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ASYMPTOTIC NORMALITY OF CERTAIN FUNCTIONS DEFINED ON A MARKOV PROCESS.

by

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Asymptotic Normality of Certain Functions Defined on a Markov Process

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0. Summary. It is proved in [1] (Theorem 7.5' p. 232) that if $\{X_n\}$, $n = 1, 2, \ldots$ is a Markov process and f and g are two real-valued, measurable functions on $(\xi_r, \mathcal{E}^{(r)})$ and $(\xi_s, \mathcal{E}^{(s)})$, respectively, then, under suitable consitions on the process and the functions f and g, each one of the sums $\Sigma_{m=1}^{n} f_m$ and $\Sigma_{m=1}^{n} g_m$, properly normalized, is asymptotically normal, where $f_m = f(X_m, \ldots, X_{m+r-1})$, $g_m = g(X_m, \ldots, X_{m+s-1})$.

In the present paper it is first proved that, under essentially the same conditions, the quotients $(\Sigma_{m=1}^{n} g_m / \Sigma_{m=1}^{n} f_m)$ and $(\Sigma_{m=1}^{n} f_m / \Sigma_{m=1}^{n} g_m)$ properly normalized, are also asymptotically normal. This generalizes Theorem 7.5' mentioned above.

Next, the functions f and g are also considered to be dependent on n-- the number of the random variables X_j, $j=1, \ldots, n$-- and asymptotic normalities similar to the ones mentioned above are established under a number of conditions.

The results obtained here are useful in statistical applications and are applied in the problem of non-parametric estimation in Markov processes.

1. Preliminaries and asymptotic normality of a certain quotient.

Let $\{X_n\}$, $n = 1, 2, \ldots$ be a stationary Markov process defined on the probability space $(\Omega, \mathcal{A}, P)$ and taking values in the Borel real line $(R, \mathcal{B})$. It will be assumed throughout that the process satisfies hypothesis (D_o) ([1], p. 221). That is,

Hypothesis (D_o).

(a) Condition (D) (Doeblin's condition) is satisfied; (b) there is only a single ergodic set and this set contains no cyclically moving subsets.

Let f and g be real-valued, measurable functions defined on $(\xi_r, \mathcal{E}^{(r)})$ and $(\xi_s, \mathcal{E}^{(s)})$--the r and s-dimensional Euclidean spaces with the corresponding Borel σ-fields--respectively. Then in [1] the following theorem, which we record here as Theorem A for later reference, is proved.

Theorem A. Let (D_o) be satisfied and f and g be as above. Assume that

$$E\left| f(X_1, \ldots, X_r) \right|^{2+\delta_1} < \infty, \quad E\left| g(X_1, \ldots, X_s) \right|^{2+\delta_2} < \infty$$

for some δ_1, $\delta_2 > 0$, and set

$$f_m = f(X_m, \ldots, X_{m+r-1}), \quad g_m = g(X_m, \ldots, X_{m+s-1}).$$

Then, as $n \to \infty$,

$$\lim E\,[n^{-\frac{1}{2}} \Sigma_{m=1}^{n} (f_m - Ef_m)]^2 = \sigma_1^2, \quad \lim E\,[\,n^{-\frac{1}{2}} \Sigma_{m=1}^{n} (g_m - Eg_m)]^2 = \sigma_2^2$$

exist; if σ_1^2, $\sigma_2^2 > 0$, then, as $n \to \infty$,

$$\mathcal{L}\,[n^{-\frac{1}{2}}\,\Sigma_{m=1}^{n}\,(f_m - Ef_m)\;\big|\;P_\pi] \to N(0,\,\sigma_1^2)$$
$$\mathcal{L}\,[n^{-\frac{1}{2}}\,\Sigma_{m=1}^{n}\,(g_m - Eg_m)\;\big|\;P_\pi] \to N(0,\,\sigma_2^2)\,,$$

for any initial distribution (of X_1) π.

It is now assumed that $Ef \neq 0$. Set

(1.1) $\quad d = -\,(Eg\,/\,Ef)\,,$

and

(1.2) $\quad \varphi_m = \varphi(X_m, \ldots, X_{m+t-1})\,,$

where

$$\varphi\,(X_m, \ldots, X_{m+t-1}) = g\,(X_m, \ldots, X_{m+s-1}) + df\,(X_m, \ldots, X_{m+r-1})$$

with $t = \max\,(r, s)$.

