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CS 638
Homework 5

Question 1: Let $x(u) = au^3 + bu^2 + cu + d$ be the curve that describes the x-coordinate curve, and let $P_i = (x_i, y_i)$ for $i = 0, 1, 2$ and 3 (Assuming a planar curve). Then

$$\begin{aligned} x_0 &= x(0) = d \\ x_1 &= x(.5) = a/8 + b/4 + c/2 + d \\ x_2 &= x'(.5) = (3/4)a + b + c \\ x_3 &= x(1) = a + b + c + d. \end{aligned}$$

Hence, in matrix form this is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1/8 & 1/4 & 1/2 & 1 \\ 3/4 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3. \end{pmatrix}$$

If we denote the 4×4 matrix by $\mathbf{A}$ and the vectors as $\mathbf{c}$ and $\mathbf{x}$, then the formula becomes $\mathbf{Ac} = \mathbf{x}$. So, if we wish to solve for the equation for $\mathbf{c}$, we need to multiply both sides of the equation by $\mathbf{A}^{-1}$ to get $\mathbf{c} = \mathbf{A}^{-1}\mathbf{x}$. In order to this explicitly, we need to find $\mathbf{A}^{-1}$. To find $\mathbf{A}^{-1}$, we simply apply transformations to $\mathbf{A}$ until we get the identity matrix and apply the same transformations to $\mathbf{I}$:

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1/8 & 1/4 & 1/2 & 1 & 0 & 1 & 0 & 0 \\ 3/4 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1/8 & 1/4 & 1/2 & 1 & 0 & 1 & 0 & 0 \\ 3/4 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1/8 & 3/8 & 7/8 & 0 & 1 & 0 & -1/8 \\ 0 & 1/4 & 1/4 & -3/4 & 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 7 & 0 & 8 & 0 & -1 \\ 0 & 1 & 1 & -3 & 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 7 & 0 & 8 & 0 & -1 \\ 0 & 0 & -2 & -10 & 0 & -8 & 4 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 7 & 0 & 8 & 0 & -1 \\ 0 & 0 & 1 & 5 & 0 & 4 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & -7 & 8 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5 & 4 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 4 & -4 & 2 & 0 \\ 0 & 1 & 0 & 0 & 8 & -4 & 6 & -4 \\ 0 & 0 & 1 & 0 & -5 & 4 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -4 & 0 & -4 & 4 \\ 0 & 1 & 0 & 0 & 8 & -4 & 6 & -4 \\ 0 & 0 & 1 & 0 & -5 & 4 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

So,

$$\mathbf{A}^{-1} = \begin{pmatrix} -4 & 0 & -4 & 4 \\ 8 & -4 & 6 & -4 \\ -5 & 4 & -2 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is the desired matrix. This allows us to calculate the canonical parameters for the curve for x, but it can be used for other coordinates as well.

Question 2: Again, let $x(u) = au^3 + bu^2 + cu + d$, and let $P_i = (x_i, y_i)$ for $i = 0, 1, 2, 3$ and 4 (Assuming a planar curve). This cubic will define the x-coordinate curve for the first half of the curve. The x-coordinate curve for the second half of the curve will be defined as $z(u) = eu^3 + fu^2 + gu + h$. The points x_i for $i = 0, 1, 2, 3$ and 4 will be defined as

$$\begin{aligned} x_0 &= x(0) = d \\ x_1 &= x'(0) = c \\ x_2 &= x''(0) = 2b \\ x_3 &= x(1) = a + b + c + d \\ x_4 &= z(1) \end{aligned}$$

Thus, for the first half of the curve we have

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3. \end{pmatrix}$$

Again, we will rewrite this as $\mathbf{Ac} = \mathbf{x}$ and solve for $\mathbf{A}^{-1}$.

