CS559 Homework 1 name: __KEY____________
Due: September 9, 2003, 2:30 pm

Cs login: ________________

Linear Algebra Refresher

Please either answer in the space provided, or attach additional sheets. Be sure to write your name AND CS login and EVERY page, and to attach multiple pages together with a staple. Remember, your course survey is also due at the same time (but don't staple it to your homework)

1. Prove that the determinant of the matrix $\begin{matrix} x1 & x2 & x3 \\ y1 & y2 & y3 \\ 1 & 1 & 1 \end{matrix}$ can be used to tell if the three 2D points ((x1,y1), (x2,y2), (x3,y3)) are co-linear. Show what value this determinant has when the points are co-linear and that it must have this value for any set of co-linear points.

There are a couple ways to prove this, either one will do.

1) If the points are collinear, then the determinant is zero.

Suppose that the points are co-linear. Then either the line is vertical (so all the x's have the same value, call it c), or the line is y=mx+b, so y1=m*x1+b and y2=m*x2+b and y3=m*x3+b.

We can plug both of these cases into the matrix determinant formula and see that they are equal to zero.

Det $\begin{matrix} c & c & c \\ y1 & y2 & y3 \\ 1 & 1 & 1 \end{matrix}$ = c y2 – c y1 + c y3 – c y2 + c y1 – c y3 = 0

And

Det $\begin{matrix} x1 & x2 & x3 \\ mx1+b & mx2+b & mx3+b \\ 1 & 1 & 1 \end{matrix}$ =

x1 (mx2+b) – x2 (mx1+b) + x2 (mx3+b) – x3 (mx2+b) + x3(mx1+b) – x1 (mx3+b)

after a little term rearrangement…

b(x1+x2+x3-x1-x2-x3) + m(x1x2 + x2x3 + x3x1 – x1x3 – x2x1 – x3x2) or 0 b + 0 m = 0

2) If the determinant is zero, then the points are collinear.

All of the line segments are connected, so to show that the points are co-linear, we need to show that the slopes are the same.
Plugging x1,y1,x2,y2,x3,y3 into the determinant formula, and setting it equal to zero, we get
$x1\ y2 - x2\ y1 + x2\ y3 - x3\ y2 + x3\ y1 - x1\ y3 = 0$
Now, a little re-arranging gives
$x1y2 - x1\ y3 + x2y3 = y1x2 - y1x3 + y2x3$
Then the clever step, subtract x2y2 from both sides, which gives
$x1y2 - x1\ y3 + x2y3 - x2y2 = y1x2 - y1x3 + y2x3 - x2y2$
Which can be factored into
$(x1-x2)\ (y2-y3) = (y1-y2)\ (x2-x3)$ **
which can be re-arranged into:
$(x1-x2)\ /\ (y1-y2) = (x2-x3)\ /\ (y2-y3)$
showing that the slopes are indeed the same for these two segments.
Except when the lines are horizontal (or vertical if you didn't put the slopes upside down as I did in the last equation), so we can't divide by (y1-y2) or (y2-y3) (since they'd be zero).
So supposed one of them (say y1-y2) is zero, Plug that into **, to get
(x1-x2) (y2-y3) = 0, which means either x1-x2 is zero or y2-y3 is zero. In the former case, 2 points are in the same place, so clearly the 3 points are collinear. In the latter case, both lines are horizontal, so their slopes are the same.

2. Suppose we have n k by k matrices (A, B, C, …) and m k-vectors (a, b, c, …) and that we want to compute the product of all of these matrices times each vector (e.g. ABCa, ABCb, ABCc, …).
   Clearly, for each of these products it is faster (in terms of the number of arithmetic operations) to do the multiplies from right to left (doing a sequence of vector/matrix multiplies) than left to right.
   However, for a large enough number of vectors (m), it will be faster to compute the product of the matrices and to multiply this matrix by each vector.
   Write a condition that determines which is faster (it should be an "if" statement involving n, m, and k.

Notice that if we have a k by k matrix, a matrix multiply is exactly the same as doing k matrix vector multiplies. This saves us from having to count individual adds, multiplies, and memory accesses.
If we do each vector multiplied by each matrix (the left to right method) the time taken (in terms of number of matrix time vector multiplies) is n*m.
If we multiply all the matrices together, that takes (n-1) * k vector matrix multiplies. Once we have this, we need to multiply it by each of the m vectors.
So, the latter is faster if (n-1)*k+m < n*m.

3. Consider the plane (in 3 space) 2x + 2y + z = 2. What point on this plane is closest to the origin? Does the unit sphere intersect this plane?

The vector normal to this plane is (2,2,1). The shortest path from a point to the plane is along the normal direction (consider the triangle inequality), so the closest point to the origin is the intersection of a line passing through the origin in this direction and the plane. So we define the line as (2t, 2t, t), plug this into the plane equation to get:
(4/9, 4/9, 2/9)

Since the distance from the origin to this point is (4/9)^2 + (4/9)^2 + (2/9)^2 < 1, this point is inside the unit sphere. Therefore, the plane must intersect the unit sphere.

4. Given 2 vectors in 3D (a,b,c) and (d,e,f), write an equation describing the set of points that includes the origin and is perpendicular to these vectors as:

The cross product of two 3D vectors returns a 3D vector that intersects the two vectors and is perpendicular to each of them. See p 27 of the Shirley book for an example of computing a cross product.

(ax + by + cz) x (dx + ey + fz) = (bf–ce)x + (cd-af)y + (ae-bd)z

   a. A parametric (explicit) equation (e.g. f(s) -> (x,y,z))

Read about 3D parametric lines on p 40 of the Shirley book. To write a parametric equation, we need to know two points on the line. Two points on the line are:
(0, 0, 0) and (bf–ce, cd-af, ae-bd)

x = (bf-ce)t
y = (cd-af)t
z = (ae-bd)t

b. An implicit equation (e.g. f(x,y,z) = s)

The implicit function (f(x,y,z) needs to have value zero if the point (x,y,z) is on the line perpendicular to the vectors, and non-zero elsewhere. Here is one way to construct such a function:

If x,y,z is perpendicular to a,b,c then (x,y,z) * (a,b,c) = 0
(using * for the dot product)
If d,e,f is perpendicular to a,b,c then (x,y,z) * (d,e,f) = 0

Now we need to combine these two functions - we can't just add them (because if one is positive and the other is negative, then we will get zero). First we square them (to make sure they are positive) and then add them. this gives us:

$f(x,y,z) = ((x,y,z) * (a,b,c))^2 + ((x,y,z) * (d,e,f))^2 = 0$

One thing to notice here: what we've basically done is compute the distance of the point to the object of interest. This is often the characteristic of an implicit function. For this reason, implicit functions are sometimes called distance fields.

5. Do problems 5 and 10 in Chapter 2 of Shirley.

5. Remember the quadratic formula from Algebra I? There are at most two possible values for x.

x = -b + (sqrt((-b)*(-b) – 4*a*c) / (2*a)) or
x = -b - (sqrt((-b)*(-b) – 4*a*c) / (2*a))

x = {0.75 - 1.299 i , 0.75 + 1.299 i }

10. To get the gradient of f(x, y, z), find the derivative for each of the x, y, and z expressions. The gradient of a 3 variable function is a 3 vector.

$f(x,y,z) = (x^2 + y - 3z^3)$
$f'(x,y,z) = (2x, 1, -9z^2)$