With this notation we prove the following lemma:

Lemma 1.1. Let hypothesis (D_o) be satisfied, and d and φ_m be defined by (1.1) and (1.2), respectively. Then, as $n \to \infty$,

$$\lim E\,[\,n^{-\frac{1}{2}}\,\Sigma_{m=1}^{n}\,(\,\varphi_m - E\,\varphi_m\,)\,]^2 = \sigma_o^2$$

exists and is given by $\sigma_o^2 = \sigma_2^2 + d^2\,\sigma_1^2 + 2d\sigma$, where

$$\sigma = E\,[\,(f-Ef)\,(g-Eg)\,] + \Sigma_{m=1}^{\infty}\,E\,[\,(f-Ef)\,(g_{m+1}-Eg_{m+1})\,]$$
$$+\,\Sigma_{m=1}^{\infty}\,E\,[\,(g-Eg)\,(f_{m+1}-Ef_{m+1})\,]\,.$$

Proof. We have

$$E\,[\,n^{-\frac{1}{2}}\,\Sigma_{m=1}^{n}\,(\varphi_m - E\,\varphi_m)\,]^2 = E\,\{\,[n^{-\frac{1}{2}}\,\Sigma_{m=1}^{n}\,(g_m-Eg_m) + d\,[\,n^{-\frac{1}{2}}\,\Sigma_{m=1}^{n}\,(f_m-Ef_m)]\}^2 = E\,[\,n^{-\frac{1}{2}}\,\Sigma_{m=1}^{n}\,(g_m-Eg_m)]^2 + d^2\,E\,[\,n^{-\frac{1}{2}}\,\Sigma_{m=1}^{n}(f_m-Ef_m)]^2$$
$$+\,2dn^{-1}E\,\{[\,\Sigma_{m=1}^{n}(g_m-Eg_m)]\,[\,\Sigma_{m=1}^{n}\,(f_m-Ef_m)]\,\}\,.$$

Then, as $n \to \infty$,

$$F_{Y_n / Z_n}(y) \to \begin{cases} F_{Y / c_o}(y) & , \text{ if } c_o > 0 \\ 1 - F_Y(c_o y) & , \text{ if } c_o < 0 , \end{cases}$$

at all continuity points of F_Y.

Remark. From the assumption that $Z_n \xrightarrow{P} c_o \neq 0$, as $n \to \infty$, it follows that, for n sufficiently large, $P[Z_n \neq 0] = 1$ and hence Y_n / Z_n is well defined.

The main result of this section is the following theorem.

Theorem 1.1. Let hypothesis (D_o) be satisfied, and also $Ef \neq 0$. Then, as $n \to \infty$,

$$\mathcal{L}\{n^{\frac{1}{2}}[(\Sigma_{m=1}^{n} g_m / \Sigma_{m=1}^{n} f_m) - (Eg / Ef)] \mid P_\pi\} \begin{cases} N(0, \sigma_o^2 (Ef)^{-2}), \text{ if } Ef > 0 \\ 1 - N(0, (\sigma_o Ef)^2), \text{ if } Ef < 0 , \end{cases}$$

in the sense of Theorem B, provided $\sigma_o^2 > 0$; σ_o^2 is given in Lemma 1.1, and the functions f_m and g_m, $m=1, 2, \ldots$ are as in Theorem A.

Proof. In the first place, $(\Sigma_{m=1}^{n} g_m / \Sigma_{m=1}^{n} f_m)$ is well defined because for sufficiently large n,

$P[\Sigma_{m=1}^{n} f_m \neq 0] = P[n^{-1} \Sigma_{m=1}^{n} f_m \neq 0] = 1$, since

$n^{-1} \Sigma_{m=1}^{n} f_m \xrightarrow{a.s.} Ef$, as $n \to \infty$, and $Ef \neq 0$.

Next,

$$n^{\frac{1}{2}} [(\Sigma_{m=1}^{n} g_m / \Sigma_{m=1}^{n} f_m) - (Eg / Ef)] = (n^{-1} \Sigma_{m=1}^{n} f_m)^{-1} [n^{-\frac{1}{2}} \cdot \Sigma_{m=1}^{n}(g_m - Eg_m) + dn^{-\frac{1}{2}} \Sigma_{m=1}^{n}(f_m - Ef_m)] \quad .$$

Thus, by Theorem B, it suffices to prove asymptotic normality for the second factor on the right side above. But

$$n^{-\frac{1}{2}} \Sigma_{m=1}^{n} (g_m - Eg_m) + dn^{-\frac{1}{2}} \Sigma_{m=1}^{n} (f_m - Ef_m) = n^{-\frac{1}{2}} \Sigma_{m=1}^{n} (\varphi_m - E\varphi_m),$$

and, by means of Minkowski inequality,

$$E^{\frac{1}{\lambda}} |\varphi|^{\lambda} = E^{\frac{1}{\lambda}} |g+df|^{\lambda} \leqq E^{\frac{1}{\lambda}} |g|^{\lambda} + |d| E^{\frac{1}{\lambda}} |f|^{\lambda} < \infty ,$$

if $\lambda = 2 + \delta$ with $\delta = \min(\delta_1, \delta_2)$.

