Department of Statistics

The University of Wisconsin
Madison, Wisconsin

TECHNICAL REPORT NO. 122

August 1967

MODELS FOR PREDICTION AND CONTROL

XI DESIGN OF SIMPLE DISCRETE CONTROL SCHEMES

by

G.E.P. Box and G. M. Jenkins*

*Department of Systems Engineering, University of Lancaster, England

This research was supported by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force under AFSOR Grant No. AF-AFOSR-1158-66.

CHAPTER 11

List of Contents

11.1 Feedforward Control

11.1.1 Feedforward control to minimize mean square error at the output

11.1.2 An Example. Control of specific gravity of an intermediate product

11.1.3 A Nomogram for feedforward control

11.1.4 Feedforward control with multiple inputs

11.2 Feedback Control

11.2.1 Feedback control to minimize output mean equare error

11.2.2 Application of the control equation: relation with three term controller

11.2.3 Some examples of discrete feedback control

11.3 Feedforward-Feedback Control

11.3.1 Feedforward-feedback control to minimize output mean square error

11.3.2 An example of feedforward-feedback control

11.3.3 Advantages and disadvantages of feedforward and of feedback control

11.4 Fitting dynamic-stochastic parameters using operating data

11.4.1 Iterative model building

11.4.2 Estimation from operating data

11.4.3 An example.

11.5 Effect of added noise in feedback schemes

11.5.1 Effect of ignoring added noise -- rounded schemes

11.5.2 Optimal action when there are observational errors in adjustment x_t.

11.5.3 Transference of the noise origin

11.6 Feedback control schemes where the adjustment variance is restricted

11.6.1 Derivation of optimal adjustment

11.6.2 A constrained scheme for the viscosity/gas rate example

11.7 Choice of the sampling interval

11.7.1 An illustration of the effect of reducing sampling frequency

11.7.2 Sampling an I.M.A. process of order (0,1,1)

Figure 11.17 Control of viscosity by varying gas rate values of $\frac{g\sigma_x}{\sigma_a}$, k_o, k_1 for a range of values of $\frac{\sigma_\varepsilon}{\sigma_a}$

Figure 11.18 Behavior of unconstrained and constrained control schemes for viscosity/gas rate example

Figure 11.19 Sampling of I.M.A. (0,1,1) process. Parameter θ_h plotted against log h

CHAPTER 11

DESIGN OF SIMPLE DISCRETE CONTROL SCHEMES

A common control problem is that of how to maintain some output variable as close as possible to a target value in a system subject to disturbances. We now discuss this problem using the previously discussed stochastic and dynamic models to describe disturbances and system dynamics.

We shall continue to assume that data is available at discrete equispaced time intervals when opportunity can also be taken to make adjustments. We shall also assume that no appreciable extra cost is associated with corrective action. This is the case for most chemical processes subject to manual or automatic control. It is then sensible to seek control schemes which minimize some overall measure of error at the output. The overall error measure we use is the mean square error.

In some instances one or more sources of disturbance may be measured and these measurements used to compensate potential deviations in the output. Such action is called feedforward control. In other situations the only evidence we have of the existence of the disturbance is the deviation from target it produces in the output. When this deviation itself is used as a basis for adjustment we have feedback control. In some instances a combination of these two modes of control is desirable and this we call feedforward-feedback control.

We first show how one can design control schemes of the various types to yield minimum mean square error at the output. The effect of input errors are then considered. We later show how process data and, in particular, data collected during the operation of a pilot control scheme may be used to obtain better estimates of the model and its parameters. This allows us to employ an interative approach in arriving at an optimal control scheme. We finally consider the design of control schemes when the variance of the input is constrained.

11.1 Feedforward Control

We now consider the design of discrete feedforward control schemes which give minimum mean square error at the output. A situation arising in the manufacture of a polymer is illustrated in Figure 11.1. The viscosity Y_t of the product is known to vary in part due to fluctuations in the feed concentration z_t which can be observed but not changed. The steam pressure X_t is a control variable which is measured, can be manipulated, and is potentially available to alter the viscosity by any desired amount and so to compensate potential deviations from target. The total effect in the output viscosity of all other sources of disturbance at time t is denoted by n_t.

11.1.1 Feedforward control to minimize mean square error at the output

We can suppose that z_t, X_t, n_t are deviations from

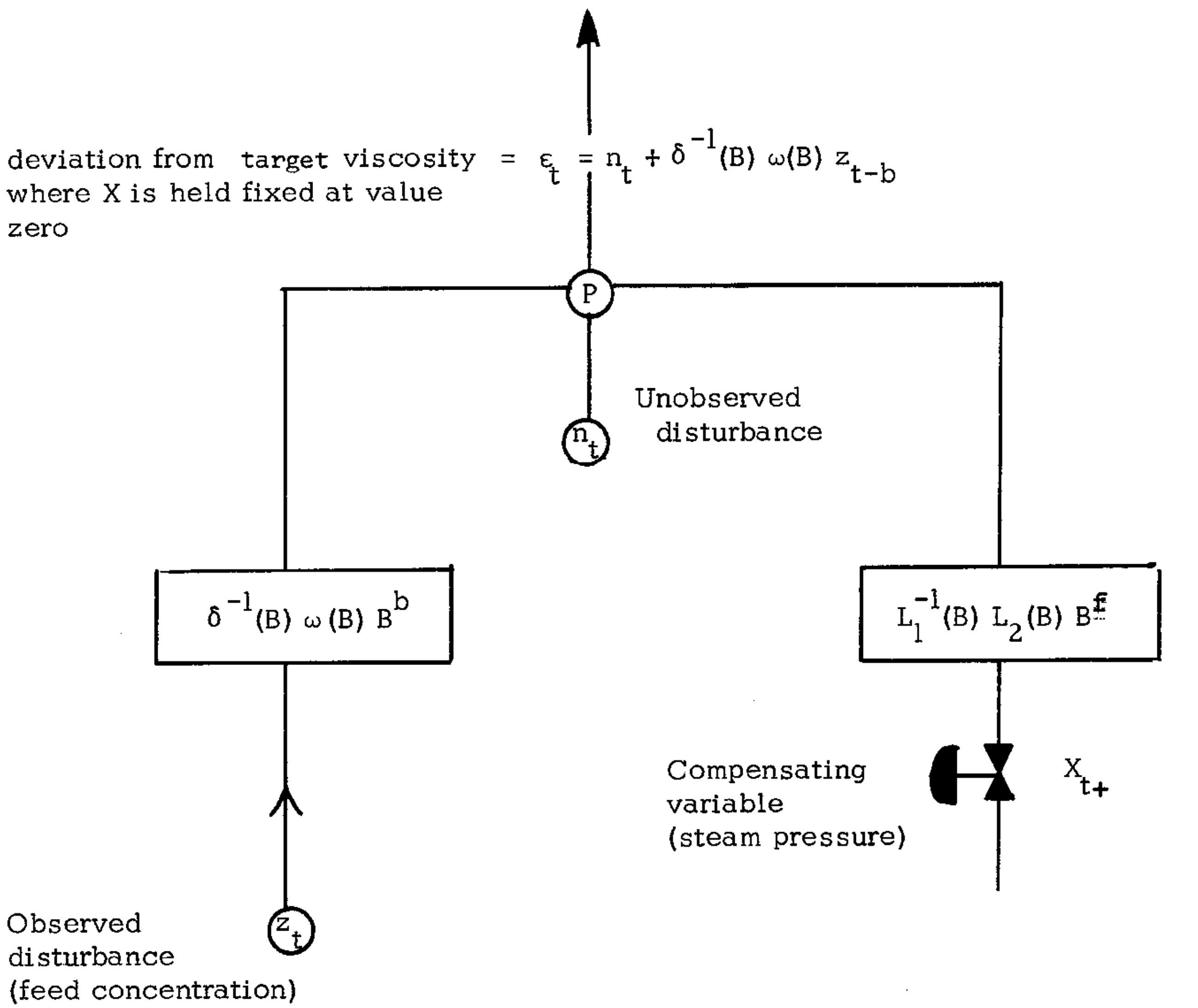


Figure 11.1 - A system at time t subject to an observed disturbance z_t and unobserved disturbance n_t with potential compensating variable X_t held fixed at $X_t = 0$

reference values which are such that if the conditions $z = 0$, $X = 0$, $n = 0$ were continuously maintained then the process would remain in an equilibrium state such that the output was exactly on the target value $Y = 0$.

The dynamic relation which connects the observed disturbance z_t (feed concentration) and the output Y_t (viscosity) is

$$Y_t = \delta^{-1}(B)\omega(B)\ B^b\ z_t\ .$$

Similarly the dynamic relation which connects the compensating variable X_t (steam pressure) and the output Y_t (viscosity) is

$$Y_t = L_1^{-1}(B)L_2(B)\ B^f\ X_t\ .$$

Then if no control is exerted (the potential compensating variable X_t is held fixed at $X_t = 0$) the total error in the output viscosity will be

$$\varepsilon_t = n_t + \delta^{-1}(B)\omega(B)\ z_{t-b}\ .$$

Clearly it ought to be possible to compensate the effect of the measured parts of the overall disturbance by manipulating X_t.

Now changes will be made in X at times t, t-1, t-2, ... immediately after the observations z_t, z_{t-1}, z_{t-2} are taken. Hence we have a "stepped" input and we denote the level of X in the interval t to t+1 by X_{t+}. At time t, and at the point P in the diagram:

the total effect of the disturbance (z) is

$\delta^{-1}(B)\omega(B)\ z_{t-b}$

the total effect of the compensation (X) is

$L_1^{-1}(B)L_2(B)\ X_{t-f+}$.

The effect of the observed disturbance z will then be cancelled if we set

$$L_1^{-1}(B)L_2(B)\ X_{t-f+} = -\delta^{-1}(B)\omega(B)\ z_{t-b}$$

that is $$L_1^{-1}(B)L_2(B)\ X_{t+} = -\delta^{-1}(B)\omega(B)\ z_{t-(b-f)}. \qquad (11.1.1)$$

Case 1 $b \geq f$

The control action (11.1.1) is directly realizable only if $(b-f) \geq 0$. This is to say if the number of whole periods of delay b between the time at which an increment of the disturbance is observed and the time it affects the process is longer than the number of whole periods of delay f before action can influence the output. The desired control action at time t is to set the manipulated variable X to the level

$$X_{t+} = -\frac{L_1(B)\omega(B)}{L_2(B)\delta(B)}\ z_{t-(b-f)} .$$

Alternatively it is often more convenient to define the control action in terms of the <u>change</u> $x_t = X_{t+} - X_{t-1+}$ which is to be made in the level of X. This is

$$x_t = -\frac{L_1(B)\omega(B)}{L_2(B)\delta(B)}\{z_{t-(b-f)} - z_{t-1-(b-f)}\}. \quad (11.1.2)$$

The situation is illustrated in Figure 11.2. The effect at P from the control action is $-\delta^{-1}(B)\omega(B)z_{t-b}$ and this exactly cancels the effect at P of the disturbance. The component of the deviation from target due to z_t is (theoretically at least) exactly eliminated and only the component n_t due to the unobserved disturbance remains.

<u>Case 2 (b-f) negative</u>

It can happen that $f > b$. This means that an observed disturbance reaches the output before it is possible for compensating action to become effective. In this case the action

$$\delta(B)L_2(B)\ X_{t+} = -L_1(B)\omega(B)\ z_{t+(f-b)} \quad (11.1.3)$$

is not realizable because at time t when the action is to be taken the relevant value $z_{t+(f-b)}$ of the disturbance is not yet available. One would usually avoid this situation if one could (if some quicker acting compensating variable could be used instead of X) but sometimes such an alternative is not available.

Now if the disturbance z_t can be represented by the linear model

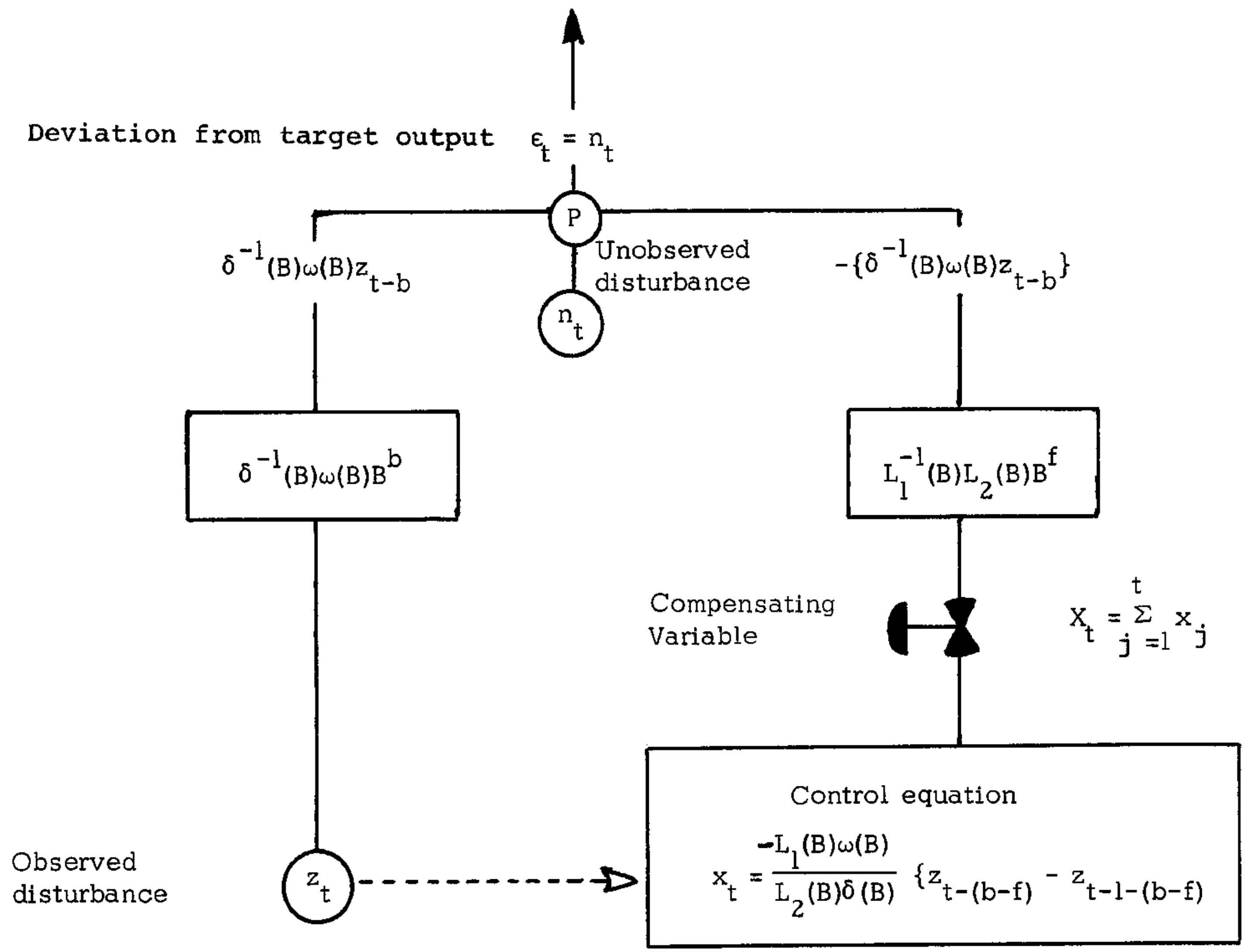


Figure 11.2 **Feedforward** control scheme at time t when $b \geq f$

$$z_t = \left\{1 + \sum_{i=1}^{\infty} \psi_i B^i \quad a_t\right\}$$

then

$$z_{t+f-b} = \hat{z}_t(f-b) + e_t(f-b) \quad .$$

In this expression

$$e_t(f-b) = a_{t+f-b} + \psi_1 a_{t+f-b-1} + \cdots + \psi_{f-b-1} a_{t+1}$$

is the forecast error. Then we can write (11.1.3) in the form

$$\delta(B)L_2(B)X_{t+} = -L_1(B)\omega(B)\hat{z}_t(f-b) - L_1(B)\omega(B)e_t(f-b) \quad .$$

Now $e_t(f-b)$ is a function of the independent random deviates a_{t+h} $(h \geq 1)$ which have not yet occurred at time t and which are independent of any variable known at time t (and so are unforecastable). It follows that the optimal action is achieved by setting

$$X_{t+} = -\frac{L_1(B)\omega(B)}{L_2(B)\delta(B)} \quad \hat{z}_t(f-b) \qquad (11.1.4)$$

that is by making the <u>change</u> in the compensating variable at time t equal to

$$x_t = -\frac{L_1(B)\omega(B)}{L_2(B)\delta(B)} \left\{\hat{z}_t(f-b) - \hat{z}_{t-1}(f-b)\right\} \qquad (11.1.5)$$

This results in an additional component in the deviation ε_t from the target, and now

$$\varepsilon_t = n_t + \delta^{-1}(B)\omega(B)\ e_{t-f}(f-b) \quad .$$

The scheme is illustrated diagramatically in Figure 11.3.

11.1.2 An Example. Control of the specific gravity of an intermediate product

In the manufacture of an intermediate use for the production of a synthetic resin the specific gravity Y_t of the product had to be maintained as close as possible to the value 1.260. This was actually achieved by a mixed scheme of feedforward and feedback control. We consider the complete scheme later and discuss here only the feedforward part. The process has rather slow dynamics and also the disturbance is known to change slowly so that observations and adjustments are made at two hourly intervals. The uncontrolled disturbance which is fed forward is the feed concentration z_t which is referred to an origin of 30 grams per litre. The relation between specific gravity and feed concentration over the range of normal operation is

$$Y_t = 0.0016\ z_t$$

where Y_t is measured from the target value 1.260.

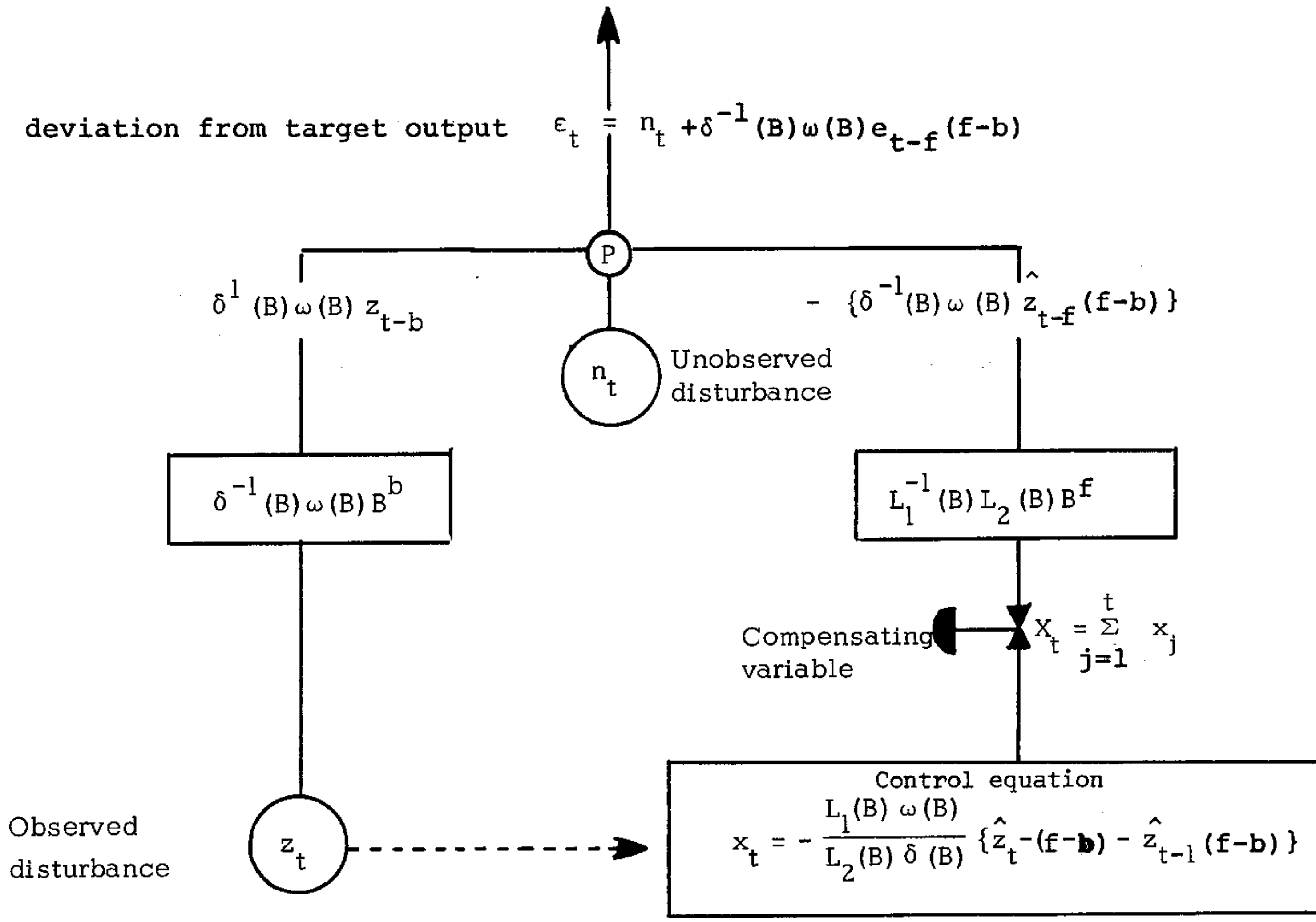


Figure 11.3 Feedforward control scheme at time t when f > b

This relation contains "no dynamics" because the feed concentration can only be measured at the inlet to the reactor so that in our general notation

$$\delta(B) = 1, \quad \omega(B) = 0.0016, \quad b = 0 \quad .$$

Control is achieved by varying pressure which is referred to a convenient origin of 25 p.s.i. The dynamic relation between specific gravity and pressure was estimated as

$$(1 - 0.7B)\ Y_t = 0.0024\ B\ X_{t+}$$

so that

$$L_1(B) = (1 - 0.7B), \quad L_2(B) = 0.0024, \quad f = 1.$$

So far as can be ascertained the effects of pressure and feed concentration are approximately additive in the region of normal operation. The control equation is therefore

$$X_t = - \frac{(1 - 0.7B)\ 0.0016}{0.0024} \quad \hat{z}_t(1) \quad .$$

Study of the feed concentration shows that it may be represented by the linear stochastic model of order (0,1,1)

$$\nabla\ z_t = (1 - \theta B)a_t, \quad \text{with} \quad \theta = 0.5.$$

For such a process

$$\hat{z}_t(1) = (1 - \theta)z_t + \theta\ \hat{z}_{t-1}(1)$$

i.e.

$$(1 - \theta B)\ \hat{z}_t(1) = (1 - \theta)\ z_t$$

$$\hat{z}_t(1) = \frac{(1-\theta)}{(1-\theta B)} z_t \quad .$$

Thus, finally the control equation can be written

$$X_{t+} = -\frac{(1-0.7B)\ 0.0016\ (0.5)}{0.0024\ (1-0.5B)} z_t \qquad (11.1.6)$$

or

$$X_{t+} = 0.5\ X_{t-1+} - 0.33\ \{z_t - 0.7\ z_{t-1}\} \quad . \qquad (11.1.7)$$

Table 11.1 shows the calculation of the first few of a series of settings of the pressure required to compensate the variations in feed concentration given the starting conditions for time $t = 0$ of $z_o = 1.6$, $X_{o+} = -0.63$

t	Concentration $z_t + 30$	z_t	X_{t+}	Pressure $X_{t+} + 25$	x_t
	31.6	1.6	-0.63	24.4	
1	31.1	1.1	-0.31	24.7	0.3
2	34.4	4.4	-1.36	23.6	-1.1
3	32.0	2.0	-0.32	24.7	1.1
4	28.2	-1.8	0.91	25.9	1.2

Table 11.1: Calculation of adjustments for feedforward control scheme

Once the calculation has been started off it is sometimes more convenient to work directly with the change x_t to be made at time t using

$$x_t = 0.5\, x_{t-1} - 0.33\, \{\nabla z_t - 0.7\, \nabla z_{t-1}\}\ . \qquad (11.1.8)$$

Figure 11.4 shows a section of the input disturbance and the corresponding output after applying feedforward control. The lower graph shows the calculated output (specific gravity) which would have resulted if no control had been applied. These values Y_t are, of course, not directly available but may be inferred from the values Y_t' which actually occurred using

$$Y_t = Y_t' + \frac{0.008\, z_t}{1 - 0.5B}$$

i.e.

$$Y_t = 0.5\, Y_{t-1} + Y_t' + 0.008\, z_t\ .$$

As a result of feedforward control the root mean square error deviation of the output from target value over the sample record shown is 0.003. Over the same period the root mean square error of the uncorrected series would have been 0.008. The improvement is marked and extremely worthwhile. However, it appears that other unidentified sources of disturbance exist in the process as evidenced by the drift away from target. This kind of tendency is frequently met with pure feedforward schemes but may be compensated by the addition of feedback control as is discussed later.

