DEPARTMENT OF STATISTICS

The University of Wisconsin
Madison, Wisconsin

TECHNICAL REPORT No. 108

March 1967

OPTIMAL CONTROL WHEN THE VARIANCE OF THE MANIPULATED VARIABLE IS CONSTRAINED

by

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This research was supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force under AFSOR Grant No. AF-AFSOR-1158-66.

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Discrete feedback control schemes discussed by Box and Jenkins [1], [2], [3], [4] were designed to produce minimum mean square error in the output. It was tacitly supposed that there was no restriction in the degree of manipulation that could be applied to x_t the input or manipulated variable to achieve this. It sometimes happens [5] that we are not able to employ these optimal schemes because the amount of variation which can be allowed in the input adjustments x_t is restricted by practical limitations. We here consider, therefore, how the previous control schemes would need to be modified if a constraint was placed on $V(x_t)$ the variance in the input. Such modified schemes are of considerable practical importance because unrestricted schemes can call for excessively large values of $V(x_t)$.

For example, consider again the important case in which the disturbance n_t at the output can be represented by a model

$$\nabla n_t = (1-\theta B)\, a_t \tag{1}$$

of order (0,1,1) [4] while the output and input are related by simple exponential dynamics so that

$$\frac{(1-\delta B)}{1-\delta}\, y_t = g\, x_{t-1} \tag{2}$$

where it will be recalled that $1-\delta$ may be interpreted as the proportion of

A constrained scheme

Consider now the situation where the models for disturbance and dynamics are again given by equations (1) and (2) but some restriction of the input variance is necessary. The unrestricted optimal schemes have the property that the errors in the output $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ are the uncorrelated random variables $a_t, a_{t-1}, a_{t-2}, \ldots$ and the variance of the output σ_ε^2 has the minimum possible value σ_a^2. With the restricted schemes the variance σ_ε^2 will necessarily be greater than σ_a^2 and the errors $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ at the output will be correlated.

We shall pose our problem in the following form: Given that σ_ε^2 be allowed to increase to some value $\sigma_\varepsilon^2 = (1+c)\sigma_a^2$, where c is a positive constant, to find that control scheme which produces the minimum value for $V(x_t)$. Making use of Lagrange's method of undetermined multipliers we may thus seek an unrestricted minimum of the function

$$f = V(x_t) + \mu \{ V(\varepsilon_t) - (1+c)\,\sigma_a^2 \} \qquad (3)$$

Derivation of optimal action

Let the optimal action expressed in terms of the a_t's be

$$x_t = -\frac{1}{g} L(B)\, a_t$$

where $L(B) = \ell_0 + \ell_1 B + \ell_2 B^2 + \ldots$

Then referring to the block diagram which compares at point D the total noise at the output with the accumulated adjustments we see that the error ε_t at the output is

Also in practice control would need to be exerted in terms of the observed output errors ε_t rather than in terms of the a_t's so that the control equation actually used would be of the form

$$x_t = -\frac{1}{g}\,\frac{L(B)}{1 + B\,\psi(B)}\,\varepsilon_t \qquad (6)$$

Equating (7) and (8)

$$\left\{\lambda - \frac{L(B)(1-\delta)}{1-\delta B}\right\} B = (1-B)\,\psi(B)\,B$$

i.e. $$(1-\delta)\,L(B) = \{\lambda - (1-B)\,\psi(B)\}\,(1-\delta B) \qquad (7)$$