Therefore φ satisfies the conditions of Theorem A, and hence, as $n \to \infty$,

$$\mathcal{L}\, [n^{-\frac{1}{2}} \Sigma_{m=1}^{n} (\varphi_m - E\varphi_m) \mid P_\pi] \to N(0, \sigma_o^2),$$

provided $\sigma_o^2 > 0$, where σ_o^2 is given in Lemma 1.1.

This completes the proof of the theorem.

The result just obtained, and those to be derived in the next section are useful in statistical applications [3].

2. More about asymptotic normality.

In this section the functions f and g of the previous section will be taken to depend also on n, the number of the random variables X_j, $j=1,\ldots,n$, and we will use the notation L_n and L_n^* for f and g, respectively. Thus, the functions we are now dealing with are $L_n(Y_j)$ and $L_n^*(Z_j)$, where we set

$$Y_j = (X_j, \ldots, X_{j+r-1}),\ Z_j = (X_j, \ldots, X_{j+s-1}),\ j = 1, 2, \ldots .$$

Before we go any further we note here that the processes $\{Y_j\}$, $\{Z_j\}$, $j = 1, 2, \ldots$ are Markov processes which also satisfy hypothesis (D_o) ([1], p. 231).

2.1. We first work with L_n and collect here some of the assumptions which will be used elsewhere.

(A1) The Markov process $\{X_n\}$, $n = 1, 2, \ldots$ satisfies hypothesis (D_o).

(A2) $\{h_n\}$, $n = 1, 2, \ldots$ is a sequence of positive constants such that $h_n \to \infty$, as $n \to \infty$.

We set

$$f_n(Y_j) = L_n(Y_j) - EL_n(Y_j)$$

and impose upon L_n and f_n the following conditions:

(A3) For $n = 1, 2, \ldots$, $\{L_n\}$ is a sequence of uniformly bounded real-valued measurable functions on $(\mathcal{E}_r, \mathcal{B}^{(r)})$ such that

(i) $E\,|L_n(Y_1)|^2$ is $O(h_n n^{-1})$

(ii) $E\,|f_n(Y_1) f_n(Y_j)|$ are $O(h_n^2 n^{-2})$ uniformly in j, $1 < j \leqq n$.

(iii) $E\,|f_n(Y_1) f_n(Y_i) f_n(Y_j)|$ are $O(h_n^3 n^{-3})$ uniformly in i and j,

$$1 < i < j \leqq n, \quad n = 2, 3, \ldots$$

(iv) $h_n^{-1} n\, \sigma^2 [L_n(Y_1)] \to \sigma_1^2$ (for some $\sigma_1^2 < \infty$), as $n \to \infty$.

From (A3) (iv) it follows that $E\,|f_n(Y_1)|^2$ is $O(h_n n^{-1})$ and hence so is also $E\,|f_n(Y_1)|^3$ by the boundedness assumption of L_n. The same boundedness assumption and (A3)(ii) imply that $E\,|f_n^2(Y_i)\, f_n(Y_j)|$ are $O(h_n^2 n^{-2})$ uniformly in i and j with $i, j = 1, \ldots, n$, $i \neq j$.

Under the regularity assumptions (A3), and an additional one which we will make, the asymptotic normality of

$$h_n^{-\frac{1}{2}} \Sigma_{j=1}^{n} f_n(Y_j) \tag{2.1.1}$$

will be established. In discussing the asymptotic normality of (2.1.1) we follow a method parallel to the one used in proving Theorem 7.5, p. 228 in [1].