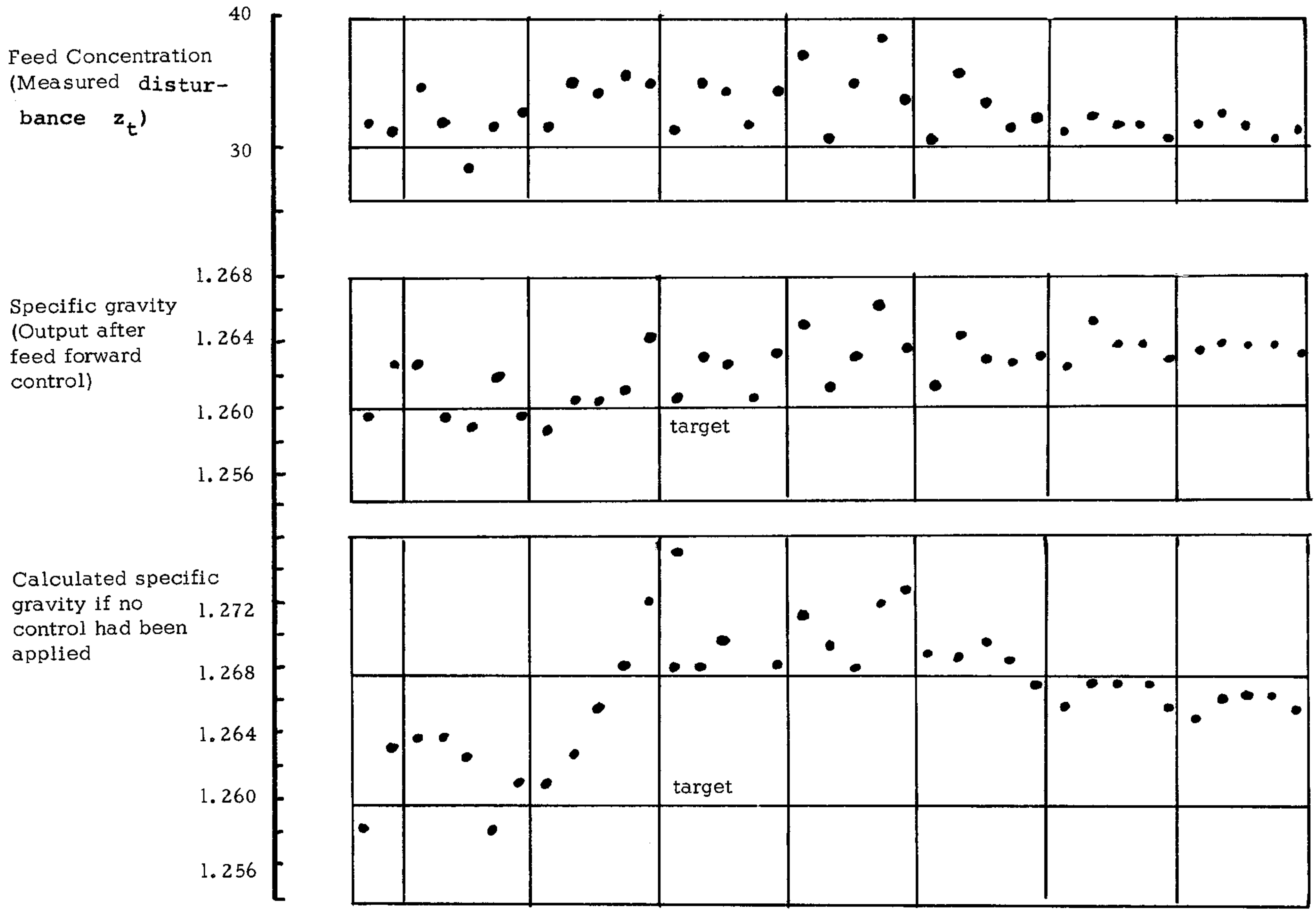


Figure 11.4 Measured disturbance and output from feedforward control scheme

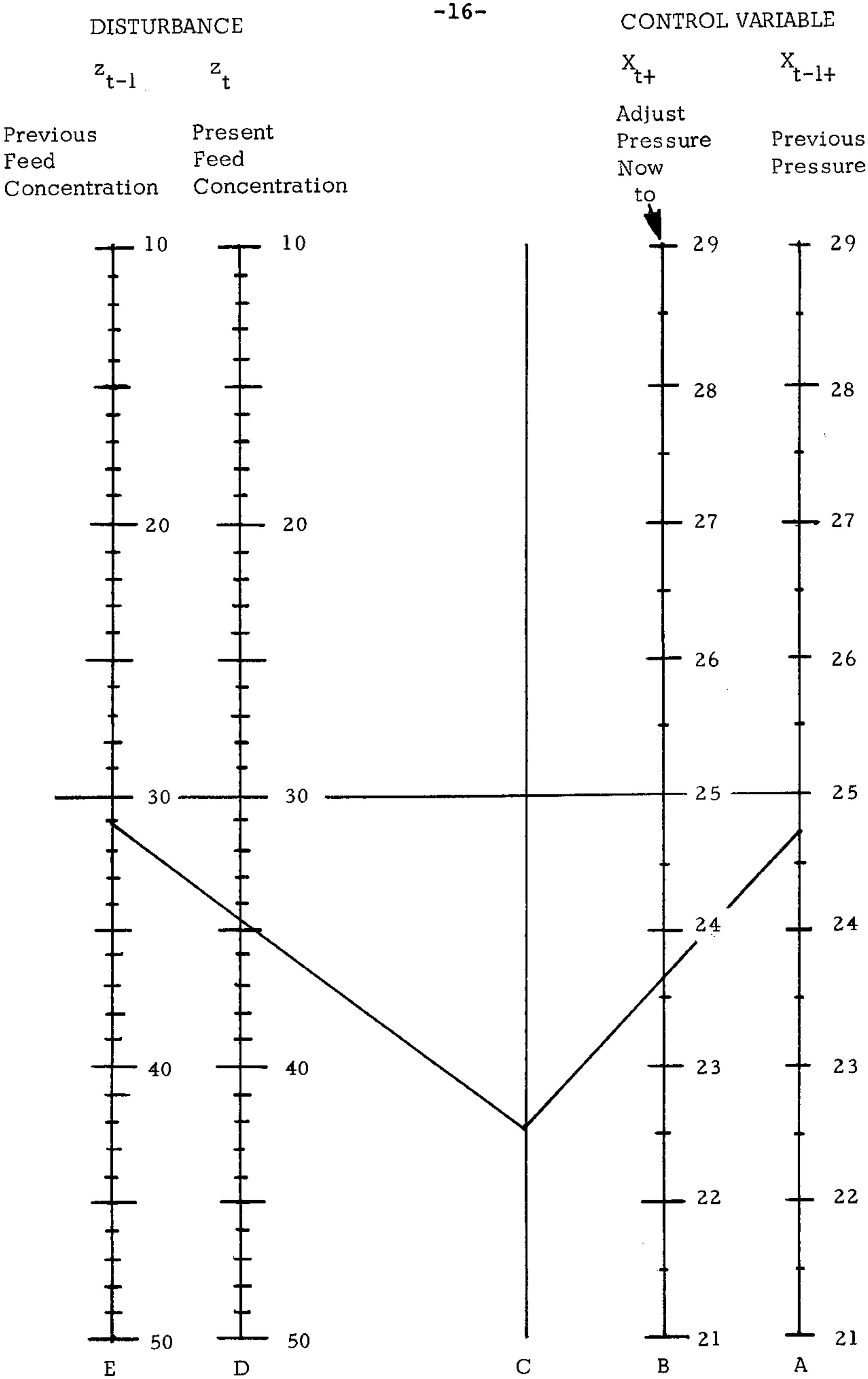


Figure 11. 5 Nomogram for feed forward control scheme

11.1.3 A nomogram for feedforward control

If changes are made fairly infrequently and if the control equation is fairly simple the theory we have outlined may be used to obtain optimal manual control. It is then convenient to use some form of control chart or nomogram which can be easily understood by the process operator.

For illustration we design a nomogram to indicate the appropriate feedforward control action for the previous example. The control equation is

$$(1 - 0.5B)\dot{X}_{t+} = -0.33\ (1 - 0.7B)\ \dot{z}_t$$

and since $(1 - \delta B) = (1 - \delta)\ (1 + \frac{\delta}{1-\delta}\nabla)$

this may be written in difference notation as

$$(1 + \nabla)\dot{X}_{t+} = -0.2\ (1 + 2.33\nabla)\dot{z}_t \qquad (11.1.9)$$

To design a nomogram which allows us to compute the value of

$$r_t = (1+\xi\nabla)X_{t+} = X_{t+} + \xi(X_{t+} - X_{t-1+})$$

we construct three vertical scales to accommodate X_{t-1+}, X_{t+} and r_t like the scales A, B, and C in Figure 11.5 and mark them off in units of X_t and space them so that BC/AB = ξ. Then by simple geometry it is evident that the value of r_t

may be obtained by projecting a line through the points corresponding to X_{t-1+} and X_{t+} on the A and B scales unto the C scale--the scale of r_t .

To achieve the control action of equation (11.1.9) we must equate two expressions of this type and we need five scales as shown in Figure 11.5. Four of these, namely A, B, D, and E, are to accommodate X_{t-1+}, X_{t+}, z_t, and z_{t-1} respectively and a further scale C allows the right side of the equation to be equated to the left. The scales are arranged

(i) so that the origins pressure = 25 p.s.i., feed concentration = 30 grams per litre are in the same horizontal line and so that 1 unit of z_t equals -0.2 units of X_t

(ii) so that $\frac{BC}{AB} = 1$, $\frac{CD}{DE} = 2.33$.

To illustrate the use of the chart we show the calculation of the action appropriate at time t = 2. We project unto the C scale at P the line joining the previous feed concentration $(z_1 + 30) = 31.1$ on the E scale and the present feed concentration $(z_2 + 30) = 34.4$ on the D scale. We then join this projected point P to the points marking the previous pressure $X_{t+} + 25 = 24.7$ on the A scale and read off the value $X_{t+} + 25 = 23.6$ to which the pressure must now be adjusted and held for the next two hours.

11.1.4 Feedforward control with multiple inputs

No difficulty arises in principle when the effect of several additive input disturbances $z_1, z_2, \ldots, z_k$ are to be compensated by changes in X using feedforward control. Suppose the combined effect in the output of all the disturbances is given by

$$Y_t = \sum_{j=1}^{k} \delta_j^{-1}(B)\omega_j(B)B^{b_j} z_{jt}$$

and as before the dynamics for the compensating variable are given by

$$Y_t = L^{-1}(B)L_2(B)B^f X_t$$

Then doing precisely as before the required control action is to change X at time t by an amount

$$x_t = -L_1(B)L_2^{-1}(B)\sum_{j=1}^{k} \delta_j^{-1}(B)\omega_j(B) \left[z_{j,t+f-b_j} - z_{j,t+f-b_j-1}\right] \tag{11.1.10}$$

where

$$z_{j,t+f-b_j} - z_{j,t+f-b_j-1} = \begin{cases} z_{j,t+f-b_j} - z_{j,t+f-b_j-1} & f-b_j \leq 0 \\ \hat{z}_{j,t}(f-b_j) - \hat{z}_{j,t-1}(f-b_j) & f-b_j > 0 \end{cases} \tag{11.1.11}$$

If, as before, n_t is an unidentified disturbance then the error at the output will be

$$\varepsilon_t = n_t + \sum_{j=1}^{k} \delta_j^{-1}(B)\omega_j(B)e_{j,t-f}(f-b_j) \qquad (11.1.12)$$

where $e_{j,t-f}(f-b_j) = 0$ if $f-b_j \leq 0$.

On the one hand this type of control allows us to take prompt action to cancel the effect of disturbing variables, and if $f-b_j \leq 0$, to anticipate completely such disturbances at least in theory. On the other hand to use feedforward control we must be able to measure the disturbing variables and possess complete knowledge--or at least a good estimate--of the relationship between each disturbing variable in the output. In practice we could never identify and measure all of the disturbances that affected the system. The remaining disturbances which we have denoted by n_t, and which are not affected by feedforward control, could of course increase the variance at the output or cause the process to wander off target as in fact happened in the example we discussed.

Clearly we should be able to prevent this from happening by using the error ε_t itself to indicate an appropriate adjustment--that is, by the use of feedback control.

11.2 Feedback Control

Consider the feedback scheme shown in Figure 11.6. Here n_t measures the joint effect at the output of unobserved disturbances and is defined as the deviation from target that would occur in the output at time t if no control action were taken. It is assumed to follow some linear stochastic process defined by

$$n_t = \phi^{-1}(B)\theta(B)a_t \tag{11.2.1}$$

or by

$$n_t = \left\{1 + \sum_{i=1}^{\infty} \psi_i B^i\right\} a_t \tag{11.2.2}$$

where $a_1, a_2, \ldots, a_t$ is a sequence of uncorrelated random variables. Arguing precisely as before we have at the point P for time t

$$\text{Total effect of disturbance} = n_t$$

$$\text{Total effect of compensation} = L_1^{-1}(B)L_2(B)\ X_{t-f+}$$

11.2.1 Feedback control to minimize output mean square error

The effect of the disturbance would be cancelled if it were possible to set

$$X_{t+} = -L_1(B)L_2^{-1}(B)\ n_{t+f}\ . \tag{11.2.3}$$

Since f is positive this is not possible, but we can obtain

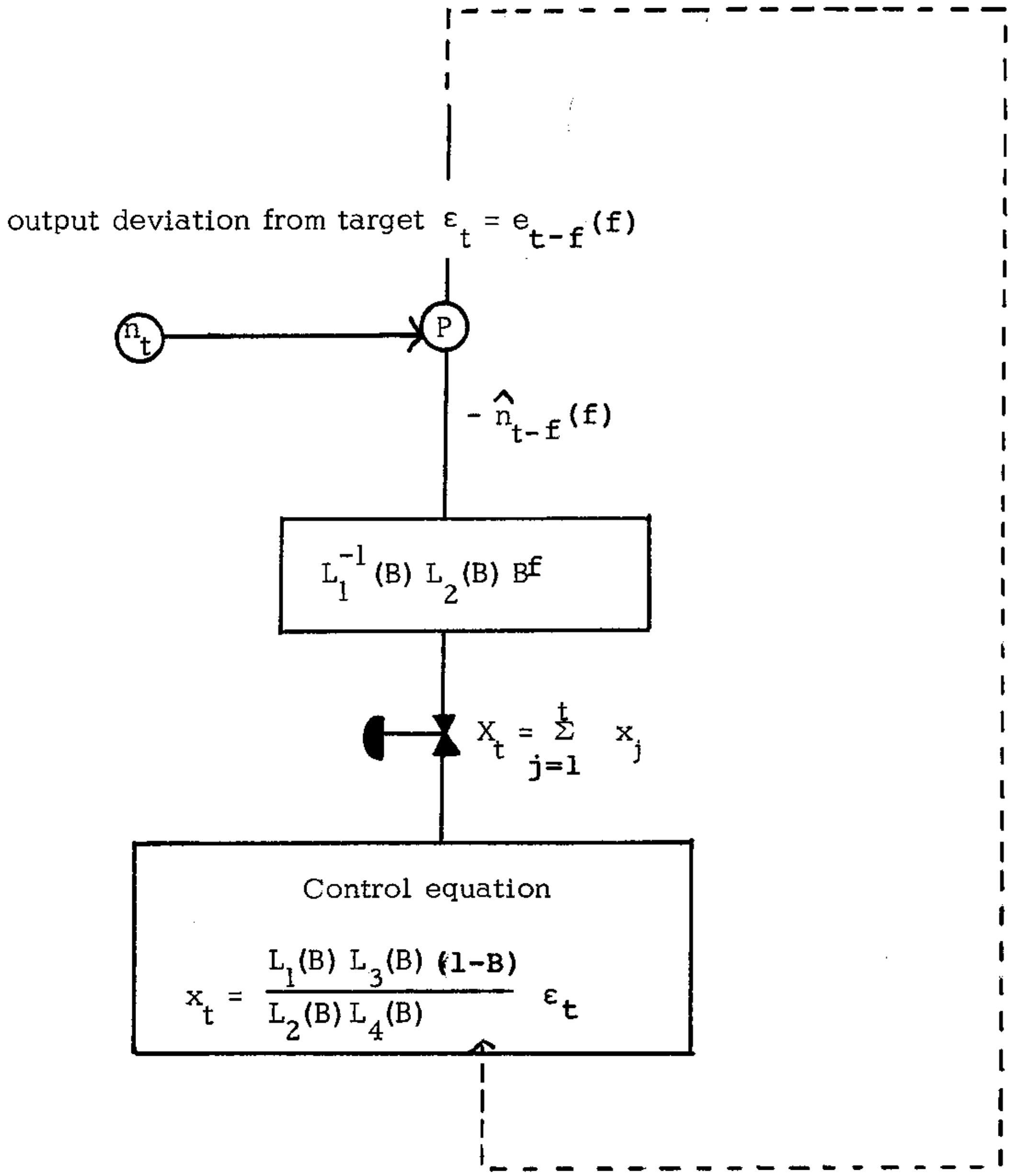


Figure 11.6 Feedback control scheme at time t

minimum mean square error controlled by replacing n_{t+f} by the forecast $\hat{n}_t(f)$; that is, by taking the control action

$$X_{t+} = -L_1(B)L_2^{-1}(B)\hat{n}_t(f) \quad .$$

The <u>change</u> or adjustment to be made in the manipulated variable is thus

$$x_t = -L_1(B)L_2^{-1}(B) \qquad \hat{n}_t(f) - \hat{n}_{t-1}(f) \qquad (11.2.4)$$

in which case the error at the output at time t will be the forecast error for lead time f for the n_t process, that is

$$\varepsilon_t = e_{t-f}(f) \quad .$$

Now $\hat{n}_t(f) - \hat{n}_{t-1}(f)$ is not known directly but it can nevertheless be deduced from the sequence ε_t, ε_{t-1}, ε_{t-2}, ... which is available.

Thus,

$$n_{t+f} = \left\{1 + \sum_{i=0}^{\infty} \psi_i B^i\right\} a_{t+f}$$

$$= \{a_{t+f}+\psi_1 a_{t+f-1}+\dots+\psi_{f-1}a_{t+1}\} + \{\psi_f a_t+\psi_{f+1}a_{t-1}+\dots\}$$

$$= e_t(f) + \hat{n}_t(f) \quad .$$

Furthermore, we can write

$$n_{t+f} = L_4(B)a_{t+f}+L_3(B)a_f \quad .$$

Knowing the model $n_t = \phi^{-1}(B)\theta(B)a_t = \psi(B)a_t$ for the stochastic process we can deduce the operators $L_3(B)$ and $L_4(B)$ in the relations

$$e_{t-f}(f) = L_4(B)a_t, \quad \hat{n}_t(f) = L_3(B)a_t \tag{11.2.5}$$

and hence the relations

$$\hat{n}_t(f) = \frac{L_3(B)}{L_4(B)} \qquad e_{t-f}(f) = \frac{L_3(B)}{L_4(B)}\varepsilon_t.$$

Finally then, the feedback control equation resulting in smallest mean square error at the output may be written

$$X_{t+} = -\frac{L_1(B)L_3(B)}{L_2(B)L_4(B)}\varepsilon_t . \tag{11.2.6}$$

Alternatively, if as is frequently convenient, we define the control action in terms of the adjustment $x_t = X_{t+} - X_{t-1+}$ to be made at time t, then

$$x_t = -\frac{L_1(B)L_3(B)(1-B)}{L_2(B)L_4(B)}\varepsilon_t \tag{11.2.7}$$

where

$$\hat{n}_t(f) - \hat{n}_{t-1}(f) = L_3(B)(1-B)\, a_t .$$

11.2.2 Application of the control equation: relation with three-term controller

In this book we are principally concerned with the derivation of the optimal control equation which indicates how the manipulated variable should be changed to maintain the controlled variable close to some target value. In practice the actual measuring and the computing and carrying out of the required action can be done in a number of ways. At one end of the scale of sophistication one may have electrical measuring instruments the results from which are fed to a computer which calculates the required control action and directly activates transducers which carry it into effect. At the other end of this scale one may have a plant operator who periodically takes a measurement, reads off the required action from a simple chart or nomogram and carries it out by hand. The theory which we have described has been used successfully in both kinds of situations. We go to some pains to describe in detail some of the manual applications because we feel that the use of elementary control ideas to assist the plant operator to do his job well has been somewhat neglected in the past. Althoughundisputably more and more schemes of automatic control are coming into use there is still a great deal of manual operation and this is likely to continue for a long period of time.

The three-term controller

A type of automatic control device which has been in use for many years is the "three-term controller." Controllers of this kind may operate through either mechanical, pneumatic, hydraulic, or electrical means and their operation is based on continuous rather than discrete measurement and adjustment. If

ε_t is the error at the output at time t, control action may be made proportional to ε itself, its integral with respect to time or its derivative with respect to time. A three-term controller uses a linear combination of all of these so that if X_t indicates the level of the manipulated variable at time t the control equation is of the form

$$X_t = k_D \frac{d\varepsilon_t}{dt} + k_P\varepsilon_t + k_I \int \varepsilon_t dt$$

where k_D, k_P, and k_I are constants.

In some situations only one or two of these three modes of action are used. Thus we find instances of simple proportional control ($k_D = 0$, $k_I = 0$), of simple integral control ($k_D = 0$, $k_P = 0$), of proportional-integral control ($k_D = 0$), and of proportional-derivative control ($k_I = 0$).

The discrete analogue of this continuous control equation is

$$X_{t+} = k_D\nabla\varepsilon_t + k_P\varepsilon_t + k_I S\varepsilon_t \qquad (11.2.8)$$

or in terms*of the adjustments to be made

$$x_t = k_D\nabla^2\varepsilon_t + k_P\nabla\varepsilon_t + k_I\varepsilon_t \quad .$$

---

* In previous papers [1],[2] we have used a different nomenclature. For example an adjustment $x_t = k_I\varepsilon_t$ was there referred to as proportional control. This control action is equivalent to $X_{t+} = k_I S\varepsilon_t$ that is to integral action in the level X of the manipulated variable. It is this latter nomenclature which has been traditionally used by control engineers and which we adopt here.

We shall find that many of the simple situations which we meet do lead to control equations containing terms of these types. for example, if the noise can be represented by a (0,1,1) process $\nabla n_t = (1-\theta B)a_t$, while the dynamics can be represented by the first order system $(1 + \xi\nabla)Y_t = g\, X_{t-1+}$, (11.2.6) reduces to

$$X_{t+} = -\frac{(1-\theta)\xi}{g}\varepsilon_t - \frac{(1-\theta)}{g} S\varepsilon_t \quad .$$

The action called for is thus the direct analogue of proportional-integral control.

It is clear, however, that by no means all control actions that might be called for by (11.2.6) could be produced by a three-term controller and rather simple examples occur where other modes of control are called for. With the present computer capability for direct digital control, there is no longer any need to restrict control to these conventional modes. We now consider some specific examples.

11.2.3 Some examples of discrete feedback control

Example 1

In a scheme to control the viscosity Y of a polymer employed in the manufacture of a synthetic fiber, the controlled variable viscosity was checked every hour and adjusted by manipulating the catalyst formulation X. The desired target value for viscosity was 47 units. The dynamic model between X and Y was adequately described by the simple first order system with no delay

$$(1 - \delta B)Y_t = (1-\delta)gX_{t-1+} \quad .$$

Furthermore, the true constant of the system was short compared with the sampling interval, δ being estimated as 0.04, so that an estimated 96% of the eventual change occurred in the sampling interval of one hour. To a sufficient approximation, therefore, we can set $\delta = 0$. Furthermore, catalyst formulation changes were by custom scaled in terms of the effect they were expected to produce. Thus, one unit of formulation change was such as would decrease viscosity by one unit. Hence $g = -1$ and the dynamic equation was taken to be

$$Y_t = gX_{t-1+} \qquad \text{with } g = -1$$

or in terms of our general dynamic model

$$L_1(B) = 1, \quad L_2(B) = g = -1, \quad b = 0 \quad .$$

The disturbance n_t at the output,which it will be recalled is defined as the variation in viscosity if no control were exerted, was adequately described by the stochastic process of order (0,1,1)

$$\nabla n_t = (1 - \theta B)a_t \qquad \text{with } \theta = 0.53, \lambda = (1-\theta) = 0.47$$

so that

$$\hat{n}_t(1) - \hat{n}_{t-1}(1) = \lambda\, a_t = (1-B)L_3(B)a_t$$

$$\varepsilon_t = e_t(1) = a_t$$

and $L_3(B)(1-B) = \lambda = 0.47, \quad L_4(B) = 1.$

The adjustment called for at time t is therefore

$$x_t = -\frac{L_1(B)L_3(B)(1-B)}{L_2(B)L_4(B)} \varepsilon_t = -\frac{\lambda}{g} \varepsilon_t \quad .$$

That is

$$x_t = 0.47 \varepsilon_t \quad \text{or} \quad X_{t+} = 0.47 S\varepsilon_t \quad .$$

In this situation then, where the inertia of the system is not large compared with the sampling interval, optimal control requires the discrete analogue of simple integral control action. We can derive the required control action for this type of example somewhat more directly as follows:

Predicted change at the output $= \hat{n}_t(1) - \hat{n}_{t-1}(1) = \lambda a_t$

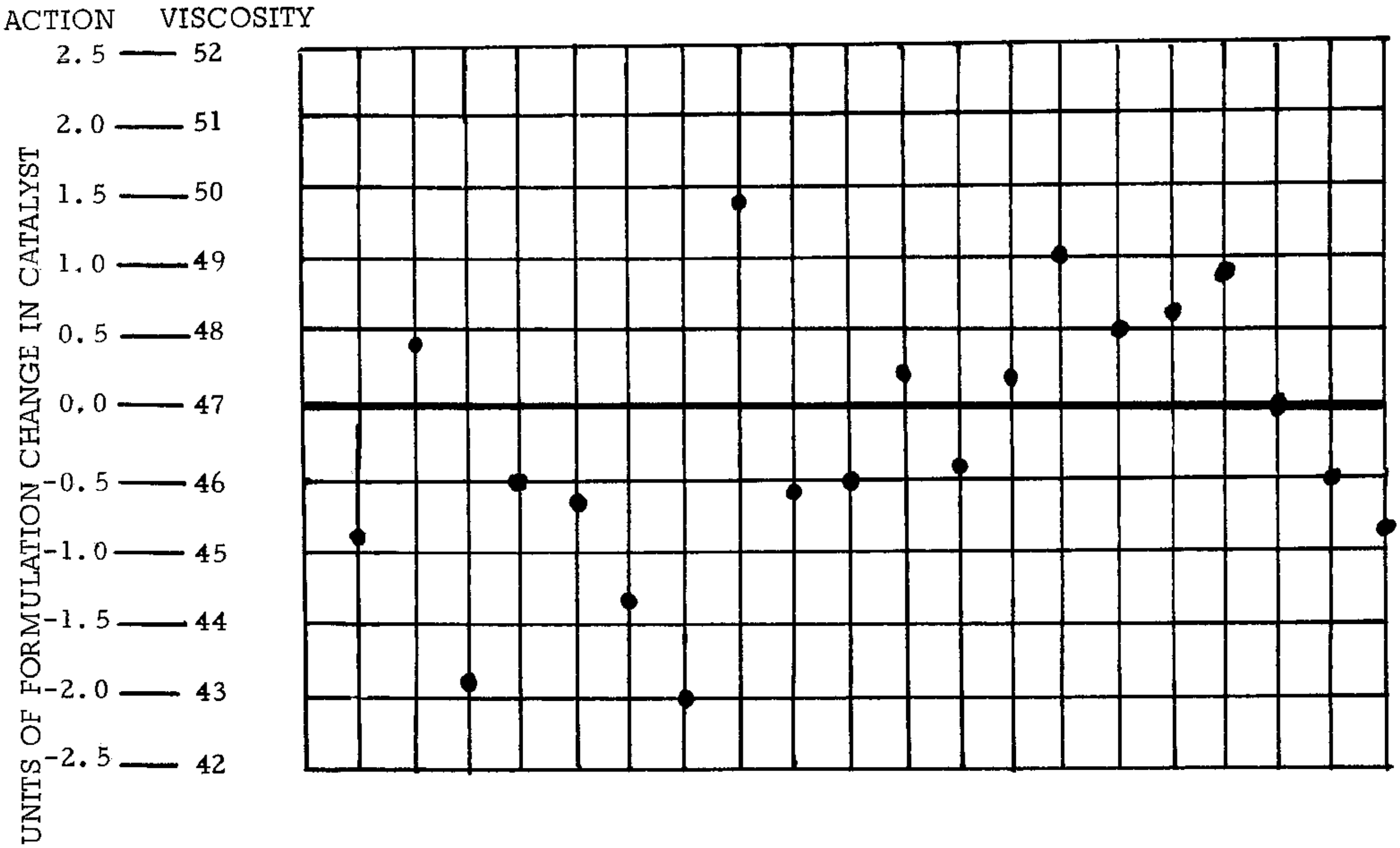
Effect of adjustment $= g x_t$.