Case where δ is negligible

As a preliminary, consider the special case where the system is sufficiently fast-acting so that δ can be taken to be zero. This is the situation where virtually all the response to a step input occurs in one time interval and we have seen that the optimal unrestricted scheme calls for proportional action

$$x_t = -\frac{\lambda}{g}\,\varepsilon_t \quad \text{with} \quad \varepsilon_t = a_t$$

and $$\frac{V(x_t)}{\sigma_a^2} = \frac{\lambda^2}{g^2}$$

For the constrained scheme equation (7) becomes

$$\begin{aligned} L(B) &= \lambda - (1-B)\,\psi(B) \\ &= \lambda - \psi_1 + (\psi_1-\psi_2)B + (\psi_2-\psi_3)B^2 + \ldots\ldots \end{aligned} \qquad (8)$$

$$(B^2 - (2+\nu) B+1) = 0$$

i.e. of $$B + B^{-1} = 2 + \nu .$$

Evidently if ρ is a root then so is ρ^{-1}. Thus the solution is of the form $\psi_j = A_1\rho^j + A_2\rho^{-j}$. Now if ρ has modulus less than 1 then ρ^{-1} has modulus greater than 1, and since $\varepsilon_t = \{ 1 + B\psi(B)\} a_t$ must have finite variance, A_2 must be zero. By substituting the solution $\psi_j = A_1\rho^j$ in (9) we find that $A_1 = \lambda$.

Finally, then $\psi_j = \lambda\rho^j$. It may be noted that since ψ_j must be real then so must the root ρ whence it follows that ν must be positive and so then must ρ. It now follows that

$$\psi(B) = \frac{\lambda\rho}{1-\rho B} \tag{11}$$

$$1 + B\psi(B) = 1 + \frac{\lambda\rho B}{1-\rho B} = \frac{1-\theta\rho B}{1-\rho B} , \qquad (\theta = 1-\lambda) \tag{12}$$

and $$\varepsilon_t = \frac{1-\theta\rho B}{1-\rho B} a_t$$

so that $$\frac{V(\varepsilon_t)}{\sigma_a^2} = 1 + \frac{\lambda^2\rho^2}{1-\rho^2} \tag{13}$$

Also using (11)

$$L(B) = \lambda - \frac{(1-B)\lambda\rho}{1-\rho B} = \frac{\lambda(1-\rho)}{1-\rho B} \tag{14}$$

$Q = c/\lambda^2$. Then $Q = \frac{\rho^2}{1-\rho^2}$ and $\rho^2 = \frac{Q}{1+Q}$ and the output variance becomes $\sigma_a^2(1+\lambda^2 Q)$.

In summary then, supposing we are prepared to tolerate an increase in variance in the output to some value $\sigma_a^2(1+\lambda^2 Q)$ then

1) we compute $\rho = \sqrt{\frac{Q}{1+Q}}$

2) optimal control will be achieved by taking action

$$x_t = (1-\lambda)\rho\, x_{t-1} - \frac{1}{g}\lambda(1-\rho)\,\varepsilon_t$$

3) the variance of the input will be reduced to

$$V(x_t) = \frac{\lambda^2}{g} \quad \frac{1-\rho}{1+\rho}$$

That is, it will reduce to a value that is R% of that for the unconstrained scheme where $R = 100\left(\frac{1-\rho}{1+\rho}\right)$.

Table 1 shows ρ and R for values of Q between 0.1 and 1.0

Q	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
ρ	0.302	0.408	0.480	0.535	0.577	0.612	0.641	0.667	0.688	0.707
R	53.7	42.0	35.1	30.3	26.8	24.0	21.9	20.0	18.5	17.2

Table 1. Values of parameters for a simple constrained control scheme.

Case where δ is not negligible

Consider now the more general situation where δ is not zero. and the system dynamics must be taken account of. Equation (7) is

$$(1-\delta)\, L(B) = \{\, \lambda - (1-B)\, \psi(B)\, \}\, (1-\delta B)$$

$$= \lambda - \psi_1 - \{\, \delta\lambda - (1+\delta)\psi_1 + \psi_2 \,\}\, B$$

$$- \{\, \delta\psi_1 - (1+\delta)\psi_2 + \psi_3 \,\}\, B^2$$

$$- \{\, \delta\psi_2 - (1+\delta)\psi_3 + \psi_4 \,\}\, B^3 \; \ldots$$

We have now to find that $\psi(B)$, and hence that $L(B)$, which minimizes f in equation (3). Equivalently we may minimize

$$f(\psi) = \frac{(1-\delta)^2 g^2\, V(x_t)}{\sigma_a^2} + \nu \left\{ \frac{V(\epsilon_t)}{\sigma_a^2} - (1 + c) \right\}$$

where ν is the undetermined multiplier.