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & -1 & -1 & -1/2 & 1 \\ 0 & 1 & 0 & 0 & | & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & 1 & 0 & 0 & 0 \end{pmatrix}$$

So,

$$\mathbf{A}^{-1} = \begin{pmatrix} -1 & -1 & -1/2 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is the matrix for the desired transformation and the parameters are given by

$$\begin{aligned} a &= -x_0 - x_1 - x_2/2 + x_3 \\ b &= x_2/2 \\ c &= x_1 \\ d &= x_0 \end{aligned} \tag{1}$$

For the second half of the curve, the requirement of $\mathcal{C}^2$ continuity gives us

$$\begin{aligned} z(0) &= x(1) = a + b + c + d \\ z'(0) &= x'(1) = 3a + 2b + c \\ z''(0) &= x''(1) = 6a + 2b \\ z(1) &= x_4 \end{aligned} \tag{2}$$

which we can now reduce to

$$\begin{aligned} z(0) &= x_3 \\ z'(0) &= -3x_0 - 2x_1 - x_2/2 + x_3 \\ z''(0) &= -6x_0 - 6x_1 - 2x_2 + 6x_3 \\ z(1) &= x_4. \end{aligned} \tag{3}$$

Substituting into the equation for $z(u)$, we get

$$\begin{aligned} h &= x_3 \\ g &= -3x_0 - 2x_1 - x_2/2 + x_3 \\ 2f &= -6x_0 - 6x_1 - 2x_2 + 6x_3 \\ e + f + g + h &= x_4. \end{aligned} \tag{4}$$

We can rewrite this into matrix form as

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -3 & -2 & -1/2 & 1 & 0 \\ -6 & -6 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4. \end{pmatrix}$$

We can rewrite this as $\mathbf{Ac} = \mathbf{Bx}$. Then multiplication of both sides by $\mathbf{A}^{-1}$ gives $\mathbf{c} = \mathbf{A}^{-1}\mathbf{Bx}$. Here $\mathbf{A}^{-1}\mathbf{B}$ is the desired matrix. Recall that we have already calculated $\mathbf{A}^{-1}$, so we merely need to multiply the matrices

$\mathbf{A}^{-1}$ and $\mathbf{B}$. The resulting equation is

$$\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1/2 & 1 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -3 & -2 & -1/2 & 1 & 0 \\ -6 & -6 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4. \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 5 & 3/2 & -5 & 1 \\ -3 & -3 & -1 & 3 & 0 \\ -3 & -2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4. \end{pmatrix}$$

and the matrix for conversion is

$$\mathbf{A}^{-1}\mathbf{B} = \begin{pmatrix} 6 & 5 & 3/2 & -5 & 1 \\ -3 & -3 & -1 & 3 & 0 \\ -3 & -2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{5}$$

The matrices for converting from the 5 points to canonical parameters can put together to get

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 6 & 5 & 3/2 & -5 & 1 \\ -3 & -3 & -1 & 3 & 0 \\ -3 & -2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4. \end{pmatrix}$$

The transformations for other coordinates are the same.

Question 3: Let $x(u) = au^5 + bu^4 + cu^3 + du^2 + eu + f$, be the x-coordinate curve, and let $P_i = (x_i, y_i)$ for $i = 0, 1, 2, 3, 4$ and 5. Then we have

$$\begin{aligned} x_0 &= x(0) = f \\ x_1 &= x'(0) = e \\ x_2 &= x(.5) = a/32 + b/16 + c/8 + d/4 + e/2 + f \\ x_3 &= x'(.5) = 5a/16 + b/2 + 3c/4 + d + e \\ x_4 &= x(1) = a + b + c + d + e + f \\ x_5 &= x'(1) = 5a + 4b + 3c + 2d + e \end{aligned}$$

