.0216	.0042	.0867	.0317	.0155	.0030
3	.1371	.0502	.0244	.0048	.1027	.0376	.0183	.0036
4	.1484	.0544	.0265	.0052	.1146	.0420	.0204	.0040
5	.1571	.0576	.0280	.0055	.1240	.0454	.0221	.0043
6	.1642	.0601	.0293	.0057	.1317	.0482	.0235	.0046
7	.1701	.0623	.0303	.0059	.1382	.0506	.0246	.0048
8	.1750	.0641	.0312	.0061	.1438	.0527	.0257	.0050
9	.1794	.0657	.0320	.0063	.1488	.0545	.0265	.0052
10	.1832	.0671	.0327	.0064	.1532	.0561	.0273	.0054
11	.1866	.0683	.0333	.0065	.1571	.0575	.0280	.0055
12	.1896	.0694	.0338	.0066	.1606	.0588	.0286	.0056
13	.1924	.0705	.0343	.0067	.1639	.0600	.0292	.0057
14	.1949	.0714	.0347	.0068	.1669	.0611	.0298	.0058
15	.1972	.0722	.0352	.0069	.1696	.0621	.0302	.0059
16	.1993	.0730	.0355	.0070	.1721	.0630	.0307	.0060
17	.2013	.0737	.0359	.0070	.1745	.0639	.0311	.0061
18	.2031	.0744	.0362	.0071	.1767	.0647	.0315	.0062
19	.2048	.0750	.0365	.0072	.1787	.0655	.0319	.0062
20	.2064	.0756	.0368	.0072	.1807	.0662	.0322	.0063
21	.2079	.0761	.0371	.0073	.1825	.0668	.0325	.0064
22	.2093	.0767	.0373	.0073	.1842	.0675	.0329	.0064
23	.2106	.0771	.0376	.0074	.1859	.0681	.0331	.0065
24	.2119	.0776	.0378	.0074	.1874	.0686	.0334	.0066
25	.2131	.0780	.0380	.0074	.1889	.0692	.0337	.0066
26	.2142	.0785	.0382	.0075	.1903	.0697	.0339	.0067
27	.2153	.0789	.0384	.0075	.1916	.0702	.0342	.0067
28	.2163	.0792	.0386	.0076	.1929	.0707	.0344	.0067
29	.2173	.0796	.0388	.0076	.1941	.0711	.0346	.0068
30	.2183	.0799	.0389	.0076	.1953	.0715	.0348	.0068
31	.2192	.0803	.0391	.0077	.1964	.0719	.0350	.0069
32	.2200	.0806	.0392	.0077	.1975	.0723	.0352	.0069
33	.2209	.0809	.0394	.0077	.1986	.0727	.0354	.0069
34	.2217	.0812	.0395	.0077	.1996	.0731	.0356	.0070
35	.2224	.0815	.0397	.0078	.2005	.0734	.0358	.0070
36	.2232	.0817	.0398	.0078	.2015	.0738	.0359	.0070
37	.2239	.0820	.0399	.0078	.2024	.0741	.0361	.0071
38	.2246	.0823	.0400	.0079	.2032	.0744	.0362	.0071
39	.2253	.0825	.0402	.0079	.2041	.0747	.0364	.0071
40	.2259	.0827	.0403	.0079	.2049	.0750	.0365	.0072
41	.2265	.0830	.0404	.0079	.2057	.0753	.0367	.0072
42	.2271	.0832	.0405	.0079	.2064	.0756	.0368	.0072
43	.2277	.0834	.0406	.0080	.2072	.0759	.0369	.0072
44	.2283	.0836	.0407	.0080	.2079	.0761	.0371	.0073
45	.2288	.0838	.0408	.0080	.2086	.0764	.0372	.0073
46	.2294	.0840	.0409	.0080	.2093	.0767	.0373	.0073
47	.2299	.0842	.0410	.0080	.2100	.0769	.0374	.0073

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Table 4.11 (Continued)

Values of K which makes the tolerance interval (4.48) β-content at confidence level γ, when sampling from the single exponential density given by (4.46)

	$\gamma = .95$				$\gamma = .99$			
n \ β	.75	.90	.95	.99	.75	.90	.95	.99
48	.2304	.0844	.0411	.0081	.2106	.0771	.0376	.0074
49	.2309	.0846	.0412	.0081	.2112	.0774	.0377	.0074
50	.2314	.0847	.0413	.0081	.2118	.0776	.0378	.0074
51	.2318	.0849	.0413	.0081	.2124	.0778	.0379	.0074
52	.2323	.0851	.0414	.0081	.2130	.0780	.0380	.0074
53	.2327	.0852	.0415	.0081	.2136	.0782	.0381	.0075
54	.2332	.0854	.0416	.0082	.2141	.0784	.0382	.0075
55	.2336	.0856	.0417	.0082	.2147	.0786	.0383	.0075
56	.2340	.0857	.0417	.0082	.2152	.0788	.0384	.0075
57	.2344	.0858	.0418	.0082	.2157	.0790	.0385	.0075
58	.2348	.0860	.0419	.0082	.2162	.0792	.0386	.0076
59	.2352	.0861	.0419	.0082	.2167	.0794	.0386	.0076
60	.2355	.0863	.0420	.0082	.2172	.0795	.0387	.0076
65	.2373	.0869	.0423	.0083	.2195	.0804	.0391	.0077
70	.2389	.0875	.0426	.0083	.2215	.0811	.0395	.0077
75	.2403	.0880	.0428	.0084	.2234	.0818	.0398	.0078
100	.2459	.0901	.0438	.0086	.2307	.0845	.0411	.0081
125	.2498	.0915	.0445	.0087	.2359	.0864	.0421	.0082
150	.2528	.0926	.0451	.0088	.2398	.0878	.0428	.0084
200	.2571	.0942	.0458	.0090	.2455	.0899	.0438	.0086
250	.2601	.0952	.0464	.0091	.2495	.0914	.0445	.0087
300	.2623	.0961	.0468	.0092	.2525	.0925	.0450	.0088
350	.2641	.0967	.0471	.0092	.2549	.0934	.0455	.0089
400	.2655	.0972	.0473	.0093	.2569	.0941	.0458	.0090
450	.2667	.0977	.0476	.0093	.2585	.0947	.0461	.0090
500	.2677	.0980	.0477	.0094	.2599	.0952	.0463	.0091
∞	.2877	.1054	.0513	.0101	.2877	.1054	.0513	.0101

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