Therefore, the adjustment required to compensate is such that $g x_t = -\lambda a_t$. But with this adjustment, the error at output is $\varepsilon_t = a_t$. Thus the optimal feedback control equation is $x_t = -\frac{\lambda}{g} \varepsilon_t$.

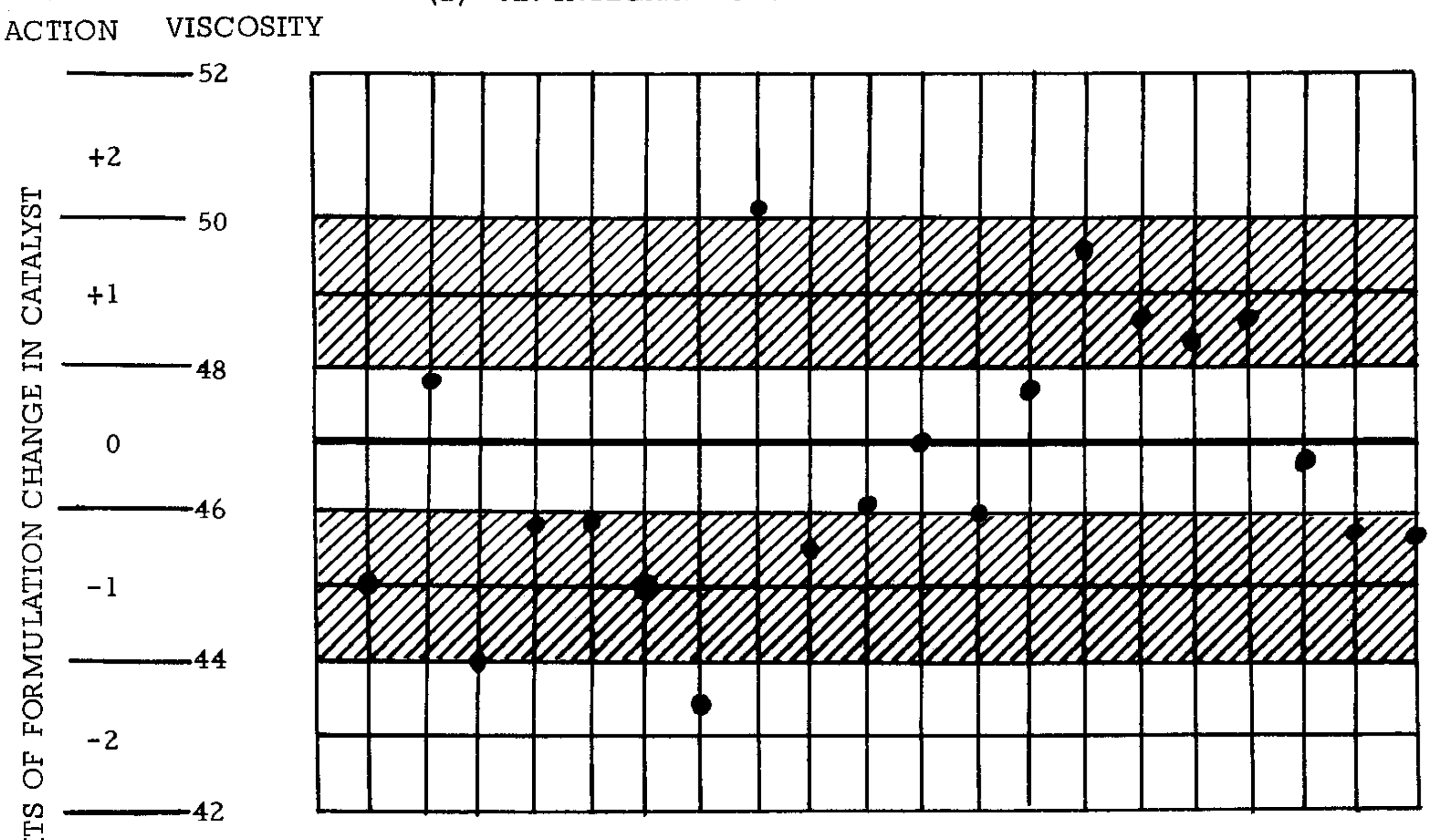
The efficiency of control action of this kind is very insensitive to moderate changes in parameter values and to a sufficient approximation we can take

$$x_t = 0.5\varepsilon_t \quad .$$

A convenient chart for use when, as in this example, manual control action was employed is shown in Figure 11.7a.



(a) AN INTEGRAL ACTION CHART



(b) A ROUNDED INTEGRAL ACTION CHART

Figure 11.7 Feedback charts for integral control action

On this particular process control had previously been carried out using a chart based somewhat arbitrarily on a sequential significance testing scheme. It had turned out in this connection that it was convenient to add or subtract from the catalyst formulation in standard steps. Possible actions were:

no action, $\pm$ one step, or $\pm$ two steps of catalyst formulation.

Significance testing procedures have little relevance in the present context. However, this scheme did have the advantages (i) that it had not been necessary to make changes every time and (ii) when changes were called for they were of one of five definite types, making the procedure easy to apply and supervise. These features can, however, very easily be included in the present control scheme with very little increase in the error by using a "rounded" action chart.

Rounded Charts

A rounded chart is easily constructed from the original chart by dividing the action scale into bands. The adjustment made when an observation falls within the band is that appropriate to the middle point of the band on an ordinary chart. Figure 11.7b shows a rounded chart in which possible action is limited to -2, -1, 0, 1, or 2 catalyst formulation changes. Figures 11.7a and b have been constructed by back calculating the values of a_t from a set of operating data and reconstructing the charts that would have resulted from using an unrounded and a rounded scheme. The increase in mean square error (less than 5% for this example) which results from using the rounded scheme is often outweighed by the convenience of working with a small number of standard adjustments. The effect of rounding is discussed in more detail in Section 11.5.

Example 2

At a later stage of manufacture of the polymer the objective was to maintain the output viscosity Y as close as possible to the target value of 92 by adjusting the gas rate. Hourly determinations of viscosity were made and as a result suitable adjustments were made. We shall discuss here the planning of the pilot control scheme for this process. We later describe how the data collected during the running of this preliminary pilot study was used to reestimate parameters and so to arrive at an improved control scheme. At this stage of the investigation some data was available which showed the variation which occurred in viscosity when no control was applied (when the gas rate was held fixed). This data came from a previous period of operations during which compensations for variations in viscosity were made at a later stage. This data is in fact the 310 observations of Series D in Chapter 4 which was identified as an I.M.A. of order (0,1,1) $\nabla n_t = (1 - \theta B)a_t$ with θ close to zero (that is $\lambda_o = 1 - \theta$ close to unity).

There was good evidence that over the range of operators the steady state relation between gas rate and viscosity was linear and that a unit change in gas rate produced 0.20 units of change in viscosity so that the steady state gain was taken to be $g = 0.20$. Experimental evidence of questioned reliability indicated simple exponential dynamics with no dead time such that about half of the eventual change occurred in one hour.

Thus we have tentatively for the dynamic relationship connecting viscosity Y and gas rate X

$$(1 - 0.5B)Y_t = 0.10\ X_{t-1+}$$

so that $L_1(B) = 1 - 0.5B$, $L_2(B) = 0.10$, $f = 1$.

Also, using the disturbance model

$$\nabla n_t = a_t$$

we have $L_4(B) = 1$, $L_3(B)(1-B) = 1$ and the appropriate feedback control equation is

$$x_t = - \frac{L_1(B) L_3(B)(1-B)}{L_2(B)\ L_4(B)} \varepsilon_t = - \frac{(1 - 0.5B)}{0.10} \varepsilon_t$$

or

$$x_t = - 10\varepsilon_t + 5\varepsilon_{t-1} ,$$

where ε_t is the output deviation from target at time t.

If the action is expressed in terms of the backward difference ∇ we have

$$x_t = -5(1+\nabla)\varepsilon_t \quad \text{or} \quad X_{t+} = -\{5\varepsilon_t + 5S\varepsilon_t\}$$

so that what we have is a combination of "integral" and proportional control.

<u>A projection chart</u>

The situation in which the disturbance n_t can be represented by a linear model of order (0,1,1)

$$\nabla n_t = (1 - \theta B)\, a_t$$

and the dynamic model is of the simple exponential form

$$Y_t = g(1 + \xi\nabla)^{-1} X_{t-1}$$

is of sufficiently common occurrence to warrant special mention. In general, the control adjustment will be

$$x_t = -\frac{(1-\theta)}{g}(1 + \xi\nabla)\,\varepsilon_t \qquad (11.2.9)$$

and

$$X_{t+} = -\left\{\frac{(1-\theta)\xi}{g}\varepsilon_t + \frac{(1-\theta)}{g} S\varepsilon_t\right\} . \qquad (11.2.10)$$

With manual control this proportional-integral action is conveniently indicated by a suitable "projection" chart. That shown in Figure 11.8a which was, in fact, used to implement the control action in the example described above will illustrate the general mode of construction. The deviation from the central target line when read on the viscosity scale will correspond to the ε_t's. A second scale is also shown indicating the control action x_t to be taken, with zero action ($x_t = 0$) aligned with the target value. The scales are arranged so that one unit in the output viscosity (ε_t) scale corresponds to $-\frac{(1-\theta)}{g}$ units on the control action scale.

We can read off the appropriate action at time t by projecting ξ time units ahead a line through ε_t and ε_{t-1} (or equivalently through the last two viscosity measurements). For the present pilot scheme $\xi = 1$ so we must project one time unit ahead. The control action at time $t = 2$, for example, is found by joining the viscosity values at time $t = 1$ and $t = 2$ by a line projecting one step ahead and reading off the value -30 on the action scale. This indicates that the gas rate should be decreased by 30 units and held at the new value until further information becomes available at time $t = 3$.

A rounded chart

As we have mentioned previously exception is sometimes taken to control schemes based on charts like the one above in that they require that action be taken after each observation. It may be felt that action ought to be taken "only when it is necessary." Two different kinds of reasoning may underlie this

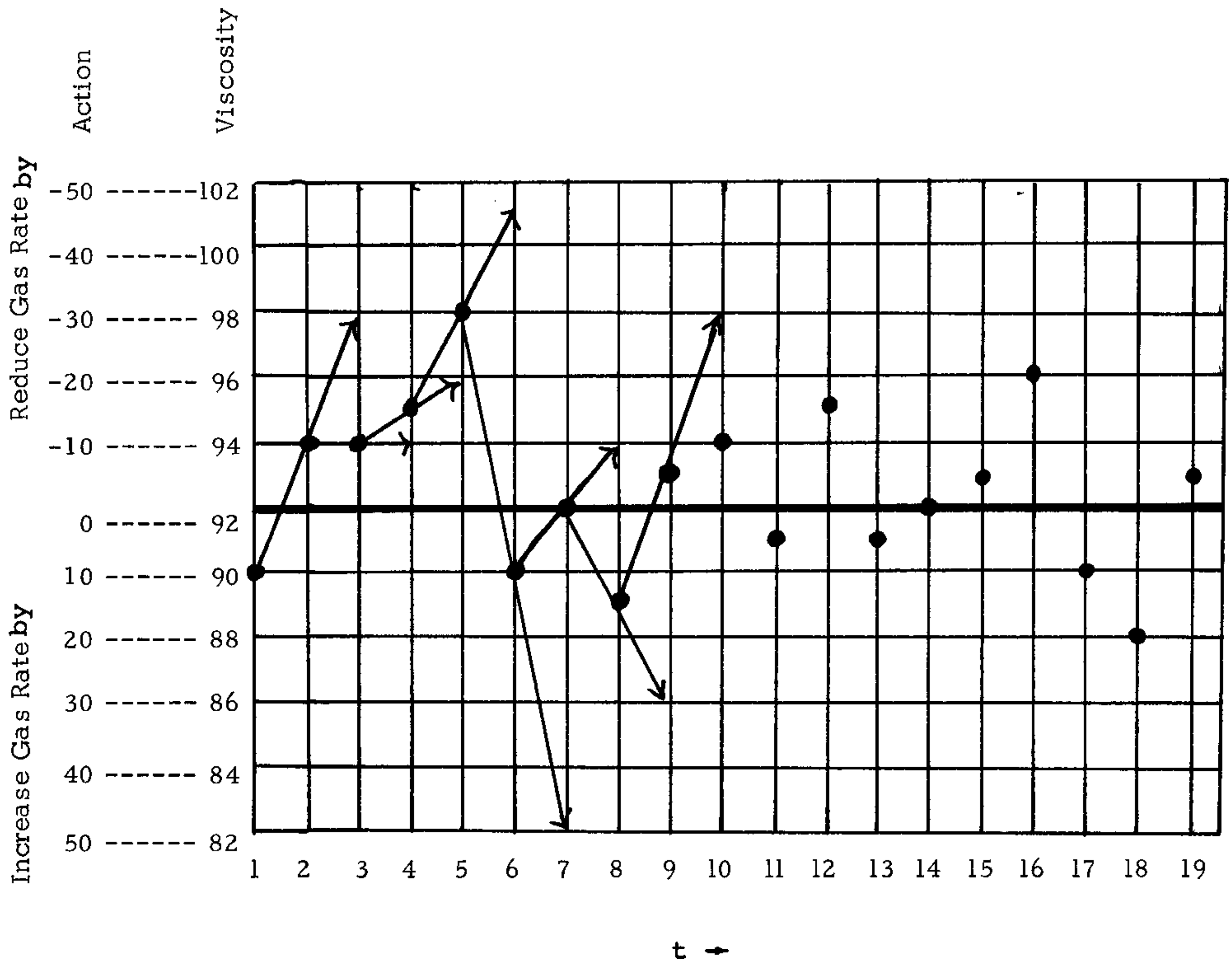


Figure 11.8 a A Proportional-integral action chart.

feeling, one having a more valid basis than the other.

(i) The practioner who is familiar with statistical significance tests and standard control charts may be persuaded that he ought to have real evidence that "the process has deviated from target" before any action is taken. When, as in the mass production metal working industries (where standard quality control procedures have traditionally been used) an additional cost is incurred every time a change is made, it is possible to justify the consequences of this thinking if not the thinking itself [2], [3]. However, in the process industries normally the process operator (or the controlling computer) is going to be on duty anyway to check the process periodically so that there is no additional cost in making a change. In this latter case it is appropriate simply to minimize some measure of deviation from target such as the mean square error and this is what we do here.

(ii) A second and more sensible argument might be that in any industrial operation it is always advantageous to simplify as much as possible the actions that the plant operator is expected to take.

> If a chart could be devised which without much loss required him to take one of a small number of distinct actions this would be an advantage.

As we have seen before, this objective is easily gained by the use of a "rounded" chart. A suitable "rounded" chart for the present example is shown in Figure 11.8b. In this chart the action scale has been divided into 5 bands each 30 gas rate units in width. The bands correspond to the 5 actions: reduce gas rate by 60, reduce gas rate by 30, no action, increase gas rate by 30, increase gas rate by 60. The viscosity is plotted and the points projected exactly as before but the action is "rounded" and corresponds to the central value of the band in which the projected point falls. The chance of a projected point falling outside the outer band is small and such points are treated as having fallen within the appropriate outer band. To put it another way the outer bands are extended to stretch to plus and minus infinity.

The result of using a rounded chart is, of course, to increase somewhat the variance of the output viscosity about target but even with such severe rounding as is illustrated the increase is usually not very great. In section 11.5 we discuss the general question of the effect of added noise in the input of the process. Using the derivation there given, it turns out that the increase in the standard deviation of viscosity about the target produced by the rounding illustrated is about 7%. The points which have

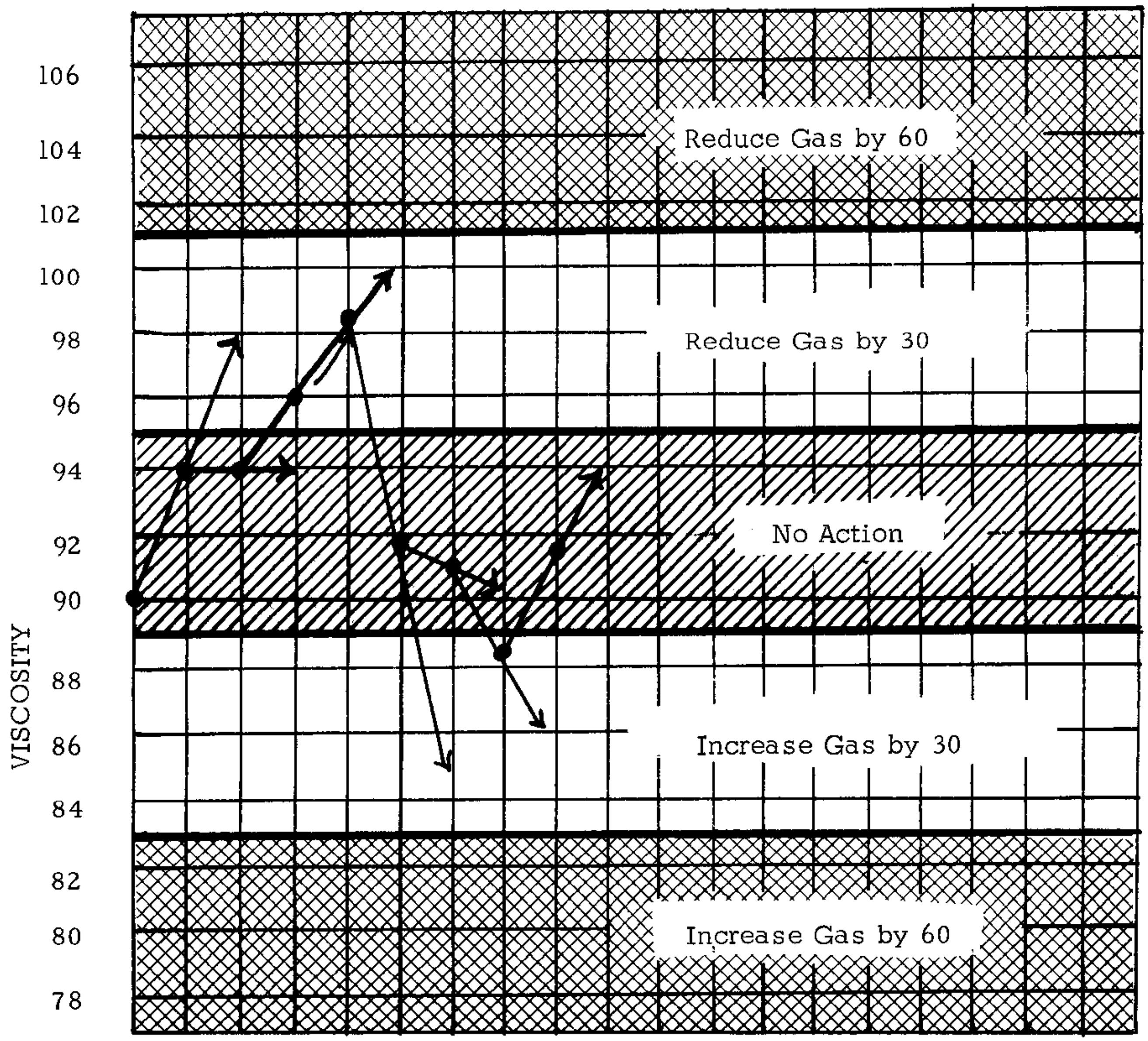


Figure 11.8 b A rounded proportional-integral action chart

been placed on the rounded chart in Figure 11.8b for illustration were, in fact, back calculated assuming that the same disturbance is present as for the unrounded chart in Figure 11.8a. It is shown later that provided δ is not too close to 1 (that is, provided the time constant of the system is not too long compared with the sampling interval) a rounding interval I as wide as one standard deviation of x may be used without causing a large increase in the variance of the output.

For the particular choice $I = \sigma_x$ and assuming a Normal distribution we would have the following distribution of actions

Zone	$-\infty$ to $-1\frac{1}{2}\sigma_x$	$-1\frac{1}{2}\sigma_x$ to $-\frac{1}{2}\sigma_x$	$-\frac{1}{2}\sigma_x$ to $+\frac{1}{2}\sigma_x$	$\frac{1}{2}\sigma_x$ to $1\frac{1}{2}\sigma_x$	$1\frac{1}{2}\sigma_x$ to ∞
Action	$-2\sigma_x$	$-\sigma_x$	0	σ_x	$2\sigma_x$
Probability %	6.7	24.2	38.3	24.2	6.7

Strictly speaking the theoretical results concerning the increase to be expected in σ_x due to rounding assume that there are also zones centered on $3\sigma_x$, $-3\sigma_x$, $4\sigma_x$, $-4\sigma_x$, and so on. However, the total probability of a point falling into these outer zones would be only 1.24% and the effect of combining them all into the $\pm 2\sigma_x$ zones is small.

Specifically it can be shown that for a scheme of the type considered here if the rounding interval is $R\sigma_x$ where σ_x^2 is the variance of the x without rounding

$$\sigma_x^2 = \frac{(1-\theta)^2(1+\delta^2)}{g^2(1-\delta)^2} \quad \sigma_a^2 \quad , \tag{11.2.11}$$

then to a close approximation the standard deviation of the output is increased by the factor F where

$$F^2 = 1 + \frac{R^2}{12} \quad \frac{(1+\theta\delta)(1-\theta)(1+\delta^2)}{(1-\theta\delta)(1+\theta)(1-\delta^2)} \quad . \tag{11.2.12}$$

For the chart in Figure 11.8b $\theta = 0$, $\delta = 0.5$, $R \approx 1$ so that $F \simeq 1.07$.

Example 3

For further illustration we consider the slightly more complicated situation which occurs when the dynamics may be represented by a first order system with dead time (delay). Thus with

$$\nabla Y_t = g(1+\xi\nabla)^{-1} \quad \{(1-\nu)x_{t-f}+\nu x_{t-f-1}\}$$

we have now

$$L_1(B)/L_2(B) = (1+\xi\nabla) \quad \{g(1-\nu\nabla)\} \quad .$$

If the disturbance n_t is represented as before by a process of order (0,1,1)

$$\nabla n_t = (1-\theta B)\, a_t \quad ,$$

we find that $\hat{n}_t(f) - \hat{n}_{t-1}(f) = (1-\theta)\, a_t$

$$\hat{e}_t(f) = 1 + (1-\theta) \quad \{1+B+B^2+\ldots+B^{f-1}\} \quad a_{t+f}$$

so that $L_3(B)(L-B) = (1-\theta)_{\scriptscriptstyle\supset}$, $L_4(B) = 1+(1-\theta)\ \{1+B+B^2+\ldots+B^{f-1}\}$.