First, $\Sigma_{j=1}^{n} f_n(Y_j)$ is split up as follows:

Define

$y_m(n) = \Sigma_j f_n(Y_j)$, where the summation extends from $(m-1)(\alpha+\beta)+1$ to $(m-1)(\alpha+\beta)+\alpha$, $m=1,\ldots,\mu$,

$y'_m(n) = \Sigma_j f_n(Y_j)$, where the summation extends from $(m-1)(\alpha+\beta)+\alpha+1$ to $m(\alpha+\beta)$, $m=1,\ldots,\mu$,

$y'_{\mu+1} = \Sigma_j f_n(Y_j)$, where the summation extends from $\mu(\alpha+\beta)+1$ to n.

The numbers α, β and μ are positive integers which tend to infinity, as $n \to \infty$, and are such that $\mu(\alpha+\beta)$ is the largest multiple of $\alpha+\beta$ which is $\leqq n$.

Clearly,

$$h_n^{-\frac{1}{2}} \Sigma_{j=1}^{n} f_n(Y_j) = h_n^{-\frac{1}{2}} \Sigma_{m=1}^{\mu} y_m(n) + h_n^{-\frac{1}{2}} \Sigma_{m=1}^{\mu+1} y'_m(n) .$$

It is first proved that

(2.1.2) $h_n^{-\frac{1}{2}} \Sigma_{m=1}^{\mu+1} y'_m(n) \to 0$, in probability, as $n \to \infty$ $(\mu \to \infty)$.

By the Tchebichev inequality, it suffices to prove that

(2.1.3) $h_n^{-1} E \mid \Sigma_{m=1}^{\mu+1} y'_m(n) \mid^2 \to 0$, as $n \to \infty$ $(\mu \to \infty)$.

Under assumption (A1), Lemma 7.1, p. 222 in [1] applies and gives

$$E \mid y'_m(n) \mid^2 \leqq \beta\sigma^2 [L_n(Y_1)] + c_1 \beta E L_n^2 (Y_1) \text{ for } m=1,\ldots,\mu$$

and

$$E\,|\,y'_{\mu+1}(n)\,|^2 \leqq [n-\mu(\alpha+\beta)]\sigma^2\,[L_n(Y_1)] + c_1\,[n-\mu(\alpha+\beta)]\,.\,E\,L_n^2(Y_1)\,,$$

where

$c_1 = 4\gamma_1^{\frac{1}{2}}\rho_1^{\frac{1}{2}}(1-\rho_1^{\frac{1}{2}})^{-1}$, the constants γ_1 and ρ_1 corresponding to the process $\{Y_j\}$, $j = 1, 2, \ldots$

The Minkowski inequality gives

$$h_n^{-\frac{1}{2}}\,E^{\frac{1}{2}}\,|\,\Sigma_{m=1}^{\mu+1}\,y'_m(n)\,|^2 \leqq h_n^{-\frac{1}{2}}\,\mu\,E^{\frac{1}{2}}\,|\,y'_1(n)\,|^2 + h_n^{-\frac{1}{2}}E^{\frac{1}{2}}|\,y'_{\mu+1}(n)\,|^2\,.$$

Using then the previous two inequalities we get

$$h_n^{-\frac{1}{2}}\,\mu\,E^{\frac{1}{2}}\,|\,y'_1(n)\,|^2 \leqq (\beta\mu^2 h_n^{-1})^{\frac{1}{2}}\,\{\sigma\,[L_n(Y_1)] + c_1^{\frac{1}{2}}\,E^{\frac{1}{2}}L_n^2(Y_1)\}$$

and

$$h_n^{-\frac{1}{2}}E^{\frac{1}{2}}\,|\,y'_{\mu+1}(n)\,|^2 \leqq [n-\mu(\alpha+\beta)]^{\frac{1}{2}}\,h_n^{-\frac{1}{2}}\,\{\sigma[L_n(Y_1)] + c_1^{\frac{1}{2}}\,E^{\frac{1}{2}}L_n^2(Y_1)\}\,.$$

Now $\beta\mu^2 n^{-1} \leqq \beta\mu\alpha^{-1}$, as is easily seen, and hence

$$\beta\mu^2 h_n^{-1} = (nh_n^{-1})\,(\beta\mu^2 n^{-1}) \leqq (nh_n^{-1})\,(\beta\mu\alpha^{-1})$$

By choosing α, β and μ to tend to infinity, as $n\to\infty$, so that

(2.1.4) $\quad \beta\mu\alpha^{-1} \to 0\,,$

we then get

(2.1.5) $\quad h_n^{-\frac{1}{2}}\mu\,E^{\frac{1}{2}}\,|\,y'_1(n)\,|^2 \to 0$, as $n\to\infty$ ($\mu\to\infty$),

by means of (A3)(i) and (A3)(iv) .