Now

$$f(\psi) = (\lambda - \psi_1)^2 + \{\, \delta\lambda - (1+\delta)\psi_1 + \psi_2 \,\}^2$$

$$+ \{\, \delta\psi_1 - (1+\delta)\psi_2 + \psi_3 \,\}^2$$

$$+ \{\, \delta\psi_2 - (1+\delta)\psi_3 + \psi_4 \,\}^2 + \ldots$$

$$+ \nu\, \{\psi_1^2 + \psi_2^2 + \psi_3^2 + \ldots\ \}$$

and on equating to zero the derivatives of $f(\psi)$ with respect to ψ_1, ψ_2, ψ_3, ...

are obtained by substitution to give

$$A_1 = \frac{\lambda\rho_1(1-\rho_2)}{\rho_1 - \rho_2} \qquad A_2 = -\frac{\lambda\rho_2(1-\rho_1)}{\rho_1 - \rho_2}$$

So that

$$\psi(B) = \frac{\lambda}{\rho_1 - \rho_2}\left\{\frac{(1-\rho_2)\rho_1^2}{(1-\rho_1 B)} - \frac{(1-\rho_1)\rho_2^2}{(1-\rho_2 B)}\right\}$$

$$= \lambda\left\{\frac{(\rho_1+\rho_2-\rho_1\rho_2) - \rho_1\rho_2 B}{1-(\rho_1+\rho_2)B+\rho_1\rho_2 B^2}\right\}$$

$$\psi(B) = \lambda\left\{\frac{k_o - k_1 B}{1-(k_o+k_1)B+k_1 B^2}\right\} \tag{20}$$

where $k_o = \rho_1 + \rho_2 - \rho_1\rho_2$ $\qquad k_1 = \rho_1\rho_2$.

and

$$1 + B\psi(B) = \frac{1-k_1 B - (1-\lambda)(k_o B - k_1 B^2)}{1-(k_o+k_1)B + k_1 B^2} \tag{21}$$

Now substituting (23) in equation (10)

$$L(B) = \frac{\lambda(1-\delta B)(1-k_o)}{(1-\delta)(1-(k_o+k_1)B + k_1 B^2)} \tag{22}$$

and $$\frac{L(B)}{1+B\,\psi(B)} = \frac{\lambda(1-\delta B)\ (1-k_o)}{(1-\delta)\ \{1 - k_1 B - (1-\lambda)(k_o B - k_1 B^2)\}}$$

Also from (22) x_t is a second order autoregressive process so that

$$\frac{V(x_t)}{\sigma_a^2} = \frac{\lambda^2}{g^2(1+\delta^2)} \quad \frac{(1-k_o)\left\{(1+\delta^2)(1+k_1)-2\delta(k_o+k_1)\right\}}{(1+k_o+2k_1)(1-k_1)} \quad . \tag{26}$$

<u>Computation of k_o and k_1</u>

It remains to compute values for k_o and k_1. The characteristic equation (19) may be written

$$B^4 - M B^3 + N B^2 - M B + 1 = 0$$

where $M = \frac{(1+\delta)^2}{\delta}$ and $N = \frac{(1+\delta)^2 + (1+\delta^2) + \nu}{\delta}$.

It may also be written in the form

$$(B^2 - TB + P)(B^2 - P^{-1}TB + P^{-1}) = 0$$

where $T = \rho_1 + \rho_2$ and $P = \rho_1\rho_2$.