Thus using (11.2.7) the optimal action is given by setting x_t so that

$$\{1-\nu\nabla\}\ \{1+(1-\theta)(B+B^2+\ldots+B^{f-1})\}x_t = -\frac{(1-\theta)}{g}\ (1+\xi\nabla)\varepsilon_t$$

$$x_t = -\ (1-\theta)(X_{t-1}-X_{t-f}) - \frac{(1-\theta)(1+\xi\nabla)}{g(1-\nu\nabla)}\ \varepsilon_t \qquad (11.2.13)$$

We notice that the introduction of delay into the dynamic model results in a mode of control in which the present adjustment depends on past <u>action</u> over the period of the delay as well as on present and past errors ε_t. In particular in the common situation where f = 1 we obtain

$$x_t = \nu\nabla x_t - \frac{(1-\theta)}{g}\ (1+\xi\nabla)\varepsilon_t \quad .$$

<u>A delay nomogram</u>

Using the same argument as before it is very easy to design a nomogram to compute the required action

$$(1-\nu\nabla)x_t = -\ \frac{(1-\theta)}{g}\ (1+\xi\nabla)\ \varepsilon_t \quad . \qquad (11.2.14)$$

Suppose we had an example with the same background as before where it was desired to maintain viscosity at the value 92 as nearly as possible. Suppose now, however, that

$$\theta = 0.5 \qquad \xi = 0.7 \qquad \nu = 0.25 \qquad g = 0.20$$

then the required adjustment is

$$x_t = 0.25\nabla x_t - 2.50\varepsilon_t - 1.75\nabla\varepsilon_t$$

that is $$x_t = -0.33x_{t-1} - 5.67\varepsilon_t + 2.33\varepsilon_{t-1} \quad .$$

This action is computed by the nomogram of Figure 11.9 with scales A,B,E,D indicating respectively ε_t, ε_{t-1}, x_t, x_{t-1} and a scale C used to equate the two sides of the control equation. The scales are arranged so that

(i) zero action and target value are aligned

(ii) one unit in the viscosity scale is equal to $-\frac{(1-\theta)}{g} = -2.5$ units in the gas rate scale.

(iii) the distances between the scales are such that $AC/AB = \xi = 0.7$, $CE/DE = \nu = 0.25$.

On the nomogram shown in Figure 11.9, a value of 92 for the viscosity has just come to hand. A straight line joining this to the previous viscosity reading of 96 is projected to cut the C scale at a point marked X. A line drawn through X and the value -32 corresponding to the previous adjustment cuts the action scale at 20. This tells us that the present optimal adjustment is to increase the gas rate by 20 units.

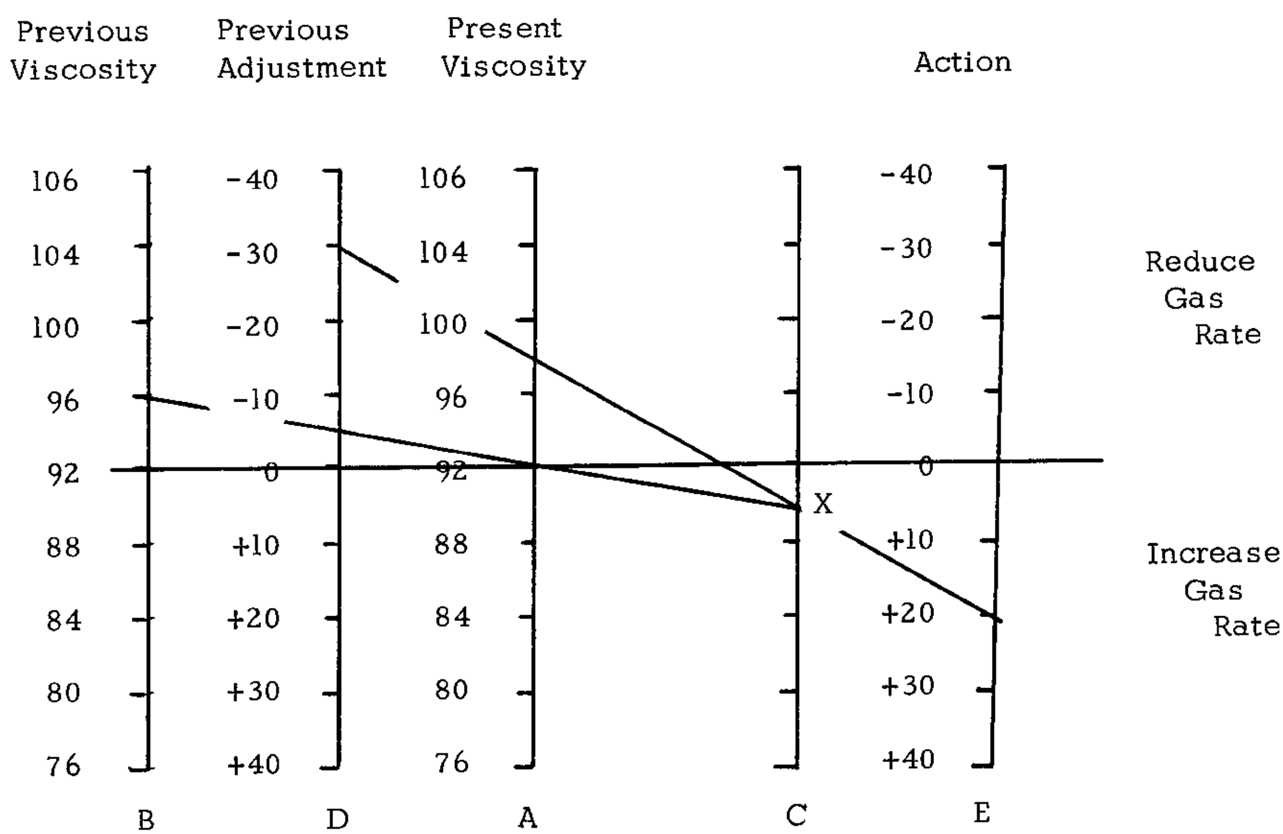


Figure 11.9 Nomogram for control when there is dead time in the system

It may be remarked that in this particular example the current value of viscosity is on target. Nevertheless, taking into account the previous behavior of the process and its dynamic-stochastic characteristics corrective action is still called for. The plant operator must increase the gas rate by 20 units if he is to follow a policy which will minimize the mean square deviation from target viscosity.

As before, if it were desired to simplify the control action a "rounded" nomogram with the action scale divided up into a suitable number of zones could be used.

11.3 Feedforward-feedback control

As we have mentioned before where possible identifiable disturbances should be eliminated by feedforward control and the remainder of the disturbance dealt with by feedback control. Figure 11.10 shows part of a combined feedforward-feedback scheme in which k identifiable disturbances $z_1, z_2, \ldots, z_k$ are fed forward. It is supposed that n_t' is a further unidentified disturbance and that

$$n_t = n_t' + \sum_{j=1}^{k} \delta_j^{-1}(B)\omega_j(B)\ e_{j,t-f}(f-b_j)$$

(with $e_{j,t-f}(f-b_j) = 0$ if $f-b_j \leq 0$) is the same noise augmented by any further noise coming from errors in forecasting the identifiable inputs and such that

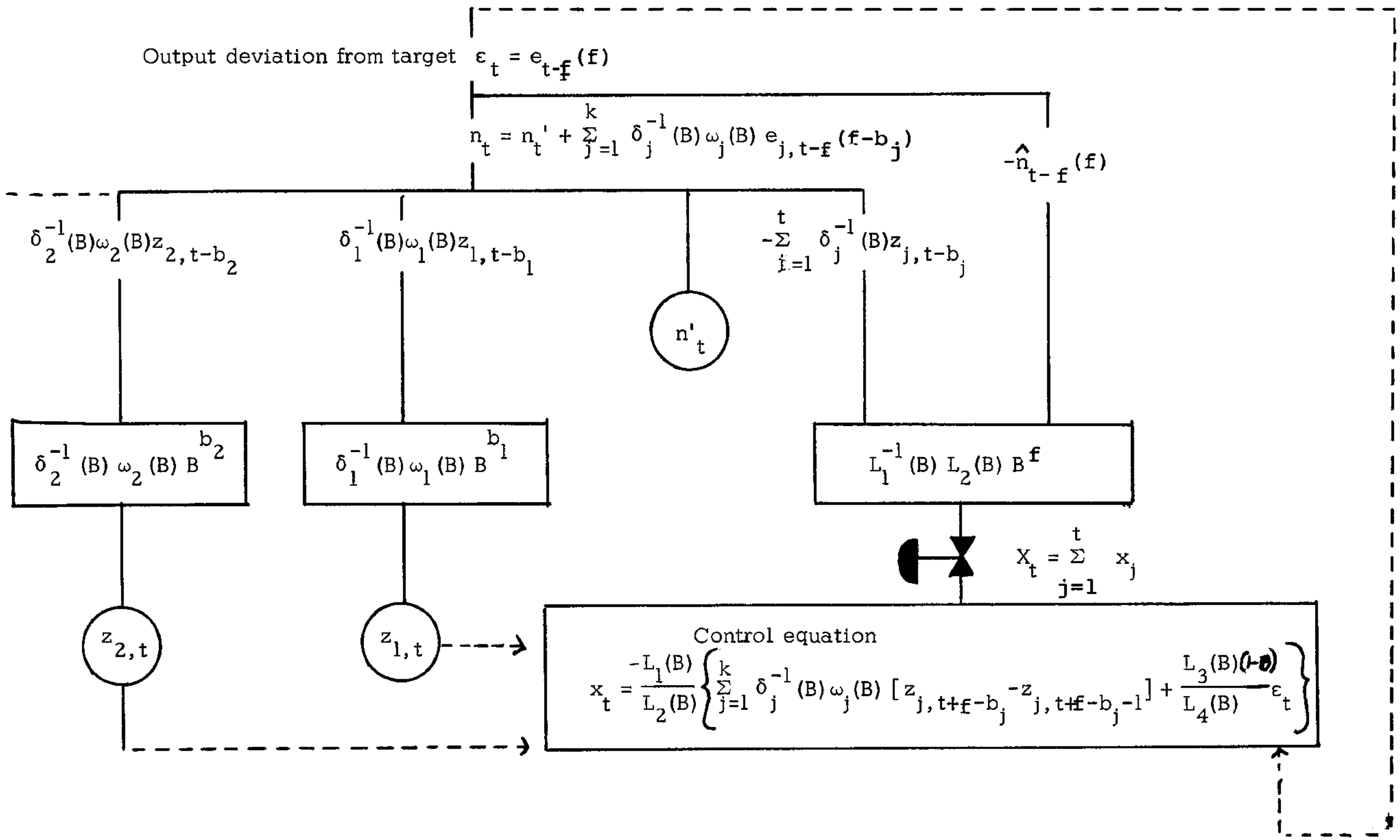


Figure 11.10 Mixed feedforward-feedback control scheme at time t

$$\frac{L_3(B)(1-B)}{L_4(B)}\varepsilon_t = \hat{n}_t(f) - \hat{n}_{t-1}(f) \quad .$$

11.3.1 Feedforward-feedback control to minimize output mean square error

Arguing as before the optimal control action is

$$x_t = -\frac{L_1(B)}{L_2(B)}\left\{\sum_{j=1}^{k} \delta_j^{-1}(B)\omega_j(B)\,[z_{j,t+f-b_j} - z_{j,t+f-b_j-1}] + \frac{L_3(B)(1-B)}{L_4(B)}\varepsilon_t\right\} \qquad (11.3.1)$$

where

$$z_{j,t+f-b_j}-z_{j,t+f-b_j-1} = \begin{cases} z_{j,t+f-b_j}-z_{j,t+f-b_j-1}, & f-b_j \leq 0 \\ z_{j,t}(f-b_j)-z_{j,t-1}(f-b_j), & f-b_j > 0 \end{cases} \qquad (11.3.2)$$

In the diagram the output for the right hand box is split into two parts only for diagnostic convenience.

11.3.2 An Example of feedforward-feedback control

We illustrate by discussing further the example used in Section 11.1.2 where it was desired to control specific gravity as close as possible to a target value 1.260. Study of the deviations from target occurring after feedforward control showed that they could be represented by the I.M.A. process of order (0,1,1)

$$\nabla n_t = (1 - 0.5B)a_t$$

where the a_t are a sequence of uncorrelated random variables.

Thus

$$\frac{L_3(B)(1-B)}{L_4(B)} a_t = \hat{n}_t(1) - \hat{n}_{t-1}(1) = 0.5a_t = e_{t-1}(1) = \varepsilon_t$$

and with the remaining parameters as before, namely

$$\delta^{-1}(B)\omega(B) = 0.0016 \qquad , \qquad b = 0$$

$$L_2^{-1}(B)L_1(B) = \frac{(1 - 0.7B)}{0.0024} \qquad , \qquad f = 1$$

and $$\hat{z}_t(1) - \hat{z}_{t-1}(1) = \frac{0.5}{1 - 0.5B} (z_t - z_{t-1}) \quad .$$

Using (11.3.1) the optimal adjustment incorporating feedforward and feedback control is

$$x_t = - \frac{\{1 - 0.7B\}}{0.0024} \left[\frac{(0.0016)(0.5)}{1 - 0.5B} \{z_t - z_{t-1}\} + 0.5\varepsilon_t \right] \quad (11.3.3)$$

i.e. $x_t = 0.5x_{t-1} - 0.33(1-0.7B)(z_t - z_{t-1}) - 208(1-0.7B)(1-0.5B)\varepsilon_t$

or $x_t = 0.5x_{t-1} - 0.33z_t + 0.56z_{t-1} - 0.23z_{t-2} - 208\varepsilon_t + 250\varepsilon_{t-1} - 73\varepsilon_{t-2}$. (11.3.4)

Figure 11.11 shows the section of record previously given in Figure 11.4 when only feedforward control was employed and the corresponding calculated variation that would have occurred if no control had been applied. This is now compared with a record from a scheme using both feedforward and feedback control. The introduction of feedback control resulted in a further substantial reduction in mean square error and corrected the tendency to drift from target which was experienced with the feedforward scheme.

Note that with a feedback scheme, the correction employs a forecast having lead time f whereas with a feedforward scheme the forecast has lead time f-b and no forecasting is involved if f-b is zero or negative. Feedforward control thus gains in the immediacy of possible adjustment whenever b is greater than zero.

The example we have quoted is exceptional in that b = 0 and consequently no advantage of immediacy is in this case gained by feedforward control. It might be true in this case that equally good control could have been obtained by feedback alone. In practice possibly because of error transmission problems the mixed scheme did rather better than the pure feedback system.

11.3.3 Advantages and disadvantages of feedforward and of feedback control

With feedback control it is the total disturbance as

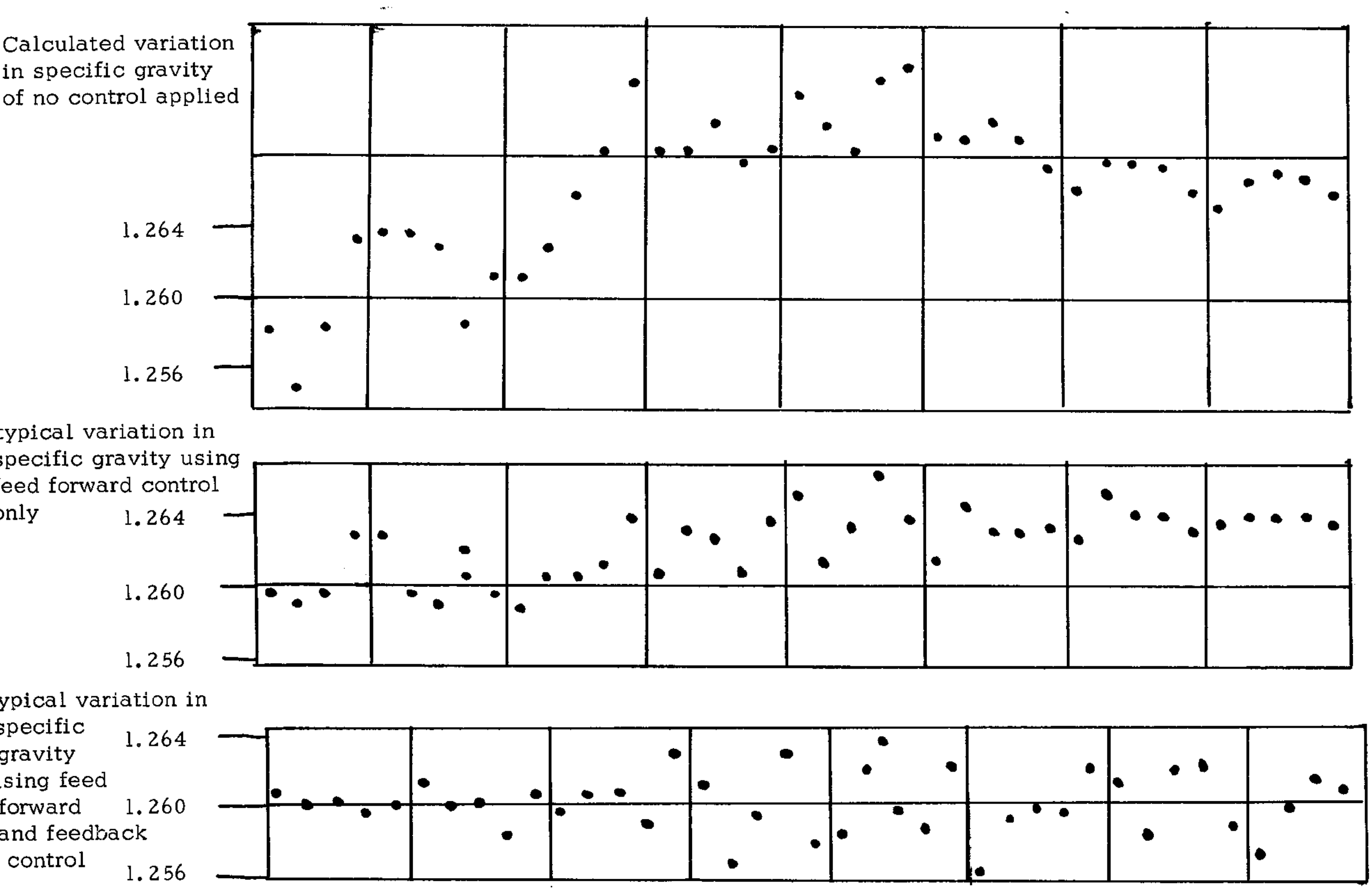


Figure 11.11 Typical variation in specific gravity with: no control, feedforward control only, and with feedforward-feedback control

evidenced by the error at the output that actuates compensation. It is, therefore, not necessary to be able to identify and measure the sources of disturbance. All that is needed is that we characterize the disturbance n_t at the output by an appropriate stochastic process. Because we are not relying on "dead reckoning," unexpected disturbances and moderate errors in estimating the system's characteristics will normally result only in greater variation about the target value and not (as may occur with feedforward control) in a consistent drift away from the target value. On the other hand especially if the delay f is large the errors about the target (since they are then the errors of a remote forecast) may be large even though they have zero mean.

Clearly if identifiable sources of disturbance can be partially or wholly got rid of by feedforward control then this should be done. Only the unidentifiable error has then to be dealt with by feedback control.

In summary then although we can design a feedback scheme which is optimal, in the sense that it is the best possible feedback scheme, it will not usually be as good as a combined feedforward-feedback scheme in which sources of eliminatable error are got rid of before the feedback loop.

11.4 Fitting dynamic-stochastic models using operating data

11.4.1 Iterative model building

It is desirable that the parameters of a control system be estimated from data collected under as nearly as possible the conditions which will apply when the control scheme is in actual operation. The calculated control action using estimates so obtained properly takes account of noise in the system which will be characterized as if it entered at the point provided for it in the model (see Section 11.5.3). This being so, it is desirable to proceed iteratively in the development of a control scheme. Using technical knowledge of the process together with whatever can be gleaned from past operating data, preliminary stochastic and dynamic models are postulated and used to design a pilot control scheme. The operation of this pilot scheme then quickly supplies further data which may be analyzed to give improved estimates of the stochastic and dynamic models which can then be used to plan an improved scheme.

11.4.2 Estimation from operating data

It will be sufficient to consider a feedforward-feedback scheme not necessarily optimal and with a single feedforward input. If we suppose at first that b-f is positive, then whether the inputs z_t and X_t it will be true that

$$\varepsilon_t = n_t + \delta^{-1}(B)\omega(B)z_{t-b} + L_1^{-1}(B)L_2(B)X_{t-f} \qquad (11.4.1)$$

where $n_t = \Phi^{-1}(B)\theta(B)a_t$. If $\Phi(B) = \phi(B)\nabla^d$

then

$$a_t = \phi(B)\theta^{-1}(B) \left\{\nabla^d\varepsilon_t - \delta^{-1}(B)\omega(B)\nabla^d z_{t-b} - L_1^{-1}(B)L_2(B)\nabla^d X_{t-f}\right\} \qquad (11.4.2)$$

In simple cases we can now investigate the estimation situation and estimate the parameters by plotting Σa_t^2 for a grid of values of the parameters.

Alternatively and more generally we can use the non-linear least squares iterative routine to obtain the estimates. This we can do following precisely the procedure described in Chapters 5 and 10.

Equation (11.4.2) allows the a_t's to be calculated for any chosen values of the parameters. As before, therefore, we need only program the recursive calculation of the a_t's and insert this sub-routine into the general non-linear estimation program which computes the derivatives numerically and automatically proceeds with the iteration.

Alternatively the appropriate Newton-Gauss algorithm can be employed explicitly. Starting with preliminary estimates for the parameters and knowing ε_t, z_t, and X_t, we can compute the quantities

$$a_t^o = \phi_o(B)\theta_o^{-1}(B) \left\{\nabla^d\varepsilon_t - \delta_o^{-1}(B)\omega_o(B)\nabla^d z_{t-b} - L_{10}^{-1}(B)L_{20}(B)\nabla^d X_{t-f}\right\} \quad (11.4.3)$$

$$\zeta_t^o = \phi_o(B)\theta_o^{-1}(B)\delta_o^{-1}(B)\omega_o(B)\nabla^d z_t \quad (11.4.4)$$

$$\xi_t^o = \phi_o(B)\theta_o^{-1}(B)L_{10}^{-1}(B)L_{20}(B)\nabla^d X_t \quad (11.4.5)$$

and then obtain improved estimates from the approximate linearization

$$\begin{aligned} a_t^o \approx & -(\phi(B)-\phi_o(B))\left\{\phi_o^{-1}(B)a_t^o\right\} + (\theta(B)-\theta_o(B))\left\{\theta_o^{-1}(B)a_t^o\right\} \\ & -(\delta(B)-\delta_o(B))\left\{\delta_o^{-1}(B)\zeta_{t-b}^o\right\} + (\omega(B)-\omega_o(B))\left\{\omega_o^{-1}(B)\zeta_{t-b}^o\right\} \\ & -(L_1(B)-L_{10}(B))\left\{L_{10}^{-1}(B)\xi_{t-f}^o\right\} + (L_2(B)-L_{20}(B)) \\ & \left\{L_{20}^{-1}(B)\xi_{t-f}^o\right\} + a_t \end{aligned} \quad (11.4.6)$$

When b-f is negative there is advantage in aggregating the noise $n_t = n_t' + e_{t-f}(f-b)$ as illustrated in Figure 11.10 and estimating an overall model for n_t.

<u>Feedback Control</u>

When we have only a feedback system as in Figure 11.6, Equation (11.4.2) simplifies to

$$a_t = \phi(B)\theta^{-1}(B) \left\{\nabla^d\varepsilon_t - L_1^{-1}(B)L_2(B)\nabla^d X_{t-f}\right\} \quad (11.4.7)$$

and the estimation proceeds exactly as before but with the term in z_t omitted.

As usual at the beginning of the recursive calculation we may need values of the various series which have occurred before the process was observed. The ways in which this problem may be dealt with are discussed and illustrated in the example that follows.

11.4.3 An Example

In the second feedback control example in Section 11.2.3 the objective was to maintain the viscosity of a polymer as close as possible to the target value of 92 by hourly readings of viscosity and adjustment of the gas rate. The previous discussion was concerned with the design of a pilot control scheme based on information of questionable accuracy. Essentially the pilot scheme assumed that the stochastic and dynamic models were

$$\nabla n_t = (1-\theta B)a_t \tag{11.4.8}$$

$$(1-\delta B)Y_t = g(1-\delta)X_{t-1+} \tag{11.4.9}$$

with $\theta = 0$, $\delta = 0.5$, $g = 0.20$.