Next,

$$[n-\mu(\alpha+\beta)]\;h_n^{-1} = (nh_n^{-1})\;[n-\mu(\alpha+\beta)]\,n^{-1} \leqq (nh_n^{-1})\;\mu^{-1}\,,$$

as is easily seen, and hence

(2.1.6) $h_n^{-\frac{1}{2}} E^{\frac{1}{2}} | y'_{\mu+1}(n)|^2 \to 0$, as $n \to \infty$ ($\mu \to \infty$),

again because of (A3)(i) and (A3)(iv).

Relations (2.1.5) and (2.1.6) taken together imply (2.1.3) and hence (2.1.2).

Next, we prove the asymptotic normality of

(2.1.7) $h_n^{-\frac{1}{2}} \Sigma_{m=1}^{\mu} y_m(n)$.

setting

$$\Phi_m(t;n) = E\{ \exp[it \Sigma_{j=1}^{n} f_n(Y_j)]\}$$

and repeating the arguments used in [1], p. 229, we get

$$E \{ \exp [it \Sigma_{m=1}^{\mu} y_m (n)] \} = \Phi_{\alpha}^{\mu}(t;n) + \zeta_{\mu}, \ | \zeta_{\mu} | < 2 \gamma_1 \mu \rho_1^{\beta+1} .$$

Again, α, β and μ are chosen so that they tend to infinity, as $n \to \infty$, and such that

(2.1.8) $\mu \rho_1^{\beta} \to 0$.

Then the characteristic function of (2.1.7) is, essentially,

(2.1.9) $\Phi_{\alpha}^{\mu}(th_n^{-\frac{1}{2}}; n)$,

since $\zeta_{\mu} \to 0$, as $\mu \to \infty$, by (2.1.8).

Now (2.1.9) is the characteristic function of $\Sigma_{m=1}^{\mu} z_m$, where z_m, $m = 1, \ldots, \mu$ are independent random variables with their common distribution that of $h_n^{-\frac{1}{2}} y_1(n)$.

Thus, the asserted normality of (2.1.7) will follow if we prove that

(2.1.10) $(C_\mu / B_\mu^{1+\frac{1}{2}}) \to 0$, as $n \to \infty$ $(\mu \to \infty)$, by Theorem 4.4, p. 141 in [1], where

$B_\mu = \Sigma_{m=1}^{\mu} E(z_m^2)$, $C_\mu = \Sigma_{m=1}^{\mu} E|z_m|^3$, $(E|z_1|^3 < \infty)$.

Now,

$$E(z_m^2) = h_n^{-1} E[\Sigma_{j=1}^{\alpha} f_n(Y_j)]^2$$

and

$$E[\Sigma_{j=1}^{\alpha} f_n(Y_j)]^2 = \alpha\sigma^2 [L_n(Y_1)] + 2\Sigma_{i<j} E[f_n(Y_i) f_n(Y_j)].$$

Thus,

$$B_\mu = (\alpha\mu n^{-1}) n h_n^{-1} \sigma^2 [L_n(Y_1)] + 2(\alpha\mu n^{-1})(\alpha h_n n^{-1}) \Sigma_{i<j} E[f_n(Y_i) f_n(Y_j)].$$

But

$$\Sigma_{i<j} E[f_n(Y_i) f_n(Y_j)] \leqq \alpha \Sigma_{j=1}^{\alpha-1} | E[f_n(Y_1) f_n(Y_{j+1})] | =$$

$$(\alpha h_n n^{-1})^2 (\alpha^{-1} h_n^{-2} n^2) \Sigma_{j=1}^{\alpha-1} | E[f_n(Y_1) f_n(Y_{j+1})] | .$$

Therefore, by means of (A3)(ii), (A3)(iv), and the fact that $\alpha\mu n^{-1} \to 1$, as $n \to \infty$ $(\mu \to \infty)$, as is easily seen, we obtain

$$B_\mu \to \sigma_1^2, \text{ as } n \to \infty \ (\mu \to \infty),$$

provided that there is a choice of α satisfying (2.1.4) and also

(2.1.11) $$\alpha h_n n^{-1} \to 0, \text{ as } n \to \infty,$$

for some choice of h_n satisfying (A2).