Equating coefficients

$$T + P^{-1}T = M \qquad \text{i.e.} \quad T = \frac{PM}{1+P}$$

$$P+P^{-1}+P^{-1}T^2 = N$$

i.e. $$P + P^{-1} + \frac{PM^2}{(1+P)^2} = N$$

An Example

We have previously [2] used for illustration an example in which viscosity was controlled to a target value of 92 by varying the gas rate. For the pilot control scheme $\lambda = 1.0$, $(\theta = 0)$, $\delta = 0.5$ so that the optimal control action would then be

$$x_t = -\frac{1}{g}\frac{1}{1-\delta}(\varepsilon_t - 0.5\,\varepsilon_{t-1}) = -\frac{1}{g}(2\varepsilon_t - \varepsilon_{t-1})$$

with $\varepsilon_t = a_t$.

The variance of x_t will then be

$$\sigma_x^2 = \frac{1}{g^2}\,5\,\sigma_a^2 \quad \text{i.e.} \quad g\frac{\sigma_x}{\sigma_a} = \sqrt{5} = 2.24$$

Figure 1 shows the reduction of $g\frac{\sigma_x}{\sigma_a}$ possible for various values of $\frac{\sigma_\varepsilon}{\sigma_a}$ with the accompanying optimal control parameters. We see in particular that for a 10% increase in the standard deviation of the output the standard deviation of the input can be halved.

Figure 2 further illustrates this point. When we previously discussed this example a set of twenty-four successive observations were shown. These values of inputs(gas rate) and outputs (viscosity) are reproduced in the left hand diagrams appropriate to the optimal unrestricted

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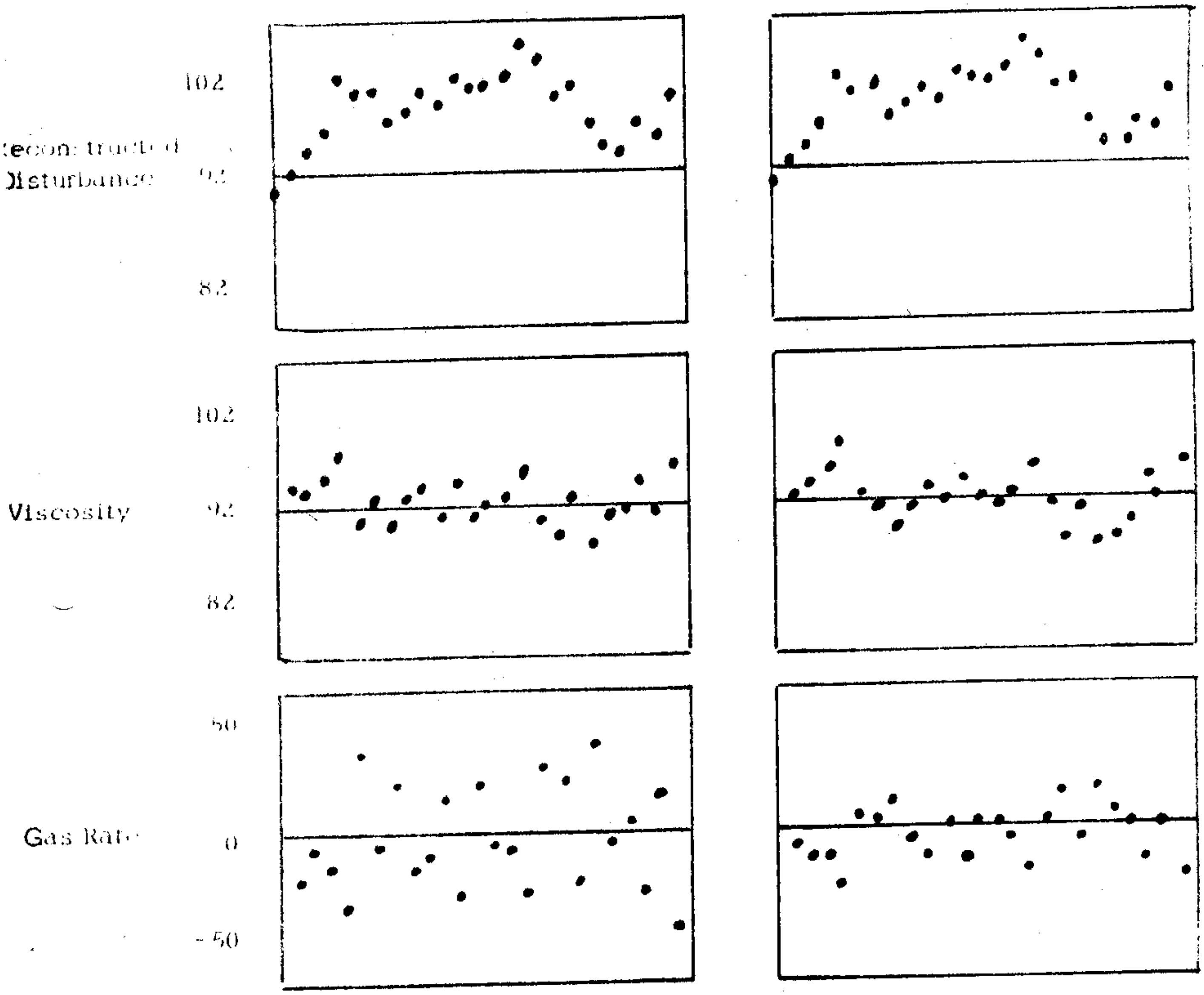
* As we have explained in [2] the correct parameters were slightly different.