These models led to the equation $x_t = -10\varepsilon_t + 5\varepsilon_{t-1}$ as defining the optimal adjustment at time t. Part of the actual operating record using this scheme is shown in Figure 11.12. The

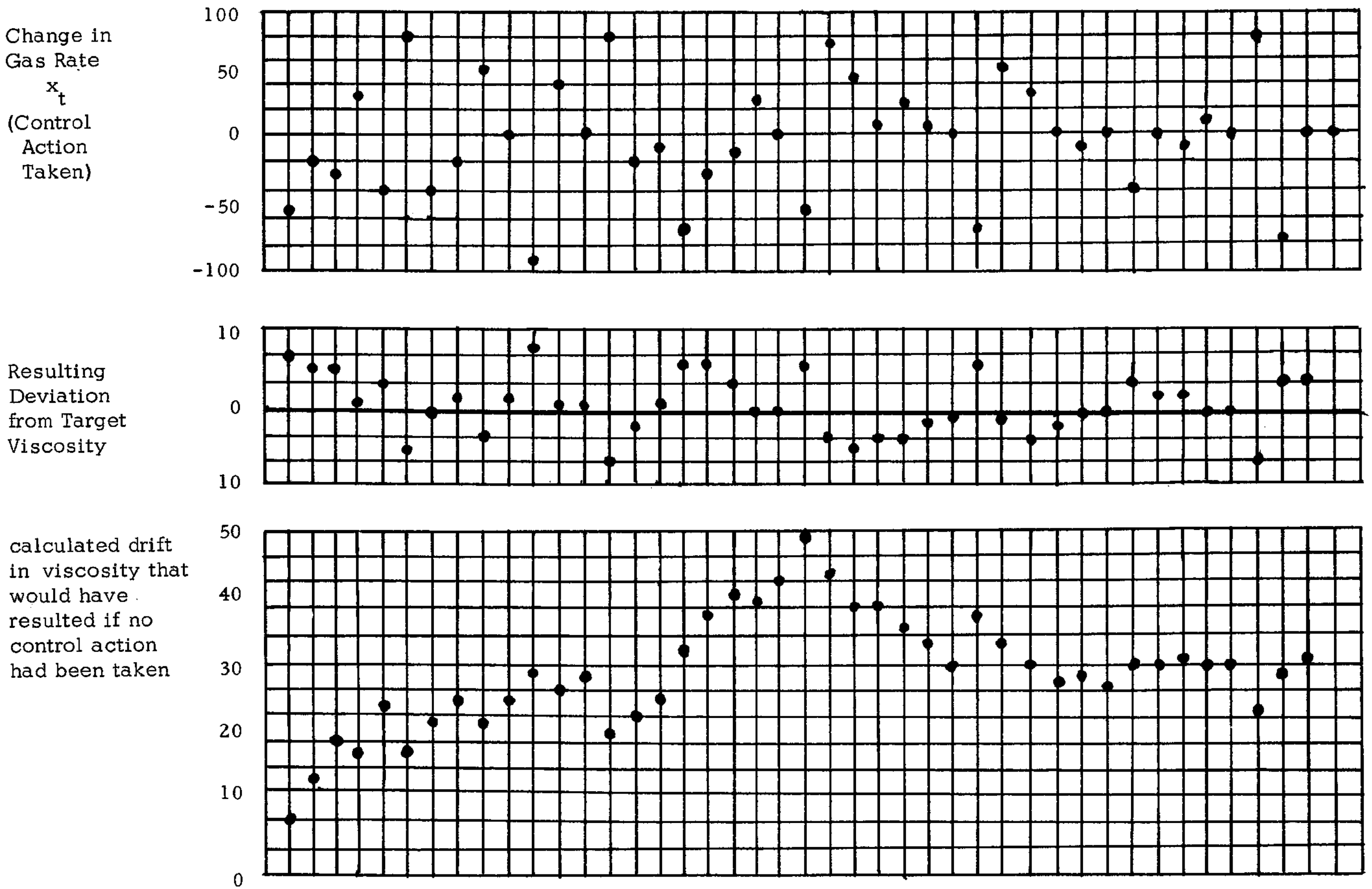


Figure 11.12 Pilot scheme for control of viscosity part of the operating record with reconstructed disturbance.

changes in gas rate x_t and the corresponding deviations from target ε_t now supply the data from which new estimates may be obtained. We proceed on the assumption that the form of model is adequate but that the estimates of the parameters θ, δ, and g may be in error. In this case (11.4.7) reduces to

$$a_t = (1-\theta B)^{-1} \nabla\varepsilon_t - (1-\delta B)^{-1}(1-\delta)gx_{t-1} \qquad (11.4.10)$$

Writing $y_t = (1-\delta B)^{-1}(1-\delta)gx_{t-1}$, the model is represented by the pair of equations

$$\begin{aligned} a_t &= \theta a_{t-1} + \nabla\varepsilon_t - y_t \\ y_t &= \delta y_{t-1} + (1-\delta)gx_{t-1} \end{aligned} \qquad (11.4.11)$$

For illustration, the set of eight pairs of values of x_t and ε_t given in Table 11.2 were taken from a series consisting of 312 observations made during 13 days of running of the pilot scheme.

t	1	2	3	4	5	6	7	8
x_t	30	0	-10	0	-40	0	-10	10
ε_t	-4	-2	0	0	4	2	2	0

Table 11.2 Eight pairs of values of (x_t, ε_t) series from pilot scheme.

The complete pair of time series is given in Appendix A11.1.

Table 11.3 shows the beginning of the recursive calculation of a_t^o for the parameter values $\theta = 0.2$, $\delta = 0.6$, $g = 0.25$. For these values equations (11.4.11) become

$$a_t^o = 0.2a_{t-1}^o + \nabla\varepsilon_t - y_t^o \quad (11.4.12)$$

$$y_t^o = 0.6y_{t-1}^o + 0.1x_{t-1} \quad (11.4.13)$$

The data are given in columns (1), (2), and (3) of Table 11.3. The entries in column (4) are obtained using (11.4.13) and represent the changes at the output which are produced by the changes x_t. Columns (5) and (6) are obtained by simple arithmetic and column (7) from (11.4.12). In this table y_1 and a_1 have been inserted for the unknown starting values. The entries in the table show the influence which the choice of these values has on subsequent calculations.

A number of points are clarified by the table.

(1) We notice that the choices of a_1 and y_1 influence only the first few values of a_t^o. This will be true more generally except for parameter values in ranges for which the weight functions for the noise model or for the dynamic model are very slow to die out.

t	x_t	ε_t	$y_t^o = 0.6y_{t-1}^o + 0.1x_{t-1}^o$	$\nabla\varepsilon_t$	$\nabla\varepsilon_t - y_t^o$	$a_t^o = 0.2a_{t-1}^o + (\nabla\varepsilon_t - y_t^o)$
1	30	-4	y_1^o			a_1^o
2	0	-2	$0.60y_1^o + 3.00$	2	$-1.00-0.60y_1^o$	$-1.00+0.20a_1^o-0.60y_1^o$
3	-10	0	$0.36y_1^o + 1.80$	2	$0.20-0.36y_1^o$	$0.04a_1^o-0.48y_1^o$
4	0	0	$0.22y_1^o + 0.08$	0	$-0.08-0.22y_1^o$	$-0.08+0.01a_1^o-0.31y_1^o$
5	-40	4	$0.13y_1^o + 0.05$	4	$3.95-0.13y_1^o$	$3.93 \quad -0.19y_1^o$
6	0	2	$0.08y_1^o - 3.97$	-2	$1.97-0.08y_1^o$	$2.76 \quad -0.12y_1^o$
7	-10	2	$0.05y_1^o - 2.38$	0	$2.38-0.05y_1^o$	$2.93 \quad -0.07y_1^o$
8	10	0	$0.03y_1^o - 2.43$	-2	$0.43-0.03y_1^o$	$1.02 \quad -0.04y_1^o$

Table 11.3 Recursive calculation of a_t^o for data from pilot scheme for parameter values $\theta = 0.2$, $\delta = 0.6$, $g = 0.25$

With the approach we adopt here the true values of the parameters are unlikely to be within these critical ranges.

(2) When, as in this example, data is cheap we can substitute guesses for a_1 and y_1 and throw away the first few values of a_t^o to allow transients arising from non optimal choice of a_1 and y_1 to die out.

(3) On the usual assumption of Normality for the a's, the maximum likelihood solution will be given by treating a_1 and y_1 as nuisance parameters where values have to be estimated. They may then be treated in exactly the same manner as are the other parameters a_1 and y_1 being set equal to guessed values and the general non-linear routine applied with derivatives determined numerically.

Alternatively the values of the a_t's with those starting values a_1 and y_1 which give a minimum sum of squares <u>conditional</u> on the choice of the "main" parameters may be computed and employed in subsequent least squares calculations. We illustrate with the data of Table 11.3 where the calculation is particularly simple. The values a_1^o and y_1^o which minimize

Σa_j^{o2} for the particular choice of parameters $\theta = 0.2$, $\delta = 0.6$, $g = 0.25$ are found by "regressing" column (a) on columns (b) and (c) in Table 11.4.

(a)	(b)	(c)
0.00	-1.00	0.00
-1.00	-0.20	0.60
0.00	-0.04	0.48
-0.08	-0.01	0.31
3.93	0.00	0.19
2.76	0.00	0.12
2.93	0.00	0.07
1.02	0.00	0.04

Table 11.4 Calculation of maximum likelihood estimates of starting values.

The elements in the table are all taken from the extreme right-hand column of Table 11.3. The elements in column (a) are the constant terms and the elements of columns (b) and (c) are the coefficients of $-a_1^o$ and $-y_1^o$ respectively. Because the coefficients in columns (b) and (c) rapidly die out, for the purpose of computing $\hat{a}_1^o$ and $\hat{y}_1^o$ we need be concerned only with the first values of the series. In fact for the particular case considered above we need only take account of the first eight entries. The normal equations are then

$$0.2008 = 1.0417a_1^o - 0.1423y_1^o$$

$$0.6990 = 0.1423a_1^o + 0.7435y_1^o$$

yielding $\hat{a}_1^o = 0.33$ $\hat{y}_1^o = 1.00$ for the starting values

The nature of the sum of squares surface for this example can be seen from Figure 11.13. The contours were obtained by interpolating in a grid of computed values. In each case starting values were obtained in the manner described above. The The approximate three dimensional 95% confidence region is indicated by the shaded region.

As an additional check the non-linear least squares routine was run using starting values employed in the pilot control scheme. The iteration proceeded as shown in Table 11.5.

Iteration	θ	$w = (1-\delta)g$	δ	Sum of Squares
0	0.01	0.10	0.50	6,247.6
1	-0.06	0.09	0.53	5,661.3
2	-0.11	0.08	0.61	5,275.9
3	-0.02	0.06	0.71	5,115.9
4	0.08	0.05	0.77	5,067.6
5	0.10	0.05	0.77	5,065.2
6	0.11	0.05	0.77	5,065.1
7	0.11	0.05	0.77	5,065.1

Table 11.5 Convergence of parameters in simultaneous fitting of dynamic and stochastic models

The sample autocorrelation function for the residual $\hat{a}_t$'s is shown in Table 11.6 together with the sample cross correlation function between the $\hat{a}_t$'s and the x_t's.

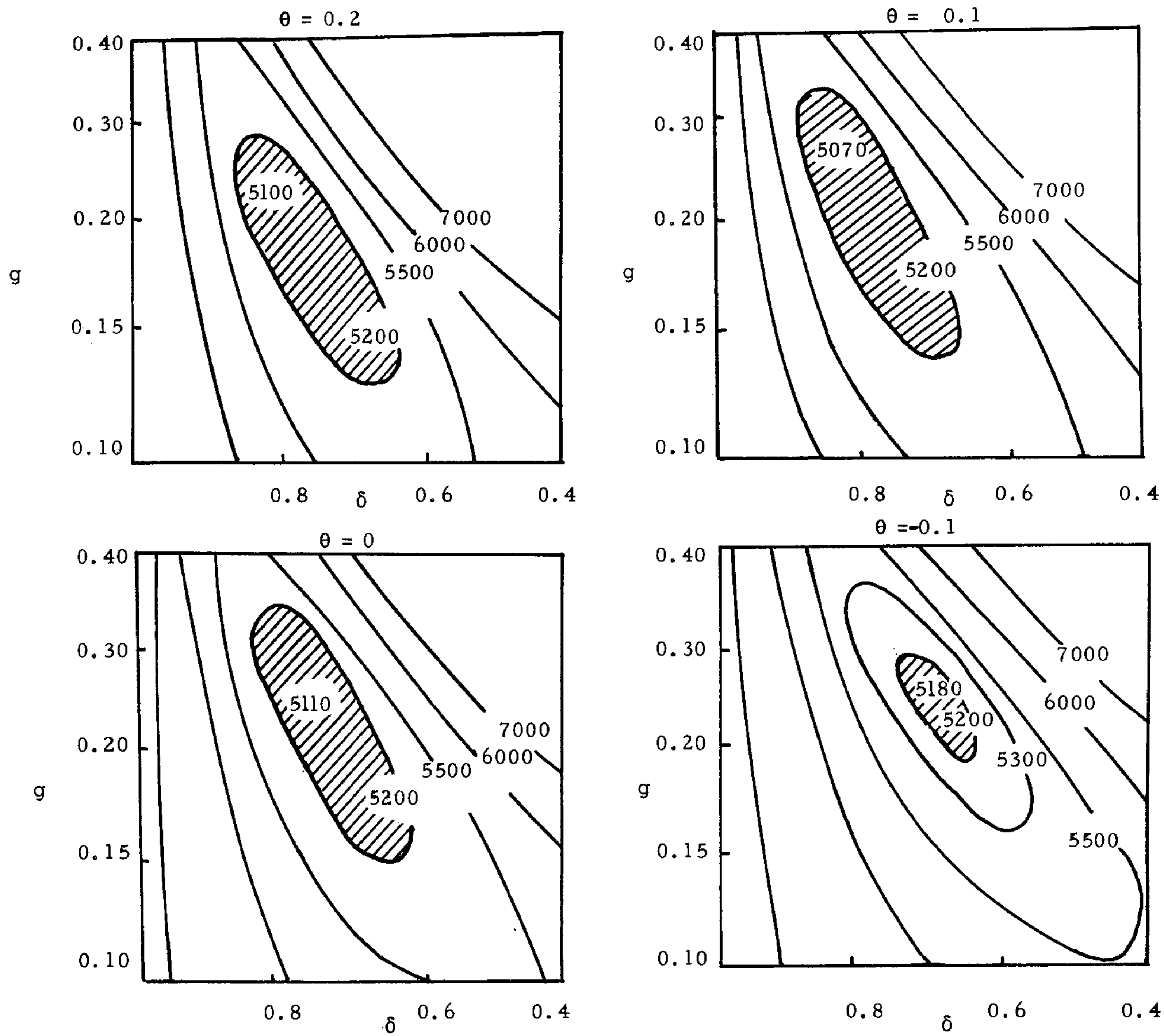


Figure 11.13 SUMS OF SQUARES CONTOURS AND APPROXIMATE 95% CONFIDENCE REGION FOR (θ, g, δ) USING DATA FROM PILOT CONTROL SCHEME

Autocorrelations

												s.e.
Lags	1-10	0.01	-0.06	-0.06	0.05	-0.02	0.06	-0.04	-0.04	0.11	0.03	± 0.06
	11-20	0.03	-0.04	0.00	0.03	0.08	-0.10	0.05	0.07	0.03	-0.04	± 0.06
	21-30	-0.07	-0.02	-0.05	0.09	0.00	0.01	0.03	0.00	-0.02	0.00	± 0.06

Crosscorrelations

Lags	1-10	-0.01	-0.05	0.09	-0.02	0.05	-0.08	0.03	0.03	-0.08	0.01
	11-20	0.00	0.08	-0.02	-0.02	-0.06	0.07	-0.13	-0.03	0.01	-0.01
	21-30	0.06	0.04	0.06	-0.08	-0.02	-0.04	0.01	-0.01	0.04	-0.03

Table 11.6 Sample autocorrelations of $\hat{a}_t$'s and crosscorrelations with x_t's.

It is clear that in this example the estimates $\theta = 0$, $g = 0.20$ used in the pilot scheme were about right but the value $\delta = 0.5$ was too high for the estimate of the dynamic parameter, a value of approximately 0.3 now being indicated. As a result of the reestimation of the parameters, the control scheme

$$x_t = -10\varepsilon_t + 5\varepsilon_{t-1}$$

was changed to

$$x_t = -22.5\varepsilon_t + 18\,\varepsilon_{t-1} \quad .$$

11.5 Effect of added noise in feedback schemes

In what has gone before we have emphasized the importance of estimating the parameters of the system under as nearly as possible the actual control conditions which will be obtained in the final scheme. The main reason for this is to ensure that all sources of noise are taken account of. If we estimate the system parameters under working control conditions then we will automatically estimate the noise as if it all originated at the source provided for it in the model. The effect of this will be that parameter estimates will be obtained which will give near optimal control action under actual working conditions.

By contrast suppose the stochastic and dynamic models were estimated "piecemeal." For example, we might use records which indicated the noise actually originating at P in Figure 11.14 to estimate the noise model for n_t. Provided the amount of additional noise was not excessive the control scheme obtained using this estimate might still be reasonably good. However, the ignoring of large additional noise sources could lead to inefficient control action.

In the sections that follow we investigate for a feedback scheme the following problems:

1) The effect of ignoring added noise
2) "Rounding" the control action as a source of added noise
3) Differences in optimal action produced by added noise
4) Effective transference of the noise origin which occurs when data are collected under operating conditions similar to that obtained in the final control scheme.

11.5.1 Effect of ignoring added noise - rounded schemes

Consider the feedback control loop of Figure 11.14 in which the noise actually originating at P is n_t and $\nabla^d n_t = \phi^{-1}(B)\theta(B)a_t$. As has already been shown, on the assumption

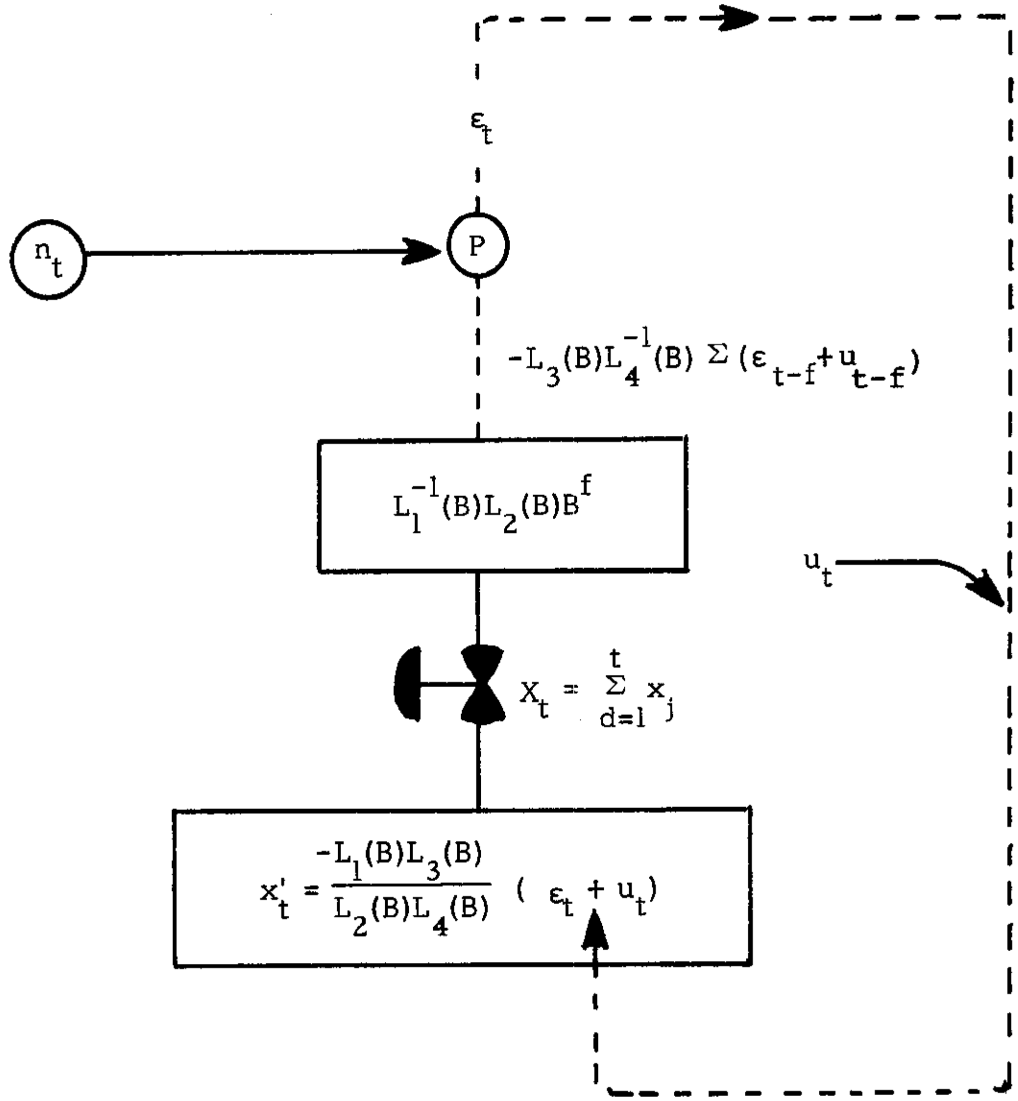


Figure 11.14 Feedback control with error in the loop

that n_t is the only noise component, optimal action results from the control equation

$$X_{t+} = - \frac{L_1(B)L_3(B)}{L_2(B)L_4(B)} \varepsilon_t \qquad (11.5.1)$$

with $\hat{n}_t(f) = L_3(B)a_t$, $e_{t-f}(f) = L_4(B)a_t$, $\varepsilon_t = e_{t-f}(f)$.

Suppose now that an additional source of noise exists so that the action actually taken is

$$X_{t+} = - \frac{L_1(B)L_3(B)}{L_2(B)L_4(B)} (\varepsilon_t + u_t') \ . \qquad (11.5.2)$$

Then at P,

$$\varepsilon_t = n_t - L_3(B)L_4^{-1}(B)B^f(\varepsilon_t+u_t')$$

$$\text{or} \quad L_3(B)L_4^{-1}(B)u_{t-f}' - n_t = - \left\{L_3(B)L_4^{-1}(B)B^f+1\right\} \varepsilon_t \ . \qquad (11.5.3)$$

$$\text{But} \qquad n_t = \hat{n}_{t-f}(f) + e_{t-f}(f)$$

$$\text{and} \qquad \hat{n}_{t-f}(f) = L_3(B)L_4^{-1}(B)B^f \ e_{t-f}(f).$$

$$\text{Hence} \qquad n_t = \left\{L_3(B)L_4^{-1}(B)B^f+1\right\} e_{t-f}(f) \ . \qquad (11.5.4)$$

Adding (11.5.3) and (11.5.4), we obtain

$$L_3(B)L_4^{-1}(B)u_{t-f}' = \left\{L_3(B)L_4^{-1}(B)B^f+1\right\} (e_{t-f}(f)-\varepsilon_t) . \qquad (11.5.5)$$

Now since

$$\nabla^d n_t = \phi^{-1}(B)\theta(B)a_t$$

$$\nabla^d n_t = \phi^{-1}(B)\theta(B)L_4^{-1}(B)\ e_{t-f}(f) \ , \qquad (11.5.6)$$

substituting (11.5.4) in (11.5.6) gives

$$\nabla^d(L_3(B)L_4^{-1}(B)B^f+1) = \phi^{-1}(B)\theta(B)L_4^{-1}(B) \ .$$

Thus (11.5.5) may be written

$$L_3(B)L_4^{-1}(B)\nabla^d u'_{t-f} = \phi^{-1}(B)\theta(B)L_4^{-1}\left\{e_{t-f}(f) - \varepsilon_t\right\}$$

$$\varepsilon_t = e_{t-f}(f)-L_3(B)\phi(B)\theta^{-1}(B)\nabla^d u'_{t-f} \qquad (11.5.7)$$

and $e_{t-f}(f) = L_4(B)a_t = a_t+\psi_1 a_{t-1}+\dots+\psi_{f-1}a_{t-f+1}$

is independent of u'_{t-f} .

If the additional noise u'_t is represented by the stochastic process

$$\phi_1(B)\nabla^{d_1} u'_t = \theta_1(B)\ b_t$$

with b'_t , b'_{t-1} , ... uncorrelated random variables, then

(11.5.7) becomes

$$\varepsilon_t = L_4(B)a_t + L_3(B)\phi(B)\theta^{-1}(B)\phi_1^{-1}(B)\theta_1(B)\nabla^{d-d_1} b'_{t-f} \qquad (11.5.8)$$

and provided $d \geq d_1$, ε_t will be a stationary process. For any choice of the parametric models for the noise at P, the additional noise in the system, and the dynamics, the variance ε_t at the output can now be calculated.

<u>Errors in x_t</u>

If we wish to think of the ignored error as occurring in the adjustments x_t we can write the control equation as

$$x_t = -\frac{L_1(B)}{L_2(B)} \; \frac{L_3(B)(1-B)}{L_4(B)} \varepsilon_t + u_t$$

where $$u_t = -\frac{L_1(B)L_3(B)(1-B)}{L_2(B)L_4(B)} u'_t \quad .$$

Equation (11.5.7) then becomes

$$\varepsilon_t = e_{t-f}(f) + L_1^{-1}(B)L_2(B)L_4(B)\phi(B)\theta^{-1}(B)\nabla u^{d-1}_{t-f}$$

and if the errors in x_t follow a stochastic process

$$\phi_2(B)\nabla^{d_2} u_t = \theta_2(B)b_t$$

$$\varepsilon_t = L_4(B)a_t + L_1^{-1}(B)L_2(B)L_4(B)\phi(B)\theta^{-1}(B)\phi_2^{-1}(B)\theta_2(B)\nabla^{d-d_2-1} b_{t-f} \qquad (11.5.9)$$

Provided then that $d > d_2$, ε_t will follow a stationary process and its variance may be calculated for any given choice of parameters.

Ignored observational errors in x_t for a simple control scheme

For illustration we now study the effect of ignored observational errors in x_t for an important but simple control scheme of the type considered before in Section 11.2. The disturbance and the dynamics are defined respectively by

$$\nabla n_t = (1 - \theta B)\, a_t \tag{11.5.10}$$

$$\nabla Y_t = g\frac{(1-\delta)}{(1-\delta B)}\, x_{t-1} \tag{11.5.11}$$

and the optimal control adjustment assuming no errors in the loop is

$$x_t = -\frac{(1-\theta)}{(1-\delta)g}(1-\delta B)\varepsilon_t \qquad \text{with } \varepsilon_t = a_t.$$

We suppose that the adjustment actually made is

$$x_t' = x_t + u_t$$

with the adjustment errors $u_t, u_{t-1}, u_{t-2}\ldots$ uncorrelated and having variance σ_u^2. Then $L_1(B)L_2^{-1}(B) = \frac{(1-\delta B)}{(1-\delta)g}$, $f = 1$,

$L_3(B)(1-B) = (1-\theta)$, $L_4(B) = 1$, $\phi^{-1}(B)\theta(B) = (1-\theta B)$, $\phi_2^{-1}(B)\theta_2(B) = 1$, $d = 1$, $d_2 = 0$. Substituting these values in (11.5.9), we obtain

$$\varepsilon_t = a_t + \frac{g(1-\delta)}{(1-\delta B)(1-\theta B)} u_{t-1}$$

and $$\sigma_\varepsilon^2 = \sigma_a^2 + \frac{g^2(1-\delta)^2(1+\theta\delta)}{(1-\theta\delta)(1-\theta^2)(1-\delta^2)} \sigma_u^2 \quad .$$

To make comparison simpler it is convenient to express σ_u as a multiple $k\sigma_x$ of the standard deviation σ_x of x when no additional noise is present.