It remains for us to prove that $C_\mu \to 0$, as $n \to \infty$ $(\mu \to \infty)$. We have

$$C_\mu = \Sigma_{m=1}^{\mu} E|z_m|^3 = \mu h_n^{-\frac{3}{2}} E|\Sigma_{j=1}^{\alpha} f_n(Y_j)|^3 \leqq \mu h_n^{-\frac{3}{2}} \{ \alpha E|f_n(Y_1)|^3 +$$

$$3\Sigma_{i,j} E|f_n^2(Y_i) f_n(Y_j)| + 6\Sigma_{i<j<k} E|f_n(Y_i) f_n(Y_j) f_n(Y_k)| \} .$$

Now,

$$\mu h_n^{-\frac{3}{2}} \alpha E |f_n(Y_1)|^3 = (\alpha \mu n^{-1}) n h_n^{-\frac{3}{2}} E |f_n(Y_1)|^3 \to 0, \text{ as } n \to \infty ,$$

by (A2) and the remark following (A3).

In a similar fashion,

$$h_n^{-\frac{3}{2}} \Sigma_{i,j} E |f_n^2 (Y_i) f_n(Y_j)| = (\alpha \mu n^{-1}) (\alpha h_n n^{-1}) h_n^{-\frac{1}{2}} .$$

$$n^2 (\alpha h_n)^{-2} \Sigma_{i,j} E |f_n^2 (Y_i) f_n(Y_j)| \to 0, \text{ as } n \to \infty ,$$

by (A2), the remark following (A3), and (2.1.11).

Finally,

$$\mu h_n^{-\frac{3}{2}} \Sigma_{i<j<k} E |f_n(Y_i) f_n(Y_j) f_n(Y_k)| = (\alpha \mu n^{-1}) (\alpha h_n n^{-1})^2 h_n^{-\frac{1}{2}} n^3 (\alpha h_n)^{-3} .$$

$$\Sigma_{i<j<k} E | f_n(Y_i) f_n(Y_j) f_n(Y_k) | \to 0, \text{ as } n \to \infty ,$$

on account of (A2), (A3) (iii), and (2.1.11).

Therefore

$$C_\mu \to 0, \text{ as } n \to \infty \quad (\mu \to \infty)$$

and this establishes (2.1.10).

Hence the following theorem has been proved.

<u>Theorem</u> 2.1.1 Let assumptions (A1) - (A3) be satisfied. We assume that a choice of α satisfying (2.1.4) can be made such that (2.1.11) is also satisfied.

Then

$$\mathcal{L}\{h_n^{-\frac{1}{2}} \Sigma_{j=1}^{n} [L_n(Y_j) - EL_n(Y_j)]\} \to N(0, \sigma_1^2), \text{ as } n \to \infty ,$$

provided $\sigma_1^2 > 0$, where $\sigma_1^2 = \lim h_n^{-1} n \sigma^2 [L_n(Y_1)]$, as $n \to \infty$.

2.2. Replacing L_n by L_n^* and f_n by g_n, where

$$g_n(Z_j) = L_n^*(Z_j) - EL_n^*(Z_j),$$

and imposing upon L_n^* and g_n the same conditions as those we used in connection with L_n and f_n, we have a theorem analogous to Theorem 2.1.1; that is,

(A3)* For $n = 1, 2, \ldots$, $\{L_n^*\}$ is a sequence of uniformly bounded real-valued measurable functions on $(\mathcal{E}_s, \mathcal{B}^{(s)})$ such that (A3) (i) - (A3) (iv) are true if L_n and f_n are replaced by L_n^* and g_n, respectively. ((A3) (iv) may be true with a difference constant $\sigma_2^2 < \infty$). Then

Theorem 2.2.1. Let assumptions (A1), (A2) and (A3)* be satisfied. We assume that a choice of α which satisfies (2.1.4) also satisfies (2.1.11). Then,

$$\mathcal{L}\{h_n^{-\frac{1}{2}} \Sigma_{j=1}^n [L_n^*(Z_j) - EL_n^*(Z_j)]\} \to N(0, \sigma_2^2), \text{ as } n \to \infty,$$

provided

$$\sigma_2^2 > 0, \text{ where } \sigma_2^2 = \lim h_n^{-1} n \sigma^2 [L_n^*(Z_1)], \text{ as } n \to \infty.$$