Control Equations

$x_t = -10(\varepsilon_t - 0.5\varepsilon_{t-1})$

$x_t = 0.15x_{t-1} - 5.5(\varepsilon_t - 0.5\varepsilon_{t-1})$

Figure 2.- Behavior of unconstrained and constrained control schemes.

Use of the table

Table 2 is provided to facilitate the selection of optimal constrained control schemes.

The vertical margin shows the value of δ the dynamic constant. The horizontal margin shows the value of Q where

$$\frac{V(\varepsilon_t)}{\sigma_a^2} = 1 + \lambda^2 Q$$

so that (equation (25))

$$Q = \frac{(k_o+k_1)^2(1-k_1) - 2k_1(k_o-k_1^2)}{(1-k_1)\{(1+k_1)^2 - (k_o+k_1)^2\}} .$$

Three entries are shown in the body of the table

(1) the % reduction R in variance of the input as compared with the unconstrained scheme

Thus (using (26))

$$R = \frac{100}{(1+\delta^2)} \quad \frac{(1-k_o)\{(1+\delta^2)(1+k_1) - 2\delta(k_o+k_1)\}}{(1+k_o+2k_1)(1-k_1)} .$$

(2) and (3) the values of k_o and k_1 for the optimal scheme.

For illustration suppose $\lambda = 0.6$, $\delta = 0.5$, $g = 1$. The optimal unconstrained control equation is then

$$x_t = -1.2(1-0.5B)\varepsilon_t$$

and $V(x_t) = 1.80 \sigma_a^2$. Suppose that this amount of variation in the input variable produces difficulties in process operation and it is desired to cut $V(x_t)$ to about $0.50\ \sigma_a^2$, that is, to about 28% of the value for the unconstrained scheme. Inspection of the table in the column labelled $\delta = 0.5$ shows that a reduction to 26.5% can be achieved by using a control scheme with constants $k_o = 0.43$, $k_1 = 0.15$; that is, by employing the control equation

$$x_t = 0.32\ x_{t-1} - 0.06\ x_{t-2} - 0.57 \times 1.2\ (1-0.5B)\ \varepsilon_t \quad .$$

This solution corresponds to a value $Q = 0.20$. The variance at the output will, therefore, be increased by a factor of $1 + \lambda^2 Q = 1+0.6^2\ 0.2 = 1.072$ that is by about 7%.