Then

$$\sigma_u^2 = k^2\sigma_x^2 = k^2 \frac{(1-\theta)^2(1+\delta^2)}{g^2(1-\delta)^2} \sigma_a^2 \qquad (11.5.12)$$

Finally, if the additional noise in x raises the variance to $(1+k^2)\sigma_x^2$ then the variance of the deviation from target output is increased according to the equation

$$\sigma_\varepsilon^2 = \sigma_a^2 \left\{1 + k^2 \frac{(1+\theta\delta)(1-\theta)(1+\delta^2)}{(1-\theta\delta)(1+\theta)\ (1-\delta^2)}\right\} \qquad (11.5.13)$$

Rounding error in the adjustment

In particular, (11.5.13) allows us to obtain approximately the effect of "rounding" the adjustments x_t as is done, for example, in the chart of Figure 11.8b. Suppose that the rounding interval is $R\sigma_x$. Very approximately we can represent the effect of rounding by adding an error u_t to x_t which is uniformly distributed over the interval $R\sigma_x$. Also, although there will be some autocorrelation among the u_t's, for most practically occurring cases this will be slight and so we assume them to be uncorrelated. With these approximations

$$\sigma_\varepsilon^2 = \sigma_a^2 \left\{1 + \frac{R^2}{12} \frac{(1+\theta\delta)(1-\theta)(1+\delta^2)}{(1-\theta\delta)(1+\theta)(1-\delta^2)}\right\} .$$

For the chart of 11.8b $\theta = 0$, $\delta = 0.5$, $R \approx 1$ so that

$$\sigma_\varepsilon^2 \approx \sigma_a^2 \left\{1 + \frac{5}{36}\right\}$$

$$\sigma_\varepsilon \approx 1.067\sigma_a \quad .$$

11.5.2 Optimal action when there are observational errors in the adjustments x_t

(11.5.9) makes it possible to calculate the effect of added noise in x_t when the optimal scheme which assumes no added noise is used. It is of interest to derive the optimal scheme for specified added noise and to see how it differs from the scheme which assumes no added noise. We use for illustration the example considered before.

Suppose the control action actually taken is

$$x_t = -\frac{(1-\theta)(1-\delta B)}{g(1-\delta)} L(B)\, \varepsilon_t + u_t$$

where again $u_t, u_{t-1}, \cdots$ are uncorrelated with variance σ_u^2 and that the disturbance n_t can be represented by an I.M.A. process of order (0,1,1). We wish to choose L(B) so as to minimize σ_ε^2.

Considering, as before, the situation at the point P in the feedback loop we obtain the equality

$$(1-B)\varepsilon_t = -(1-\theta)L(B)\varepsilon_{t-1}+(1-\theta B)a_t + g(1-\delta)(1-\delta B)^{-1}u_{t-1}$$

that is

$$(1-\delta B)\{1-B+(1-\theta)BL(B)\}\varepsilon_t = (1-\delta B)(1-\theta B)a_t+g(1-\delta)u_{t-1} \ . \quad (11.5.14)$$

Now the right hand side of (11.5.14) is a representation of a second order moving average process with added white noise and can therefore (see Section 3.4.2) be represented by another second order moving average process with representation

$$(1 - \pi_1 B - \pi_2 B^2)\ b_t$$

where $b_t, b_{t-1}, \ldots$ is a sequence of uncorrelated random variables. The problem is, therefore, reduced to that of choosing L(B) so that $\mathrm{Var}(\varepsilon_t)$ is minimized where

$$(1-\delta B)\ \{1-B+(1-\theta)BL(B)\}\varepsilon_t = (1-\pi_1 B-\pi_2 B^2)\ b_t \ .$$

Alternatively, we can write this equality in the form

$$\varepsilon_t = (1-\psi_1 B - \psi_2 B^2 \ldots)\ b_t$$

so that $\sigma_\varepsilon^2 = (1+\psi_1^2 + \psi_2^2 + \ldots)\ \sigma_b^2$

and σ_ε^2 is minimized only if $0 = \psi_1 = \psi_2 = \psi_3 = \ldots$

We require then that

$$(1-\delta B)\ \{1-B+(1-\theta)BL(B)\} = 1-\pi_1 B-\pi_2 B^2 \qquad (11.5.15)$$

That is

$$L(B) = \frac{(1+\delta-\pi_1) - (\delta+\pi_2)B}{(1-\theta)(1-\delta B)} .$$

The optimal adjustment is, therefore,

$$x_{ot} = -\left\{\frac{(1+\delta-\pi_1) - (\delta+\pi_2)B}{g(1-\delta)}\right\} \varepsilon_t . \qquad (11.5.16)$$

Now substituting (11.5.15) in (11.5.14) we have

$$(1-\pi_1 B-\pi_2 B^2)\varepsilon_t = (1-\delta B)(1-\theta B)a_t + g(1-\delta)\ u_{t-1}$$

whence π_1 and π_2 may be found by equating covariances of lags 0, 1, 2.

Writing $r = \sigma_\varepsilon^2/\sigma_a^2$ we obtain

$$\left.\begin{aligned}(1+\pi_1^2+\pi_2^2)r &= 1+(\delta+\theta)^2+(\delta\theta)^2+g^2(1-\delta)^2\frac{\sigma_u^2}{\sigma_a^2}\\ \pi_1(1-\pi_2)r &= (\delta+\theta)(1+\delta\theta)\\ -\pi_2 r &= \delta\theta.\end{aligned}\right\} \quad (11.5.17)$$

Optimal rounded control scheme

For illustration consider again the rounded chart of Figure 11.8b. Making the same approximations as before we consider what would have been the optimal control scheme given that the additional rounding error is to be taken account of.

Suppose, as before, that $\theta = 0$, $\delta = 0.5$, $R = 1$,

$$\frac{\sigma_u^2}{\sigma_a^2} = \frac{1}{12}\ \frac{(1-\theta)^2(1+\delta^2)}{g^2(1-\delta)^2}\ .$$

Then $\pi_2 = 0$, $\pi_1 = 0.5/r$

$$r+\frac{0.25}{r} = 1 + 0.5^2+\frac{1.25}{12} = 1.3542\ .$$

Hence $r = 1.134$ $\pi_1 = 0.43$ $\pi_2 = 0$.

Substituting these values in (11.5.16) we now find that the optimal control adjustment is

$$x_{ot} = 10.68\varepsilon_t - 5.00\varepsilon_{t-1} \quad \text{with } \sigma_\varepsilon = 1.065\sigma_a\ .$$

This may be compared with the scheme

$$x_t \;=\; 10.00\varepsilon_t - 5.00\varepsilon_{t-1} \qquad \text{with } \sigma_\varepsilon \;=\; 1.067\sigma_a$$

which was actually used and which is optimal on the assumption that there is no added error. Clearly in this case the choice of optimal control equation is not much effected by the added noise.

<u>Changes in the optimal adjustment induced by noise in the input</u>

If, as before, we write $\sigma_u^2 = k^2\sigma_x^2 = k^2\dfrac{(1-\theta)^2(1+\delta^2)}{g\;(1-\delta)^2}\,\sigma_a^2$

then from equations (11.5.17) we have

$$\pi_2 \;=\; -\frac{\delta\theta}{r} \tag{11.5.18a}$$

$$\pi_1 \;=\; \frac{(\delta+\theta)(1+\delta\theta)}{r+\delta\theta} \tag{11.5.18b}$$

$$1+\frac{(\delta+\theta)^2(1+\delta\theta)^2}{(r+\delta\theta)^2}+\left(\frac{\delta\theta}{r}\right)^2 \;=\; 1+(\delta+\theta)^2+(\delta\theta)^2+k^2(1-\theta)^2(1+\delta^2) \tag{11.5.18c}$$

where as before $r = \sigma_\varepsilon^2/\sigma_a^2$.

In practice when relating r to k^2 it is easiest to solve (11.5.18c) for k^2 for a series of suitably chosen values of r and then obtain the corresponding values of π_1 and π_2 by substituting in (11.5.18a) and (11.5.18b).

In the example above a moderate amount of additional noise (due to severe rounding) did not greatly increase σ_ε^2 nor was the optimal scheme which took account of the added noise much better than the scheme which ignored it. This kind of conclusion applies for moderate added noise levels over wide ranges of the parameters. It does not apply, however, when δ approaches unity (the system has a time constant which is large compared with the sampling interval) and for very large components of added noise in the loop. To throw some further light on these questions we consider some examples. In each case we take $k^2 = \frac{R^2}{12}$ with R = 1 so that $\frac{\sigma_u}{\sigma_x} = 0.29$. This then corresponds to adding noise u with σ_u the same as that of rounding error with the rounding interval equal to σ_x (where σ_x is the standard deviation of x for the no noise case).

We now consider the two cases:

Case (1) $g = 1$ $\theta = 0.5$ $\delta = 0.5$

Case (2) $g = 1$ $\theta = 0.5$ $\delta = 0.9$

The optimal control schemes corresponding to these parameters are summarized in Table 11.7. To further appreciate the result of Table 11.7 we notice that if instead of writing the control equation in terms of the adjustment $x_t = X_{t+} - X_{t-1+}$ we write it in terms of th level X_{t+} at which the manipulated variable is maintained from time t to t+1, then all of the

Case (1) $g = 1,\ \theta = 0.5,\ \delta = 0.5$

	Control equation for adjustment x_t	Variance at output σ_ε^2
Optimal scheme. No added noise	$-x_t = 1.00\varepsilon_t - 0.50\varepsilon_{t-1}$ $= 0.50(1+1.00\nabla)\varepsilon_t$	$1.000\sigma_a^2$
Effect on "no added noise" scheme of noise at input with $\frac{\sigma_u}{\sigma_x} = 0.29$	as above	$1.077\sigma_a^2$
Optimal scheme with added noise	$-x_t = 1.11\varepsilon_t - 0.53\varepsilon_{t-1}$ $= 0.58\ (1 + 0.92\nabla)\varepsilon_t$	$1.072\sigma_a^2$

Case (2) $g = 1,\ \theta = 0.5,\ \delta = 0.9$

	Control equation for adjustment x_t	Variance at output σ_ε^2
Optimal scheme. No added noise	$-x_t = 5.00\varepsilon_t - 4.50\varepsilon_{t-11}$ $= 0.50\ \{1+9.00\nabla\}\varepsilon_t$	$1.000\sigma_a^2$
Effect on "no added noise" scheme of noise at input with $\frac{\sigma_u}{\sigma_x} = 0.29$	as above	$1.697\sigma_a^2$
Optimal scheme with added noise	$-x_t = 7.25\varepsilon_t - 5.50\varepsilon_{t-1}$ $= 1.77(1+3.1\nabla)\varepsilon_t$	$1.278\sigma_a^2$

Table 11.7 Behavior of particular control schemes with added noise at the input

schemes above would be of the form

$$X_{t+} = k_P\varepsilon_t + k_I S\varepsilon_t$$

calling for proportional-integral control action.

The adjustment equation is then

$$x_t = k_I \left\{1 + \frac{k_P}{k_I}\nabla\right\} \varepsilon_t \,. \qquad (11.5.19)$$

We see from the table that with $\delta = 0.5$ (the time constant of the system of moderate size compared with the sampling interval) the ratio of proportional to integral control $\frac{k_P}{k_I} = 1.0$ and the introduction of the noise does not change the nature of the optimal control very much. However, when $\delta = 0.9$ (so that the time constant of the system is very large compared with the sampling interval) the ratio of proportional to integral control is large $\left(\frac{k_P}{k_I} = 9.0\right)$. The optimal scheme accommodates to the added noise by increasing the amount k_I and drastically cutting back on the ratio $\frac{k_P}{k_I}$ of proportional to integral control. We can use the ratio

$$E = \frac{\text{Variance of optimal "added noise" scheme}}{\text{Variance of optimal "no added noise" scheme}} \times 100$$

to increase the efficiency of the optimal "no added noise" scheme in the noisy situation. Thus for the schemes considered above

$$E = 99.54\% \text{ for } \delta = 0.5$$

and

$$E = 75.31\% \text{ for } \delta = 0.9 \quad .$$

For further illustration Figures 11.15 (a) and (b) show the changes in the efficiency factor E and the values of k_I and k_P/k_I as more and more noise is introduced into the loop for the two cases ($\theta = 0.5$, $\delta = 0.5$) and ($\theta = 0.5$, $\delta = 0.9$) previously considered. In inspecting these graphs it should be borne in mind that

(i) in industrial control applications even a 10% error in the input might be rather unusual and certainly in the range $0 < \frac{100\sigma_u}{\sigma_x} < 10$ even with $\delta = 0.9$ the efficiency of the scheme which assumes no ignored noise is quite good.

(ii) if the parameters are estimated from operating data the added noise will have already been taken account of in the basic scheme.

Nevertheless, if the parameters had not been estimated in this way and if there was a great deal of added noise in the input which had been ignored in designing the scheme then control could be very inefficient. For these examples the optimal schemes for added noise involve a greater use of integral action and a small ratio of proportional to integral action.

$\theta = 0.5 \qquad \delta = 0.5$

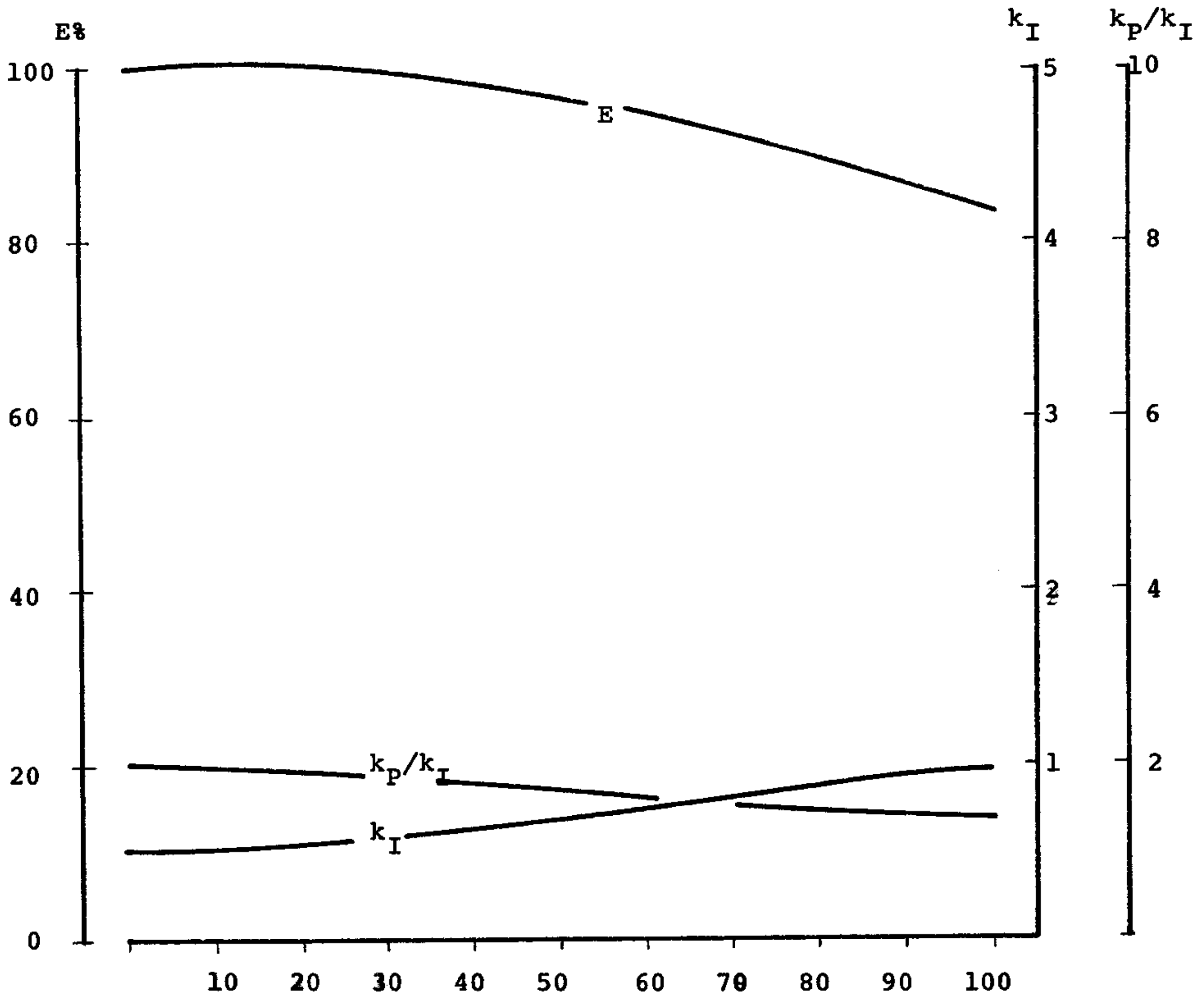


Figure 11.15 a

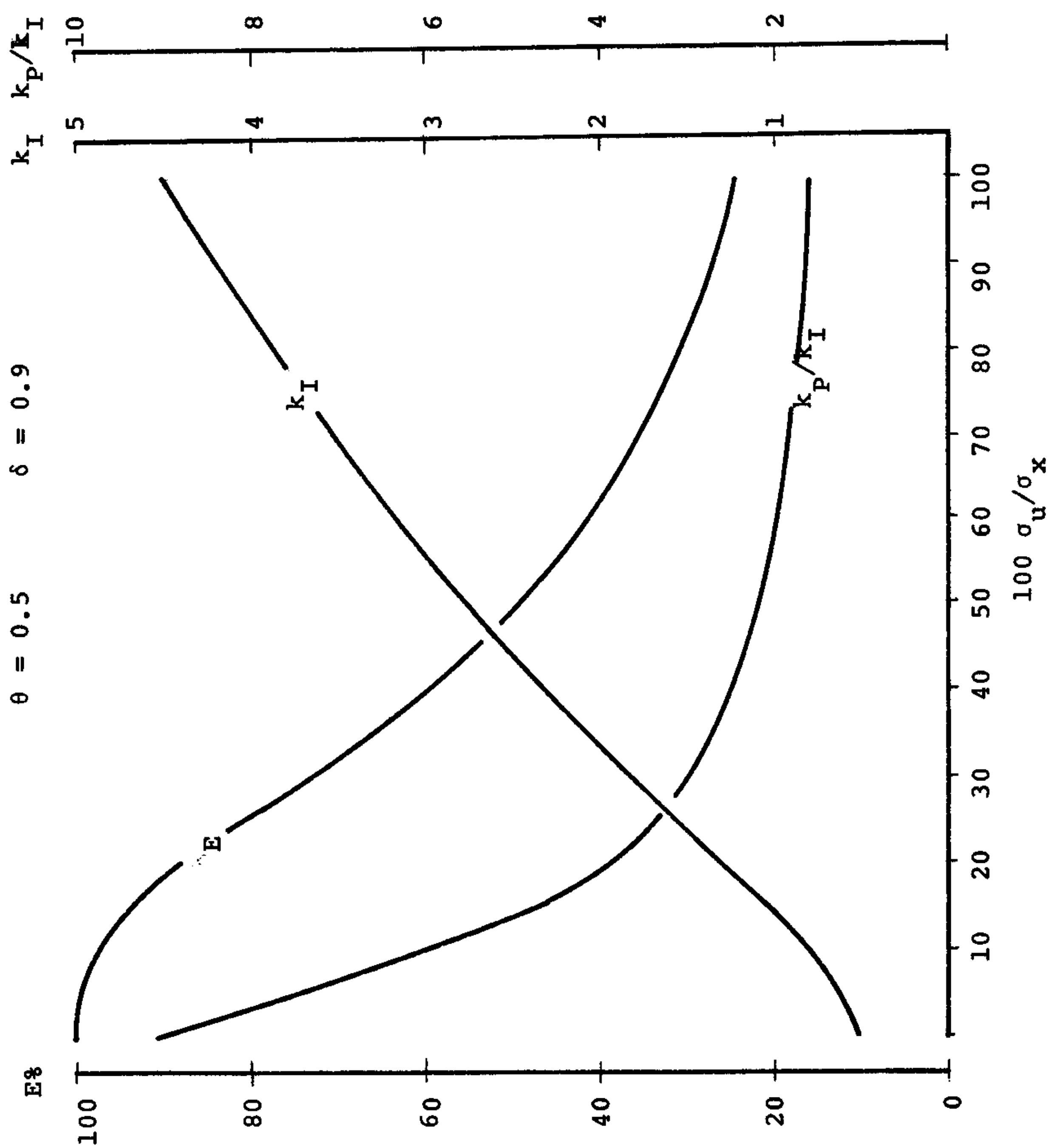


Figure 11.15 b

11.5.3 Transference of the noise origin

It is instructive to consider the derivation of (11.5.15) in the previous section from a different point of view. We have there supposed that although the intended action was

$$x_{ot} = -\frac{(1-\theta)(1-\delta B)}{g(1-\delta)} L(B)\varepsilon_t$$

because of the error u_t the action actually taken was

$$x_t = -\frac{(1-\theta)(1-\delta B)}{g\ (1-\delta)} L(B)\varepsilon_t + u_t \quad .$$

We have derived the appropriate operator L(B) to give optimal control in these circumstances.

Now the effect of the additional noise u_t is that after being acted upon by the dynamics of the process an additional component $\frac{g(1-\delta)}{1-\delta B} \sum_{j=1}^{t-1} u_j$ is produced at P. We could equally well regard this component as part of the noise source at P. In fact the situation is as if the noise entering at P was

$$n_t' = n_t + \frac{g(1-\delta)}{1-\delta B} \sum_{j=1}^{t-1} u_j$$

In that case

$$\nabla n_t' = (1-\theta B)a_t + \frac{g(1-\delta)}{1-\delta B} u_{t-1}$$

$$(1-\delta B)\ \nabla n_t' = (1-\delta B)(1-\theta B)a_t + g(1-\delta)\ u_{t-1}$$

$$= (1-\pi_1 B - \pi_2 B^2)\ b_t$$

where π_1, π_2, and b_t are defined precisely as before.

We can now apply the general equation (11.2.7) for optimal adjustment

$$x_t = \frac{L_1(B)L_3(B)}{L_2(B)L_4(B)} (1-B)\ \varepsilon_t \quad \text{with} \quad \varepsilon_t = e_{t-f}(f).$$

The total noise at P is now represented by the process of order (1,1,2)

$$\nabla n'_{t+1} = \frac{(1-\pi_1 B - \pi_2 B^2)}{1-\delta B} \quad b_{t+1}$$

so that

$$\hat{n}_t(1)-\hat{n}_{t-1}(1) = \left\{\frac{(1-\pi_1 B - \pi_2 B^2)}{1-\delta B} - (1-B)\right\} \quad b_{t+1}$$

$$= \left\{\frac{(1+\delta-\pi_1)-(\delta+\pi_2)B}{1-\delta B}\right\} b_t = \frac{L_3(B)(1-B)}{L_4(B)}\ b_t$$

and f = 1 so that $\varepsilon_t = b_t$.

Also $L_1(B)/L_2(B) = (1-\delta B)/g(1-\delta)$.

Optimal adjustment is thus obtained as before by setting

$$x_{ot} = -\left\{\frac{(1+\delta-\pi_1) - (\delta+\pi_2)B}{g(1-\delta)}\right\} \varepsilon_t .$$

This device of transference of the noise origin can be applied more generally to obtain optimal control action with additional noise of any kind entering the system at any point.

Implications for estimates of dynamic-stochastic model

The fact that the noise origin can be transferred in the manner described above has a very important practical implication which has already been referred to. This is that provided the parameters of the system are estimated from actual operating records when closed loop control is being applied, the estimates will automatically take account of added noise and a control scheme based on these parameters will be optimal for the actual situation in which added noise occurs. On the other hand a scheme based on estimating the actual noise n_t which really originates at the point P in Figure 11.6 could fail to give optimal control. For example, consider again the simple scheme with added noise in the input x discussed in Section 11.5.2. In practice to use such a scheme we would need to know the form of the appropriate stochastic and dynamic models and have estimates of the parameters.

If we were successful in characterizing the actual noise at P by, for example, performing an experiment in which the process was run with the manipulated variable X held fixed we would be led to the noise model $\nabla n_t = (1-\theta B)a_t$. If under normal operating conditions there was really a great deal of additional noise entering the system from observational errors in x which were not present under the conditions of the experiment then the scheme ignoring this additional noise could be rather inefficient. On the other hand if data collected during the actual running of a closed loop control scheme not necessarily optimal were used to estimate parameters, added white noise u_t in the adjustments x_t would lead to the noise at P being estimated as

$$\nabla n'_t = (1-\delta B)^{-1}(1-\pi_1 B - \pi_2 B^2) \; b_t$$

and would lead to the design of an optimal scheme.

11.6 Feedback control schemes where the adjustment variance is restricted

The discrete feedback control schemes previously discussed were designed to produce minimum mean square error at the output. It was tacitly supposed that there was no restriction in the amount of adjustment x_t that could be tolerated to achieve this. It sometimes happens that we are not able to employ these schemes because the amount of variation which can be allowed in x_t is restricted by practical limitations. Therefore, we consider how a particular class of feedback control schemes would need to be modified if a constraint was placed on $\mathrm{Var}(\dot{x}_t)$.

We consider again the important case in which the disturbance n_t at the output can be represented by a model

$$\nabla n_t = (1-\theta B)\, a_t \qquad (11.6.1)$$

of order (0,1,1) while the output and input are related by simple exponential dynamics so that

$$\frac{(1-\delta B)}{1-\delta}\, y_t = g\, x_{t-1} \qquad (11.6.2)$$

where it will be recalled that $1-\delta$ may be interpreted as the proportion of the total response to a step input that occurs in the first time interval. As we have seen the control equation yielding minimum output variance is

$$x_t = -\frac{\lambda}{g}\,\frac{(1-\delta B)}{1-\delta}\,\varepsilon_t \qquad (11.6.3)$$

where $\lambda = 1-\theta$ and $\varepsilon_t = a_t$.

If δ is negligibly small, optimal control is obtained from $x_t = -\frac{\lambda}{g}\varepsilon_t$ and then $V(x_t) = \frac{\lambda^2}{g^2}\sigma_a^2 = k$ say.