Remarks: In the various derivations in proving Theorem 2.2.1 we will use the constant c_2 rather than c_1, where $c_2 = 4\gamma_2^{\frac{1}{2}} \rho_2^{\frac{1}{2}} (1 - \rho_2^{\frac{1}{2}})^{-1}$, the constants γ_2 and ρ_2 corresponding to the process $\{Z_j\}$, $j = 1, 2, \ldots$ There is always a choice of α, β and μ with the property that α, β and μ are positive integers tending to infinity with n, such that $\mu(\alpha + \beta)$ is the largest multiple of $\alpha + \beta$ which is $\leqq$ n, and for which both conditions (2.1.4) and (2.1.8) (and the corresponding property: $\mu\rho_2^{\beta} \to 0$, as $n \to \infty$ $(\mu \to \infty)$) are satisfied. This is explained in [1], p. 230. That is, it suffices to take β to be the largest integer which is $\leqq n^{\frac{1}{4}}$ and $\alpha = \beta^3$. It follows then that μ is approximately β, and all required conditions are satisfied.

We now proceed in proving asymptotic normality for a certain quotient. For this purpose it is assumed that $EL_n(Y_1) \neq 0$, $n = 1, 2, \ldots$ and

$$h_n^{-1} \Sigma_{j=1}^{n} L_n(Y_j) \to \ell \ (\neq 0 \text{ constant}), \text{ in probability, as } n \to \infty .$$

Then

$$h_n^{\frac{1}{2}} \{[\Sigma_{j=1}^{n} L_n^*(Z_j) / \Sigma_{j=1}^{n} L_n(Y_j)] - [EL_n^*(Z_1) / EL_n(Y_1)]\}$$

is well defined and we intend to prove its asymptotic normality, under some additional assumptions. It is easily seen that

$$h_n^{\frac{1}{2}} \{[\Sigma_{j=1}^{n} L_n^*(Z_j) / \Sigma_{j=1}^{n} (Y_j)] - [EL_n^*(Z_1) / EL_n(Y_1)]\}$$

$$= [h_n^{-1} \Sigma_{j=1}^{n} L_n(Y_j)]^{-1} \{h_n^{-\frac{1}{2}} \Sigma_{j=1}^{n} [L_n^*(Z_j) - EL_n^*(Z_j)]$$

$$+ v_n h_n^{-\frac{1}{2}} \Sigma_{j=1}^{n} [L_n(Y_j) - EL_n(Y_j)]\}$$

$$= [h_n^{-1} \Sigma_{j=1}^{n} L_n(Y_j)]^{-1} h_n^{-\frac{1}{2}} \Sigma_{j=1}^{n} [\varphi_n(W_j) - E\varphi_n(W_j)]$$

$$= [h_n^{-1} \Sigma_{j=1}^{n} L_n(Y_j)]^{-1} h_n^{-\frac{1}{2}} \Sigma_{j=1}^{n} \Psi_n(W_j) ,$$

where

$$v_n = -[EL_n^*(Z_1)][EL_n(Y_1)]^{-1}$$

$$\varphi_n(W_j) = L_n^*(Z_j) + v_n L_n(Y_j)$$

$$\Psi_n(W_j) = \varphi_n(W_j) - E\,\varphi_n(W_j)$$

$$W_j = (X_j, \ldots, X_{j+t-1}) \quad (t = \max(r, s)) \ .$$

By Theorem B it suffices then to prove asymptotic normality for

$$h_n^{-\frac{1}{2}} \Sigma_{j=1}^{n} [\varphi_n(W_j) - E\,\varphi_n(W_j)] \ .$$

This last expression will clearly be asymptotically normal, provided φ_n and Ψ_n satisfy a condition analogous to (A3). Below, a theorem referring to the asymptotic normality of the expression in question is formulated, and a set of sufficient conditions for this theorem to be true is given. The conditions to be used in this subsection are

(A2)** (i) $EL_n(Y_1) \neq 0, \quad n = 1, 2, \ldots$

(ii) $h_n^{-1} \Sigma_{j=1}^{n} L_n(Y_j) \to \ell$ ($\neq 0$ constant), in probability, as $n \to \infty$.

(A3)** For n = 1, 2, ..., $\{\varphi_n\}$ is a sequence of uniformly bounded real-valued measurable functions on $(\mathcal{E}_t, \mathcal{B}^{(t)})$ such that the relations we get if L_n and f_n are replaced by φ_n and Ψ_n, respectively, in (A3) are true. (The relation corresponding to (A3) (iv) may be valid with a different constant $\sigma_o^2 < \infty$) .

(A4)** (i) Both (A3) (ii) and (A3) (iii) remain true if any one or two f's are replaced by the corresponding g's.

(ii) $[EL^*_n(Z_1)][EL_n(Y_1)]^{-1} = -v_n \to -v$(finite), as $n \to \infty$.