When δ is not negligible, however, $V(x_t) = k\,\frac{1+\delta^2}{(1-\delta)^2}$.

If δ is near its upper limit of unity $V(x_t)$ can become very large. For example, if $\delta = .9$ (so that only one tenth of the eventual change produced by a step input is experiences in the first interval) then $V(x_t) = 181\ k$. The reason for this large variance is, of course, that as δ approaches unity the control action

$$x_t = -\frac{\lambda}{g(1-\delta)}(\varepsilon_t - \delta\varepsilon_{t-1})$$

takes on more and more of an "alternating" character, the adjustment made at time t reversing a substantial portion of the adjustment made at time t-1. Now, in fact, a value of $\delta = 0.9$ corresponds to a time constant for the system of about 22 sampling intervals. The occurrence of such a value would immediately raise the question as to whether the sampling interval were being taken too short. Whether in fact the inertia of the process was so large that little would be lost by less frequent surveillance.

Now (see Section 11.7) the question of the choice of sampling interval must depend on the nature of the disturbance which infects the system. Because the properties of the disturbance usually also effect system inertia, in many cases it would be concluded that the sampling interval should be increased. Nevertheless, cases have occurred in practice [4] where a sensible sampling interval has been used and yet the excessive size of $\mathrm{Var}(x_t)$ has rendered a scheme which minimizes output variance impossible to operate.

Consider now the situation where the models for disturbance and dynamics are again given by (11.6.1) and (11.6.2) but some restriction of the input variance is necessary. The unrestricted optimal scheme has the property that the errors in the output ε_t, ε_{t-1}, ε_{t-2}, ... are the uncorrelated random deviates a_t, a_{t-1}, a_{t-2}, ... and the variance of the output

σ_ε^2 has the minimum possible value σ_a^2. With the restricted schemes the variance σ_ε^2 will necessarily be greater than σ_a^2 and the errors ε_t, ε_{t-1}, ε_{t-2}, ... at the output will be correlated.

We shall pose our problem in the following form: Given that σ_ε^2 be allowed to increase to some value $\sigma_\varepsilon^2 = (1+c)\ \sigma_a^2$, where c is a positive constant, to find that control scheme which produces the minimum value for $\mathrm{Var}(x_t)$.

11.6.1 Derivation of optimal adjustment

Let the optimal adjustment expressed in terms of the a_t's be

$$x_t = -\frac{1}{g} L(B)\ a_t \qquad (11.6.4)$$

where $L(B) = \ell_o + \ell_1 B + \ell_2 B^2 + \ldots$

Then referring to Figure 11.16 we see that the error ε_t at the output is given by

$$\varepsilon_t = a_t + \left\{\lambda - \frac{L(B)(1-\delta)}{1-\delta B}\right\} S\ a_{t-1} \ . \qquad (11.6.5)$$

The coefficient of a_t in this expression is unity so that we can write

$$\varepsilon_t = \{1 + B\mu(B)\}\ a_t \qquad (11.6.6)$$

where $\mu(B) = \mu_1 + \mu_2 B + \mu_3 B^2 + \ldots$

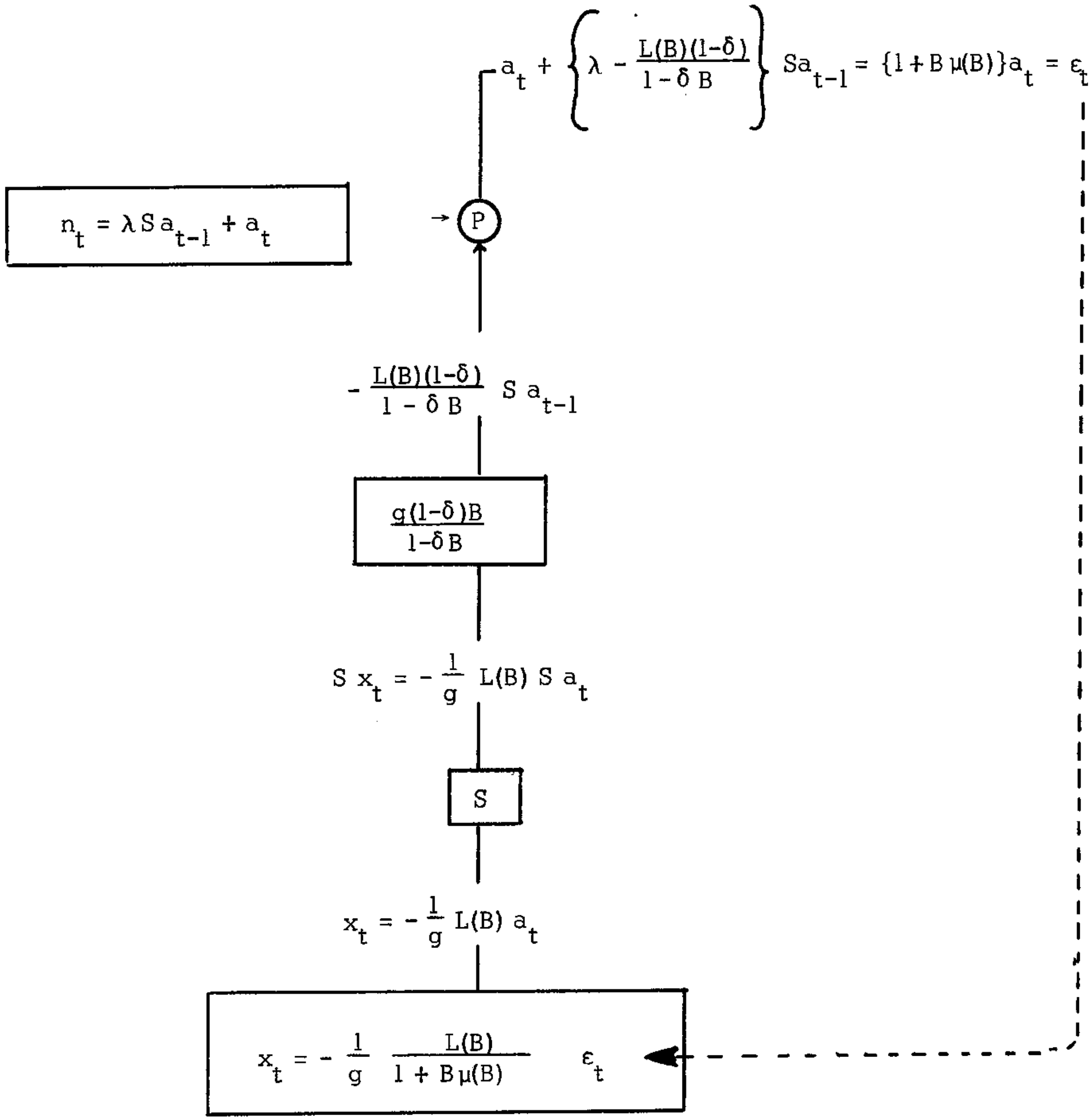


Figure 11.16 A feedback scheme for exponential dynamics and a (0,1,1) disturbance model

Also, in practice control would need to be exerted in terms of the observed output errors ε_t rather than in terms of the a_t's so that the control equation actually used would be of the form

$$x_t = -\frac{1}{g} \frac{L(B)}{1 + B\mu(B)} \varepsilon_t . \tag{11.6.7}$$

Equating (11.6.5) and (11.6.6) we obtain

$$(1-\delta)L(B) = \{\lambda - (1-B)\mu(B)\} \quad (1-\delta B) \tag{11.6.8}$$

Since δ, g, and σ_a^2 are constants we can proceed conveniently by finding an unrestricted minimum of

$$F(\mu) = \frac{(1-\delta)^2 g^2 V(x_t)}{\sigma_a^2} + \nu \left\{ \frac{V(\varepsilon_t)}{\sigma_a^2} - (1+c) \right\} \tag{11.6.9}$$

Equivalently, using covariance generating functions we require an unrestricted minimum of the coefficient of $B^o = 1$ in the expression

$$G(B) = (1-\delta)^2 L(B)L(F) + \nu\{1+B\mu(B)\}\{1+F\mu(F)\}$$

that is, in

$$G(B) = (1-\delta B)(1-\delta F)\{\lambda-(1-B)\mu(B)\}\{\lambda-(1-F)\mu(F)\}$$
$$+ \nu \{1+B\mu(B)\} \{1+F\mu(F)\} \tag{11.6.10}$$

where $F = B^{-1}$. This we can obtain by differentiating $G(B)$ with respect to each μ_i ($i = 1,2,\ldots$), selecting the coefficients of $B^{o} = 1$ in the resulting expression, equating them to zero and solving the resulting equations. We have

$$\frac{\partial}{\partial\mu_i} G(B) = (1-\delta B)(1-\delta F)\Big[-\lambda\{(1-B)B^{i-1}+(1-F)F^{i-1}\}$$

$$+(1-B)(1-F)\{\mu(B)F^{i-1}+\mu(F)B^{i-1}\}\Big] \qquad (11.6.11)$$

$$+\nu\left[B^{i}+F^{i}+B^{i-1}\mu(F)+F^{i-1}\mu(B)\right].$$

After picking out the coefficients of $B^{o} = 1$ for $i = 1, 2, 3, \ldots$ and setting each of these equal to zero we obtain the following equations

$$(i=1) \quad -\lambda(1+\delta+\delta^2)+2(1+\delta+\delta^2)\mu_1-(1+\delta)^2\mu_2+\delta\mu_3+\nu\mu_1 = 0 \qquad (11.6.12)$$

$$(i=2) \quad \lambda\delta-(1+\delta)^2\mu_1+2(1+\delta+\delta^2)\mu_2-(1+\delta)^2\mu_3+\delta\mu_4+\nu\mu_2 = 0 \qquad (11.6.13)$$

$$i>2 \quad \{\delta B^2-(1+\delta)^2B+2(1+\delta+\delta^2)-(1+\delta)^2F+\delta F+\nu\}\mu_i = 0 \qquad (11.6.14)$$

<u>The case where δ is negligible</u>

Consider first the simpler case where δ is small and can be set equal to zero. Then the equations can be written

$$(i=1) \qquad -(\lambda-\mu_1) + (\mu_1-\mu_2) + \nu\mu_1 = 0 \qquad (11.6.15)$$

$$(i>1) \qquad \{B-(2+\nu)+F\}\ \mu_j = 0 \quad . \qquad (11.6.16)$$

These difference equations have a solution of the form

$$\mu_j = A_1\rho_1^j + A_2\rho_2^j$$

where ρ_1 and ρ_2 are the roots of the characteristic equation

$$(B^2-(2+\nu)B+1) = 0 \qquad (11.6.17)$$

i.e. of $B + B^{-1} = 2 + \nu$.

Evidently if ρ is a root then so is ρ^{-1}. Thus the solution is of the form $\mu_j = A_1\rho^j + A_2\rho^{-j}$. Now if ρ has modulus less than or equal to 1 then ρ^{-1} has modulus greater than or equal to 1, and since $\varepsilon_t = \{1+B\mu(B)\}a_t$ must have finite variance, A_2 must be zero with $|\rho| < 1$. By substituting the solution $\mu_j = A_1\rho^j$ in (11.6.15) we find that $A_1 = \lambda$.

Finally, then $\mu_j = \lambda\rho^j$ and since μ_j and λ must be real then so must the root ρ. It follows that ν must be positive and so then must ρ. We have then that

$$\mu(B) = \frac{\lambda\rho}{1-\rho B} \qquad 0 < \rho < 1 \qquad (11.6.18)$$

$$1+B\mu(B) = 1 + \frac{\lambda\rho B}{1-\rho B} = \frac{1-\theta\rho B}{1-\rho B}, \quad (\theta = 1-\lambda) \qquad (11.6.19)$$

and

$$\varepsilon_t = \frac{1-\theta\rho B}{1-\rho B}\, a_t$$

so that $$\frac{V(\varepsilon_t)}{\sigma_a^2} = 1 + \frac{\lambda^2\rho^2}{1-\rho^2} \quad . \tag{11.6.20}$$

Also using (11.6.8) with $\delta = 0$

$$L(B) = \lambda - \frac{(1-B)\lambda\rho}{1-\rho B} = \frac{\lambda(1-\rho)}{1-\rho B} \quad . \tag{11.6.21}$$

Thus $$x_t = -\frac{1}{g}\frac{(1-\rho)}{1-\rho B}\, a_t \qquad \text{and}$$

$$\frac{V(x_t)}{\sigma_a^2} = \frac{\lambda^2}{g^2}\,\frac{(1-\rho)^2}{1-\rho^2} = \frac{\lambda^2}{g^2}\,\frac{(1-\rho)}{(1+\rho)} \; . \tag{11.6.22}$$

Using (11.6.7) with (11.6.19) and (11.6.21), we now find that the optimal control action in terms of the observed output error ε_t is

$$x_t = -\frac{1}{g}\,\frac{\lambda(1-\rho)}{1-\theta\rho B}\,\varepsilon_t$$

i.e. $$x_t = (1-\lambda)\rho x_{t-1} - \frac{1}{g}\lambda(1-\rho)\,\varepsilon_t \quad . \tag{11.6.23}$$

Note that the constrained control equation differs from the unconstrained one in two respects

(i) a new factor $(1-\lambda)\rho\, x_{t-1}$ is introduced thus making present action depend partly on previous action

(ii) the constant determining the amount of proportional control is reduced by a factor $1-\rho$.

We have supposed that the output variance be allowed to increase to some value $\sigma_a^2(1+c)$. It follows from (11.6.20) that

$$c = \frac{\lambda^2\rho^2}{1-\rho^2} \qquad \text{i.e. } \rho^2 = \frac{c}{\lambda^2+c}$$

$$\rho = \sqrt{\frac{c}{\lambda^2+c}}$$

where the positive square root is to be taken. It is convenient to write $Q = c/\lambda^2$. Then $Q = \frac{\rho^2}{1-\rho^2}$ and $\rho^2 = \frac{Q}{1+Q}$ and the output variance becomes $\sigma_a^2(1+\lambda^2Q)$.

In summary then, supposing we are prepared to tolerate an increase in variance in the output to some value $\sigma_a^2(1+\lambda^2Q)$, then

1) we compute $\rho = \sqrt{\frac{Q}{1+Q}}$

2) optimal control will be achieved by taking action

$$x_t = (1-\lambda)\rho\, x_{t-1} - \frac{1}{g}\lambda(1-\rho)\varepsilon_t$$

3) the variance of the input will be reduced to

$$V(x_t) = \frac{\lambda^2}{g^2}\,\frac{1-\rho}{1+\rho}\,\sigma_a^2$$

That is, it will reduce to a value that is R% of that for the unconstrained scheme where

$$R = 100\left(\frac{1-\rho}{1+\rho}\right).$$

Table 11.8 shows ρ and R for values of Q between 0.1 and 1.0.

$\frac{c}{\lambda^2}=Q$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
ρ	0.302	0.408	0.480	0.535	0.577	0.612	0.641	0.667	0.688	0.707
R	53.7	42.0	35.1	30.3	26.8	24.0	21.9	20.0	18.5	17.2

Table 11.8 Values of parameters for a simple constrained control scheme.

For illustration suppose $\lambda = 0.4$ then the optimal unconstrained scheme will employ the control action

$$x_t = -\frac{0.4}{g}\varepsilon_t$$

with $\varepsilon_t = a_t$. The variance of x_t would be $V(x_t) = \frac{\sigma_a^2}{g^2}0.16$.

Suppose it was desired to reduce this by a factor of four to the value $\frac{\sigma_a^2}{g^2}0.04$. Thus we require R to be 25%. The table shows that a reduction of the input variance to 24% of its unconstrained value is possible with $Q = 0.60$ and $\rho = 0.612$. If we use this scheme the output variance will be

$$\sigma_\varepsilon^2 = \sigma_a^2\{1 + 0.16 \times 0.60\} = 1.10\sigma_a^2 \quad .$$

Thus by the use of the control action

$$x_t = 0.37\, x_{t-1} - \frac{1}{g}0.16\,\varepsilon_t$$

instead of

$$x_t = -\frac{0.4}{g}\varepsilon_t$$

the variance of the input is reduced to about a quarter of its previous value whilst the variance of the output is increased by only 10%.

Case where δ is not negligible

Consider now the more general situation where δ is not negligible and the system dynamics must be taken account of. The j^{th} difference equation of (11.6.14) is of the form

$$(\alpha B^{-2}+\beta B^{-1}+\gamma+\beta B+\alpha B^2)\mu_j = 0$$

and if ρ is a root of the characteristic equation then so is ρ^{-1}. Suppose the roots are ρ_1, ρ_2, ρ_1^{-1}, ρ_2^{-1} and suppose that ρ_1 and ρ_2 are a pair of roots with modulus < 1. Then in the solution

$$\mu_j = A_1\rho_1^j + A_2\rho_2^j + A_3\rho_1^{-j} + A_4\rho_2^{-j} \quad ,$$

A_3 and A_4 must be zero because ε_t is required to have a finite variance.

The solution is then of the form

$$\mu_j = A_1\rho_1^j + A_2\rho_2^j \quad , |\rho_1| < 1 \quad |\rho_2| < 1.$$

The A's satisfying the initial conditions defined by (11.6.12) and (11.6.13) are obtained by substitution to give

$$A_1 = \frac{\lambda\rho_1(1-\rho_2)}{\rho_1 - \rho_2} \qquad A_2 = -\frac{\lambda\rho_2(1-\rho_1)}{\rho_1 - \rho_2} .$$

If we write $k_o = \rho_1 + \rho_2 - \rho_1\rho_2 \qquad k_1 = \rho_1\rho_2$ then

$$\mu(B) = \lambda\left\{\frac{k_o - k_1 B}{1-(k_o+k_1)B+k_1B^2}\right\} \tag{11.6.24}$$

and

$$1+B\mu(B) = \frac{1-k_1B-(1-\lambda)(k_oB-k_1B^2)}{1-(k_o+k_1)B+k_1B^2} . \tag{11.6.25}$$

Now substituting (11.6.24) in (11.6.8),

$$L(B) = \frac{\lambda(1-\delta B)(1-k_o)}{(1-\delta)(1-(k_o+k_1)B+k_1B^2)} \tag{11.6.26}$$

and $$\frac{L(B)}{1+B\mu(B)} = \frac{\lambda(1-\delta B)\ (1-k_o)}{(1-\delta)\{1-k_1B-(1-\lambda)(k_oB-k_1B^2)\}} .$$

Using (11.6.7) we find, therefore, that the optimal control action in terms of the error ε_t is

$$x_t = -\frac{\lambda}{g}\ \frac{(1-\delta B)(1-k_o)}{(1-\delta)\{1-k_1B-(1-\lambda)(k_oB-k_1B^2)\}}\varepsilon_t \tag{11.6.27}$$

$$\text{or} \quad x_t = (k_1+(1-\lambda)k_o)x_{t-1}-(1-\lambda)k_1x_{t-2} - \frac{\lambda(1-k_o)(1-\delta B)}{g(1-\delta)}\varepsilon_t \,. \quad (11.6.28)$$

The modified control scheme thus makes x_t depend on both x_{t-1} and x_{t-2} (only on x_{t-1} if $\lambda = 1$) and reduces the standard integral and proportional action by a factor $1-k_o$.

<u>The variances of output and input</u>

The actual variances for the output and input are readily found for

$$\varepsilon_t = a_t + \lambda \frac{(k_o-k_1B)}{1-(k_o+k_1)B+k_1B^2} a_{t-1} \,.$$

The second term on the right defines a mixed autoregressive moving average process of order (2,0,1) the variance for which is readily obtained to give

$$\frac{V(\varepsilon_t)}{\sigma_a^2} = 1 + \lambda^2 \left\{\frac{(k_o+k_1)^2(1-k_1) - 2k_1(k_o-k_1{}^2)}{(1-k_1)\ \{(1+k_1)^2 - (k_o+k_1)^2\}}\right\} = 1+\lambda^2Q \,.$$

(11.6.29)

Also from (11.6.26) x_t is a second order autoregressive process so that

$$\frac{V(x_t)}{\sigma_a^2} = \frac{\lambda^2}{g^2(1+\delta)^2}\ \frac{(1-k_o)\ \{(1+\delta^2)(1+k_1)-2\delta(k_o+k_1)\}}{(1+k_o+2k_1)(1-k_1)} \,. \quad (11.6.30)$$

Computation of k_o and k_1

Returning to the difference equations (11.6.14) the characteristic equation may be written

$$B^4 - M\,B^3 + N\,B^2 - M\,B + 1 = 0$$

where $M = \frac{(1+\delta)^2}{\delta}$ and $N = \frac{(1+\delta)^2 + (1+\delta^2) + \nu}{\delta}$.

It may also be written in the form

$$(B^2 - TB + P)(B^2 - P^{-1}TB + P^{-1}) = 0$$

where $T = \rho_1 + \rho_2$ and $P = \rho_1\rho_2$.

Equating coefficients

$$T + P^{-1}T = M \qquad \text{i.e.} \quad T = \frac{PM}{1+P}$$

$$P + P^{-1} + P^{-1}T^2 = N$$

Thus $$P + P^{-1} + \frac{PM^2}{(1+P)^2} = N$$

i.e. $$(P+2+P^{-1})\,(P+P^{-1}) + M^2 = N(P+2+P^{-1})$$

$$(P+P^{-1})^2 + (2-N)(P+P^{-1}) + M^2 - 2N = 0 \quad .$$

For suitable values of ν this quadratic equation will have two real roots

$$u_1 = \rho_1\rho_2 + \rho_1^{-1}\rho_2^{-1} \quad \text{and} \quad u_2 = \rho_1\rho_2^{-1} + \rho_1^{-1}\rho_2$$

the root u_1 being the larger. The required quantity P is now the smallest root of the quadratic equation

$$P^2 - u_1 P + 1 = 0$$

and T is given by

$$T = \{P(u_2+2)\}^{\frac{1}{2}} .$$

Table of optimal values for constrained schemes

Construction of the table.

Table 11.9 is provided to facilitate the selection of an optimal control scheme. The tabled values may be obtained as follows for each chosen value of the dynamic constants δ.

1) Compute $M = \frac{(1+\delta)^2}{\delta}$ and $N = \frac{(1+\delta)^2+(1+\delta^2)+\nu}{\delta}$

for a series of values of ν chosen to provide a suitable range for Q.

2) Compute $u_1 = \frac{1}{2}(N-2)+\left\{\left[\left(\frac{N-2}{2}\right)^2 + 2N-M^2\right]\right\}^{\frac{1}{2}}$

and $u_2 = \frac{1}{2}(n-2)-\left\{\left[\left(\frac{N-2}{2}\right)^2 +2N-M^2\right]\right\}^{\frac{1}{2}}$

3) Compute $k_1 = P = \frac{1}{2} u_1 - \left\{\left(\frac{1}{2}u_1\right)^2 - 1\right\}^{\frac{1}{2}}$

$$k_o = T-P = \left\{k_1(u_2+2)\right\}^{\frac{1}{2}} - k_1$$

4) Compute $Q = \dfrac{(k_o+k_1)^2(1-k_1)-2k_1(k_o-k_1^2)}{(1-k_1)\ \{(1+k_1)^2+(k_o+k_1)^2\}}$

5) Compute $R = \dfrac{(1-k_o)\{(1+\delta^2)(1+k_1)-2\delta(k_o+k_1)\}}{(1+k_o+2k_1)(1-k_1)(1+\delta^2)}$

6) Interpolate among the R, k_o, k_1 values at convenient values of Q.

Use of the table

Table 11.9 may be used as follows. The value of δ is entered in the vertical margin. Using the fact that $V(\varepsilon_t) = (1+\lambda^2 Q)\sigma_a^2$, the percentage increase in output variance is $100Q\lambda^2$. A suitable value of Q is entered in the horizontal margin. The entries in the table are then

(i) 100R the % reduction in the variance of x_t

(ii) k_o

(iii) k_1 .

δ		100Q 20	40	60	80	100
0.9	100R	21.7	11.3	6.7	4.5	3.1
	k_o	0.44	0.585	0.68	0.74	0.78
	k_1	0.18	0.27	0.34	0.39	0.44
0.8	100R	22.0	11.7	7.2	4.8	3.4
	k_o	0.44	0.585	0.68	0.74	0.78
	k_1	0.18	0.27	0.33	0.38	0.43
0.7	100R	22.7	12.4	8.0	5.6	4.1
	k_o	0.44	0.585	0.68	0.74	0.78
	k_1	0.17	0.25	0.32	0.36	0.40
0.6	100R	24.1	13.6	9.0	6.6	5.0
	k_o	0.44	0.58	0.67	0.73	0.78
	k_1	0.16	0.24	0.29	0.33	0.365
0.5	100R	26.5	15.5	10.5	7.9	6.2
	k_o	0.43	0.58	0.67	0.72	0.77
	k_1	0.15	0.21	0.26	0.29	0.32
0.4	100R	28.5	17.7	12.7	9.8	7.9
	k_o	0.43	0.57	0.66	0.72	0.76
	k_1	0.13	0.18	0.22	0.245	0.265
0.3	100R	31.5	20.5	15.2	12.0	9.9
	k_o	0.43	0.57	0.65	0.71	0.75
	k_1	0.105	0.145	0.17	0.19	0.20
0.2	100R	34.8	23.6	18.0	14.5	12.2
	k_o	0.42	0.56	0.64	0.69	0.73
	k_1	0.07	0.10	0.12	0.13	0.14
0.1	100R	38.2	26.7	21.0	17.3	14.6
	k_o	0.42	0.55	0.63	0.68	0.72
	k_1	0.04	0.05	0.06	0.065	0.07

Table 11. 9 Table to facilitate the calculation of optimal constrained control schemes

For illustration suppose $\lambda = 0.6$, $\delta = 0.5$, $g = 1$. The optimal unconstrained control equation is then

$$x_t = -1.2(1-0.5B)\,\varepsilon_t$$

and $V(x_t) = 1.80\sigma_a^2$. Suppose that this amount of variation in the input variable produces difficulties in process operation and it is desired to cut $V(x_t)$ to about $0.50\sigma_a^2$, that is, to about 28% of the value for the unconstrained scheme. Inspection of the table in the column labelled $\delta = 0.5$ shows that a reduction to 26.5% can be achieved by using a control scheme with constants $k_o = 0.43$, $k_1 = 0.15$; that is, by employing the control equation

$$x_t = 0.32x_{t-1} - 0.06x_{t-2} - 0.57 \times 1.2(1-0.5B)\varepsilon_t \quad .$$

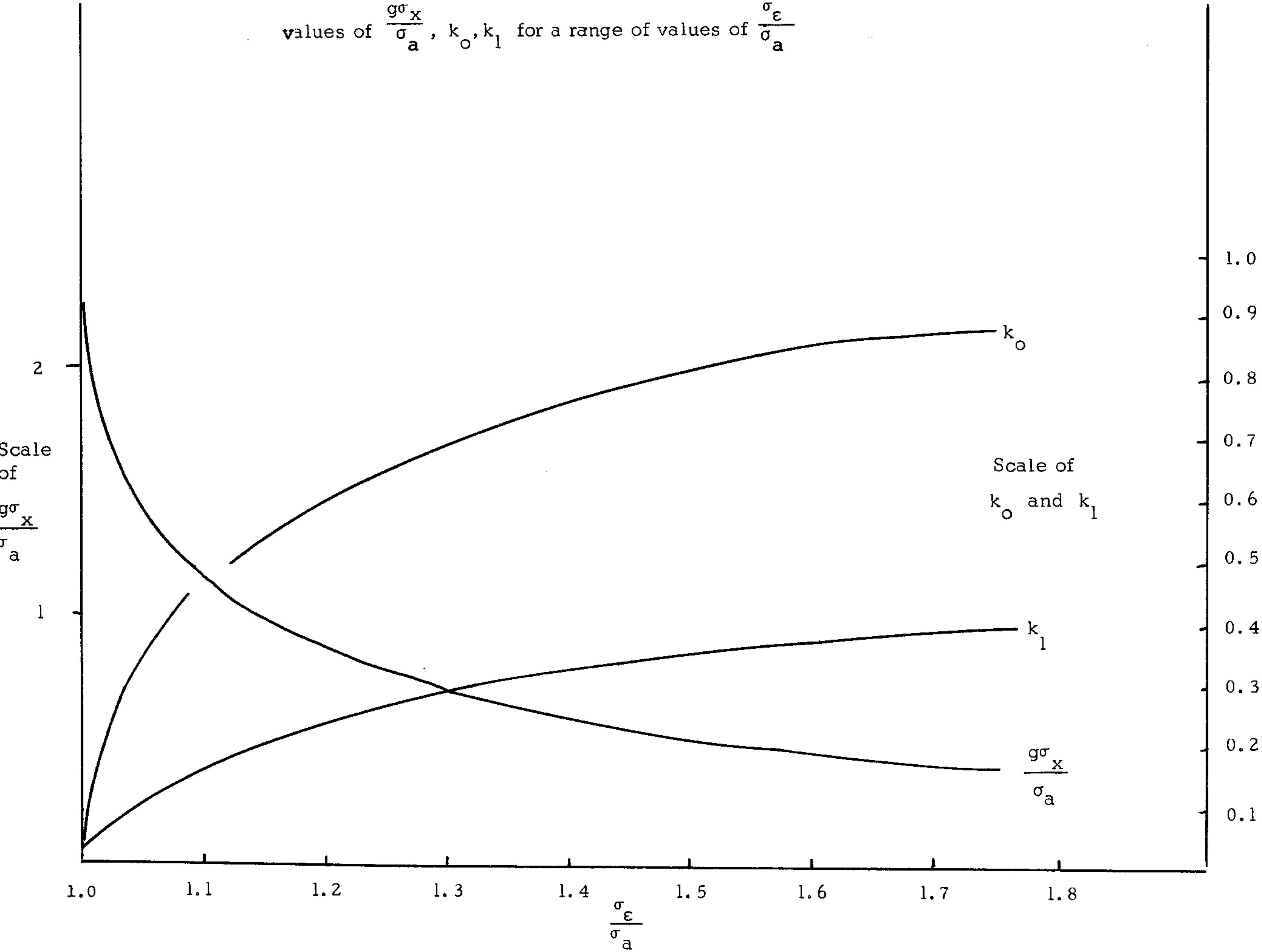
This solution corresponds to a value $Q = 0.20$. The variance at the output will, therefore, be increased by a factor of $1+\lambda^2 Q = 1+0.6^2 0.2 = 1.072$ that is by about 7%.

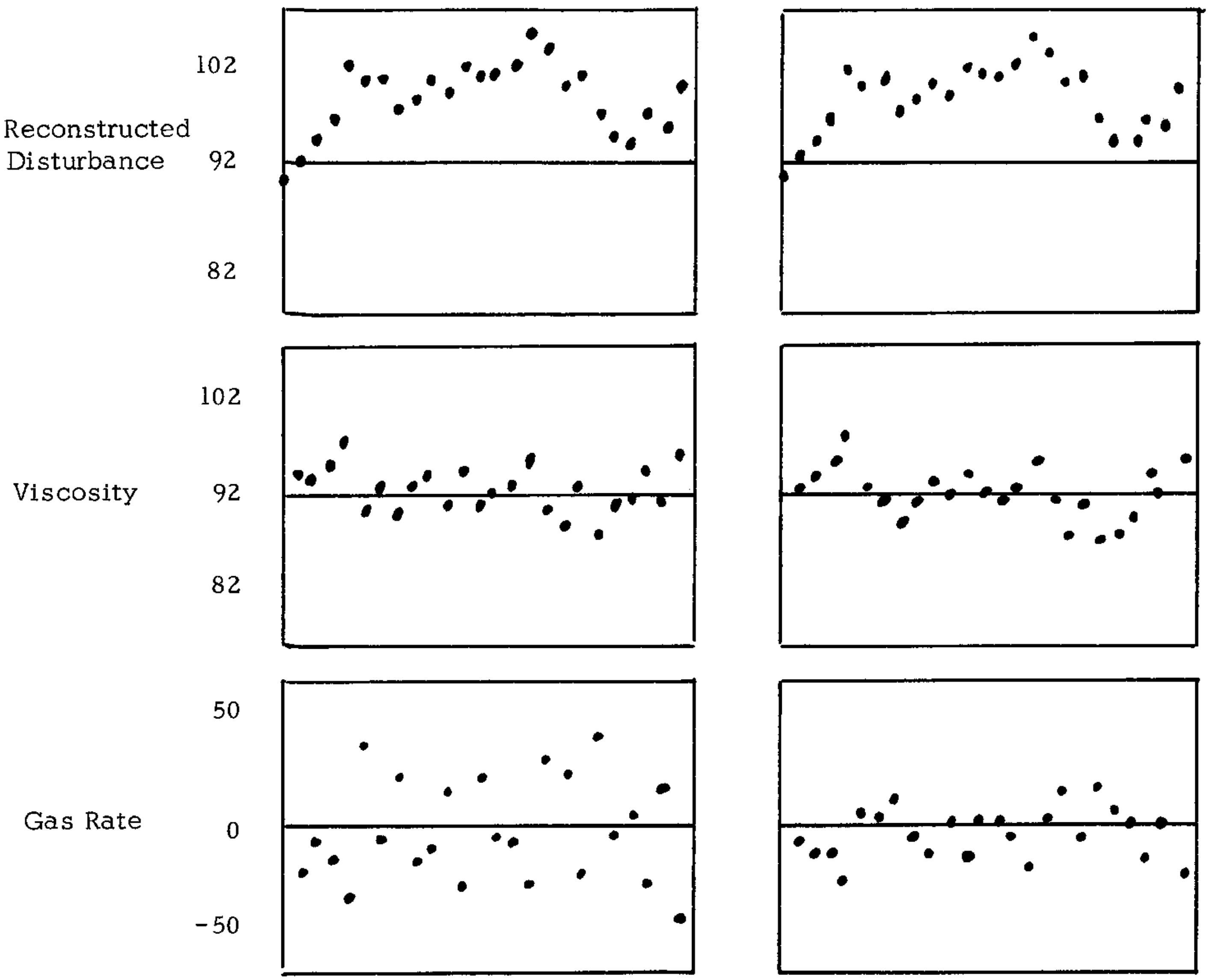
11.6.2 A constrained scheme for the viscosity/gas rate example

In the second example in Section 11.2 we considered a chemical process in which viscosity was controlled to a target value of 92 by varying the gas rate. For the pilot control scheme $\lambda = 1.0$, $(\theta = 0)$, $\delta = 0.5$ so that the optimal control action was

Figure 11.17 Control of viscosity by varying gas rate

values of $\frac{g\sigma_x}{\sigma_a}$, k_o, k_1 for a range of values of $\frac{\sigma_\varepsilon}{\sigma_a}$





Control Equations

$x_t = -10(\varepsilon_t - 0.5\varepsilon_{t-1})$

$x_t = 0.15x_{t-1} - 5.5(\varepsilon_t - 0.5\varepsilon_{t-1})$

Figure 11.18 Behavior of unconstrained and constrained control schemes for viscosity/gas rate example.

11.7 Choice of the sampling interval

In comparison with continuous systems discrete systems of control such as are disucssed here can be very efficient provided that the sampling interval is suitably chosen. Roughly speaking we want the interval to be such that not too much change can occur during the sampling interval. Usually the behavior of the disturbance which has to pass through all or part of the system reflects the inertia or dynamic properties of the system so that the sampling interval will often be chosen tacitly or explicitly to be proportional to the time constant or constants of the system. In chemical processes involving reaction and mixing of liquids where time constants of 2 or 3 hours are common ,rather infrequent sampling, say at hourly intervals and possibly with operator surviellance and manual adjustment, will be sufficient. By contrast where reactions between gases are involved a suitable sampling interval may be measured in seconds and automatic monitoring and adjustment may be essential.

In some cases experimentation may be needed to arrive at a satisfactory sampling interval and in others rather simple calculations will show how the choise of sampling interval will effect the degree of control that is possible.

11.7.1 An illustration of the effect of reducing sampling frequency

To illustrate the kind of calculation that is helpful, suppose again that we have a simple system in which using a particular sampling interval the disturbance is represented by an I.M.A. process of order (0,1,1) $\nabla n_t = (1-\theta B)a_t$ and the dynamics by the first order system $(1-\delta B)y_t = g(1-\delta)\, x_{t-1}$. In this case if we employ the optimal adjustment

$$x_t = -\frac{(1-\theta)}{g(1-\delta)}(1-\delta B)\varepsilon_t \qquad (11.7.1)$$

then the deviation from target is $\varepsilon_t = a_t$ and has variance $\sigma_a^2 = \sigma_1^2$ say.

In practice the question has often arisen: How much worse off would we be if we took samples less frequently? To answer this question we must consider the effect of sampling the stochastic process involved.

11.7.2 Sampling an I.M.A. process of order (0,1,1)

Suppose that with observations being made at some "unit" interval we have a process

$$\nabla n_t = (1-\theta_1 B)\, a_t \quad \text{with Var}\,(a_t) = \sigma_a^2 = \sigma_1^2$$

where the subscript 1 is used in this context to denote the choice of sampling interval. Then, for the differences ∇n_t the autocovariances γ_o, γ_1, etc. are given by

$$\gamma_o = (1+\theta_1^2)\sigma_1^2$$

$$\gamma_1 = -\theta_1\sigma_1^2 \qquad (11.7.2)$$

$$\gamma_j = 0 \qquad j = 2,3,\ldots \infty$$

Writing $\quad \zeta = (\gamma_o+2\gamma_1)/\gamma_1$

we have $\quad \zeta = -(1-\theta_1)^2/\theta_1$

so that given γ_o and γ_1 the parameter λ of the I.M.A. process may be obtained by solving the quadratic equation

$$(1-\theta_1)^2 - \zeta(1-\theta_1) + \zeta = 0$$

selecting that root for which $-1 < \theta_1 < 1$.

Also $\quad \sigma_1^2 = -\gamma_1/\theta_1 \quad . \qquad (11.7.3)$

Suppose now the process n_t is observed at intervals of h units (where h is a positive integer) and the resulting process is denoted by m_t. Then

$$\nabla m_t = n_t - n_{t-h} = (a_t+a_{t-1}+\ldots+a_{t-h+1}) - \theta(a_{t-1}+a_{t-2}+\ldots+a_{t-h})$$

$$\nabla m_{t-1} = n_{t-h} - n_{t-2h} = (a_{t-h}+a_{t-h-1}+\ldots+a_{t-2h+1}) - \theta(a_{t-h-1}+\ldots+a_{t-2h})$$

etc.

Then for the differences ∇m_t the auto covariances $\gamma_o(h)$, $\gamma_1(h)$, etc. are

$$\gamma_o(h) = \{(1+\theta_1^2)+(h-1)(1-\theta_1)^2\} \quad \sigma_1^2$$

$$\gamma_1(h) = -\theta\sigma_1^2 \qquad (11.7.4)$$

$$\gamma_j(h) = 0 \qquad (j = 2,3,\ldots\infty)$$

It follows that the process m_t is also an I.M.A. process of order (0,1,1)

$$\nabla m_t = (1-\theta_h B)e_t$$

where e_t, e_{t-1}, ... are a sequence of uncorrelated deviates with variance σ_h^2.

Now
$$\frac{\gamma_o(h)+2\gamma_1(h)}{\gamma_1(h)} = -\frac{h(1-\theta)^2}{\theta}$$

so that

$$\frac{h(1-\theta)^2}{\theta} = \frac{(1-\theta_h)^2}{\theta_h} \qquad (11.7.5)$$

Also since $\gamma_1(h) = -\theta_h\sigma_h^2 = -\theta\sigma_1^2$

$$\frac{\sigma_h^2}{\sigma_1^2} = \frac{\theta}{\theta_h} \quad . \qquad (11.7.6)$$

We have shown, therefore, that the sampling of an I.M.A. process of order (0,1,1) at interval h produces another I.M.A. process of order (0,1,1). From (11.7.5) we can obtain the value of the parameter θ_h for the sampled process and from (11.7.6) we can obtain the variance σ_h^2 of that process in terms of the parameters θ_1 and σ_1^2 of the original process.

In Figure 11.19 θ_h is plotted against log h, a scale of h being appended. The graph enables one to find the effect of increasing the sampling interval of a (0,1,1) process by any given multiple. For illustration suppose we have a process for which $\theta_1 = 0.5$ and $\sigma_1^2 = 1$. Let us use the graph to find the values of the corresponding parameters θ_2, θ_4, σ_2^2, σ_4^2 when the sampling interval is (a) doubled (b) quadrupled. Marking on the edge of a piece of paper the points $h = 1$, $h = 2$, $h = 4$ from the scale on the graph we set the paper horizontally and so that $h = 1$ corresponds to the point on the curve for which $\theta_1 = 0.5$. We then read off the ordinates for θ_2 and θ_4 corresponding to $h = 2$ and $h = 4$. We find

$$\theta_1 = 0.5 \qquad \theta_2 = 0.38 \qquad \theta_4 = 0.27 \quad .$$

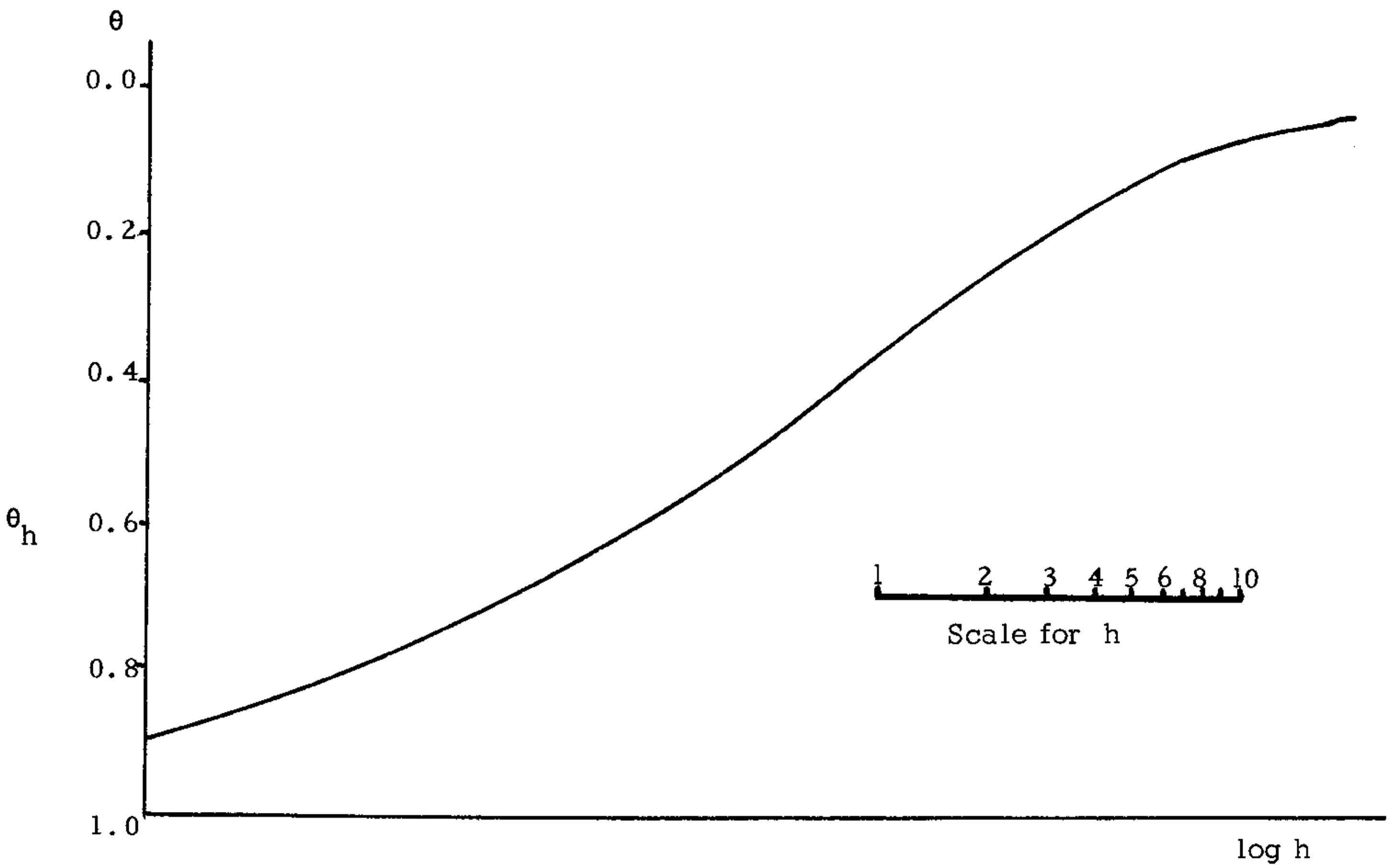


Figure 11.19 Sampling of I.M.A. (0,1,1) process
Parameter θ_h plotted against log h

Using (11.7.6) the variances are in inverse proportion to the values of θ so that

$$\sigma_1^2 = 1.00 \quad , \quad \sigma_2^2 = 1.32 \quad , \quad \sigma_4^2 = 2.17 \quad .$$

Suppose now that for the original scheme with unit interval the dynamic constant was δ_1 (again we will use the subscript to denote the sampling interval). Then since in real time the same fixed time constant $T = -h/\ln \delta$ applies to all the schemes we have

$$\delta_2 = \delta_1^2 \quad , \quad \delta_4 = \delta_1^4 \quad .$$

The scheme giving minimum mean square error for a <u>particular</u> sampling interval h would be

$$x_t(h) = \frac{(1-\theta_h)}{g(1-\delta_1^h)} (1-\delta_1^h B)\ \varepsilon_t(h)$$

or
$$x_t(h) = -\frac{(1-\theta_h)}{g}\left\{1+\frac{\delta_1^h}{1-\delta_1^h}\nabla\right\}\varepsilon_t(h) \quad . \tag{11.7.7}$$

Suppose, for example, with $\theta_1 = 0.5$ as above, $\delta_1 = 0.8$ so that $\delta_2 = 0.64$, $\delta_4 = 0.41$, then the optimal schemes would be

$h = 1 \quad x_t(1) = -\frac{0.5}{g}(1+4\nabla)\varepsilon_t(1), \ \sigma_\varepsilon^2 = 1.00 \ , \ g^2\sigma_x^2 = 6.50$

$h = 2 \quad x_t(2) = -\frac{0.62}{g}(1+1.78\nabla)\varepsilon_t(2), \ \sigma_\varepsilon^2 = 1.32 \ , \ g^2\sigma_x^2 = 5.50$

$h = 4 \quad x_t(4) = -\frac{0.73}{g}(1+0.69\nabla)\ \varepsilon_t(4), \sigma_\varepsilon^2 = 2.17, g^2\sigma_x^2 = 3.84.$

In accordance with expectations, as the sampling interval is increased and the dynamics of the system have relatively less importance the amount of "integral" control is increased and the ratio of proportional to integral control is markedly reduced. We have noted earlier that in some cases an excessively large adjustment variance σ_x^2 would be a disadvantage. The values of $g\sigma_x^2$ are indicated to show how the schemes differ in this respect. The smaller value for σ_x^2 would not of itself, of course, justify the choice h = 4. Using an optimal constrained scheme such as is described in Section 11.6 with h = 1 a very large reduction in σ_x would be produced with only a small increase in the output variance. For example, entering Table 11.5 with $\delta = 0.8$, $100Q = 20$, we find that for a 5% increase of output variance to the value $(1+\lambda^2 Q)\sigma_1^2 = 1.05\sigma_1^2$ the input variance for the scheme with h = 1 could be reduced to 22% of its unconstrained value so that $g^2\sigma_x^2 = 6.50 \times 0.22 = 1.43$.

Using (11.6.28) we obtain h = 1 (constrained scheme)

$$x_t = 0.40x_{t-1} - 0.09x_{t-2} - 0.56 \left\{\frac{0.5}{g}(1+4\nabla)\right\} \varepsilon_t(1) ,$$

$$\sigma_\varepsilon^2 = 1.05 , \quad g^2\sigma_x^2 = 1.43 .$$

In practice various alternative schemes could be set out with their accompanying characteristics and an economic choice made to suit the particular problem. In particular the increase in output variance which comes with the larger interval would have to be balanced off against the economic advantage of less frequent surveillance.

REFERENCES

[1] Box, G.E.P. and Jenkins, G.M., Some statistical aspects of adaptive optimisation and control, J.R.Statist.Soc., B 24 297, 1962.

[2] Box, G.E.P. and Jenkins, G.M., Further contributions to adaptive quality control: simultaneous estimation of dynamics: non-zero costs, Bulletin of the International Statistical Institute, 34th Session, Ottawa, Canada, 1963.

[3] Bather, J.A., Control charts and minimization of costs, J.R.Statist.Soc.,B 25, 1963.

[4] Reilly, P.M., Personal communication, 1966.

[5] Whittle, P., Prediction and Regulation, English U.P., London, 1963.

APPENDIX A11.1

Pilot Scheme Data

t	x_t	ε_t	t	x_t	ε_t	t	x_t	ε_t
1	0	0	46	0	0	91	-50	8
2	0	0	47	0	0	92	40	0
3	-40	4	48	40	-4	93	0	0
4	20	0	49	0	-2	94	0	0
5	0	0	50	50	-6	95	-20	2
6	0	0	51	-40	0	96	-30	4
7	20	-2	52	-50	3	97	-60	8
8	-10	0	53	-60	6	98	-20	6
9	20	-2	54	50	-2	99	-30	6
10	50	-6	55	-10	0	100	30	0
11	-10	-2	56	40	-4	101	-40	4
12	-55	4	57	40	-6	102	80	-6
13	0	2	58	-30	0	103	-40	0
14	10	0	59	20	-2	104	-20	2
15	0	-2	60	-30	2	105	55	-4
16	10	-2	61	10	0	106	0	2
17	-70	6	62	-20	2	107	-90	8
18	30	0	63	30	-2	108	40	0
19	-20	2	64	-50	4	109	0	0
20	10	0	65	10	-2	110	80	-8
21	0	0	66	10	-2	111	-20	-2
22	0	0	67	10	-2	112	-10	0
23	20	-2	68	-30	0	113	-70	6
24	30	-4	69	0	0	114	-30	6
25	0	-2	70	-10	2	115	-10	4
26	-10	0	71	-10	3	116	30	-1
27	-20	2	72	15	0	117	-5	0
28	-30	4	73	20	-2	118	-60	6
29	0	2	74	-50	4	119	70	-4
30	10	0	75	20	0	120	40	-6
31	20	-2	76	0	0	121	10	-4
32	-10	0	77	0	0	122	20	-4
33	0	0	78	0	0	123	10	-3
34	20	-2	79	0	0	124	0	-2
35	10	-2	80	-40	4	125	-70	6
36	-10	0	81	-100	12	126	50	-2
37	0	0	82	0	8	127	30	-4
38	0	0	83	0	-12	128	0	-2
39	0	0	84	50	-15	129	-10	0
40	0	0	85	85	-15	130	0	0
41	0	0	86	5	-12	131	-40	4
42	0	0	87	40	-14	132	0	2
43	20	-2	88	10	-8	133	-10	2
44	-50	4	89	-60	2	134	10	0
45	20	0	90	-50	6	135	0	0

286	-5	-1
287	-15	1
288	45	-4
289	40	-6
290	-50	2
291	-10	2
292	-50	6
293	20	1
294	5	0
295	-40	4
296	0	6
297	-60	8
298	40	0
299	-20	2
300	130	-12
301	-20	-4
302	0	-2
303	30	-4
304	-20	0
305	60	6
306	10	-4
307	-10	-1
308	-25	2
309	0	1
310	15	-1
311	-5	0
312	0	0