(iii) $h_n^{-1}\ n\ E[f_n(Y_1)g_n(Z_1)] \to \sigma$ (finite), as $n \to \infty$.

<u>Theorem</u> 2.3.1. Let assumptions (A1), (A2), (A2)**, and (A3)** be satisfied. We assume that a choice of α which satisfies (2.1.4) also satisfies (2.1.11).

Then, as $n \to \infty$, the law of

$$h_n^{\frac{1}{2}} \{[\Sigma_{j=1}^n L_n^*(Z_j) / \Sigma_{j=1}^n L_n(Y_j)] - [EL_n^*(Z_1) / EL_n(Y_1)]\} \to \begin{cases} N(0, \sigma_o^2 \ell^{-2}), & \text{if } \ell > 0 \\ 1-N(0, (\sigma_o \ell)^2), & \text{if } \ell < 0 , \end{cases}$$

in the sense of Theorem B, provided $\sigma_o^2 > 0$, where $\sigma_o^2 = \lim h_n^{-1} n$.

$\sigma^2[\varphi_n(Z_1)]$, as $n \to \infty$. Furthermore (A4)**, (A3), and (A3)* make up a set of sufficient conditions for (A3)** to be true, and therefore under (A1), (A3), (A3)*, (A2)**, (A4)**, and a choice of α satisfying both (2.1.4) and (2.1.11) the theorem is true. In this case $\sigma_o^2 = \sigma_2^2 + v^2\sigma_1^2 + 2v\sigma$.

<u>Proof</u>. Clearly, for the first part of the theorem there is nothing to be proved. As for the second part, we have to show that (A3), (A3)*, and (A4)** imply (A3)**. The uniform boundedness of $\{\varphi_n\}$, n = 1, 2,... follows from the uniform boundedness of $\{L_n\}$, $\{L_n^*\}$, n = 1, 2,... and (A4)** (ii). Next, $E|\varphi_n(W_1)|^2$ is $O(h_n n^{-1})$ by Minkowski inequality,

(A3) (i), (A3)* (i), and (A4)** (ii). We also have

$$E[\Psi_n(W_1)\Psi_n(W_j)] = E[g_n(Z_1)g_n(Z_j)] + v_n^2 E[f_n(Y_1)f_n(Y_j)]$$
$$+ v_n E[g_n(Z_1)f_n(Y_j)] + v_n E[f_n(Y_1)g_n(Z_j)]$$

from which if follows that $E[\Psi_n(W_1)\Psi_n(W_j)]$ are $O(h_n^2 n^{-2})$ uniformly in j, $1 < j \leqq n$, by means of (A3) (ii), (A3)* (ii), the first part of (A4)** (i), and (A4)** (ii). In a similar fashion replacing the Ψ_n's by what they are equal to in $E[\Psi_n(W_i)\Psi_n(W_j)]$ and using (A3) (iii), (A3)* (iii), the second part of (A4)** (i), and (A4)** (ii), we see that $E[\Psi_n(W_1)\Psi_n(W_i)\Psi_n(W_j)]$ are $O(h_n^3 n^{-3})$ uniformly in i and j, $1 < i < j \leqq n$.

Finally,

$$h_n^{-1} n\sigma^2[\varphi_n(W_1)] = h_n^{-1} n\sigma^2[L_n^*(Z_1)] + v_n^2 h_n^{-1} n\sigma^2[L_n(Y_1)]$$
$$+ 2v_n h_n^{-1} n E[f_n(Y_1)g_n(Z_1)]$$

and this converges to $\sigma_2^2 + v^2\sigma_1^2 + 2v\sigma$, as $n \to \infty$, by (A3) (iv), (A3)* (iv), (A4)** (ii), and (A4)** (iii). This completes the proof of the theorem.

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13. Abstract. In the present paper it is first proved that, under essentially the same conditions, the quotients

$$(\Sigma_{m=1}^{n} g_m / \Sigma_{m=1}^{n} f_m) \text{ and } (\Sigma_{m=1}^{n} f_m / \Sigma_{m=1}^{n} g_m)$$

properly normalized, are also asymptotically normal. This generalizes Theorem 7.5' mentioned above. Next, the functions f and g are also considered to be indepedent on n--the number of the random variables X_j, j=1,...,n--and asymptotic normalities similar to the ones mentioned above are established under a number of conditions. The results obtained here are useful in statistical applications and are applied in the problem of non-parametric estimation in Markov processes.