We can write this in matrix form as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/32 & 1/16 & 1/8 & 1/4 & 1/2 & 1 \\ 5/16 & 1/2 & 3/4 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Denote the 6×6 matrix as $\mathbf{A}$ and the column matrices by $\mathbf{c}$ and $\mathbf{x}$ as before, and we get $\mathbf{Ac} = \mathbf{x}$. Multiplication by $\mathbf{A}^{-1}$ on both sides gives $\mathbf{c} = \mathbf{A}^{-1}\mathbf{x}$. So, we need to solve for $\mathbf{A}^{-1}$. As before,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/32 & 1/16 & 1/8 & 1/4 & 1/2 & 1 \\ 5/16 & 1/2 & 3/4 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ - & - & - & - & - & - \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 1/32 & 1/16 & 1/8 & 1/4 & 1/2 & 1 \\ 5/16 & 1/2 & 3/4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 1/32 & 3/32 & 7/32 & 15/32 & 31/32 \\ 0 & 3/16 & 7/16 & 11/16 & 11/16 & -5/16 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 1 & 0 & -1/32 & 0 \\ 0 & 0 & 0 & 1 & -5/16 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 7 & 15 & 31 \\ 0 & 3 & 7 & 11 & 11 & -5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & -1 \\ 0 & 0 & 32 & 0 & -1 & 0 \\ 0 & 0 & 0 & 16 & -5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 4 & 11 & 26 \\ 0 & 0 & 1 & 2 & -1 & -20 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & -1 \\ 0 & 0 & 32 & 0 & -6 & 1 \\ 0 & 0 & 0 & 16 & -20 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 4 & 11 & 26 \\ 0 & 0 & 0 & -2 & -12 & -46 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & -1 \\ 0 & 0 & 32 & 0 & -6 & 1 \\ 0 & 0 & -32 & 16 & -14 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 4 & 11 & 26 \\ 0 & 0 & 0 & 1 & 6 & 23 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & -1 \\ 0 & 0 & 32 & 0 & -6 & 1 \\ 0 & 0 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 4 & 11 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 0 & 5 & -1 \\ -26 & 0 & 32 & 0 & -6 & 1 \\ -23 & 0 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -5 & -4 & 0 & 0 & 5 & -1 \\ -26 & -11 & 32 & 0 & -6 & 1 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 22 & 5 & -16 & 8 & -6 & 1 \\ 64 & 14 & -48 & 24 & -16 & 2 \\ 66 & 13 & -32 & 32 & -34 & 5 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ -44 & -8 & 16 & -24 & 28 & -4 \\ -68 & -12 & 16 & -40 & 52 & -8 \\ 66 & 13 & -32 & 32 & -34 & 5 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ - & - & - & - & - & - \\ 24 & 4 & 0 & 16 & -24 & 4 \\ -68 & -12 & 16 & -40 & 52 & -8 \\ 66 & 13 & -32 & 32 & -34 & 5 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So, the desired matrix is

$$\mathbf{A}^{-1} = \begin{pmatrix} 24 & 4 & 0 & 16 & -24 & 4 \\ -68 & -12 & 16 & -40 & 52 & -8 \\ 66 & 13 & -32 & 32 & -34 & 5 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 4: The transformation matrix for Hermite cubic is

$$\mathbf{H}^{-1} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

From the last problem, we have the matrix

$$\mathbf{Q}^{-1} = \begin{pmatrix} 24 & 4 & 0 & 16 & -24 & 4 \\ -68 & -12 & 16 & -40 & 52 & -8 \\ 66 & 13 & -32 & 32 & -34 & 5 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for the quintic.

The difference between the two curves occurs at the middle points (every other point) of the specified curves. At the middle point, the quintic is $\mathcal{C}^\infty$ whereas the cubic is only $\mathcal{C}^1$. So, the quintic is much smoother at intermediate points.

For example, consider the quintic specified by

$$\begin{aligned} x(0) &= 0 \\ x'(0) &= 0 \\ x(.5) &= 0 \\ x'(.5) &= 0 \\ x(1) &= 0 \\ x'(1) &= 1. \end{aligned}$$

The cubic polynomials corresponding to these constraints are given by

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &= \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} &= \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z(0) \\ z(1) \\ z'(0) \\ z'(1), \end{pmatrix} \end{aligned}$$

where $x(u) = au^3 + bu^2 + cu + d$ and $z(u) = eu^3 + fu^2 + gu + h$. After multiplication, we have $x(u) = 0$ and $z(u) = u^3 - u^2$. Notice that the first polynomial is a constant. The corresponding quintic can not be zero on any interval since it is a holomorphic nonconstant function and any such function must have isolated zeros.

The corresponding quintic is

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 24 & 4 & 0 & 16 & -24 & 4 \\ -68 & -12 & 16 & -40 & 52 & -8 \\ 66 & 13 & -32 & 32 & -34 & 5 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 4 & 0 & 16 & -24 & 4 \\ -68 & -12 & 16 & -40 & 52 & -8 \\ 66 & 13 & -32 & 32 & -34 & 5 \\ -23 & -6 & 16 & -8 & 7 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v(0) \\ v'(0) \\ v(.5) \\ v'(.5) \\ v(1) \\ v'(1), \end{pmatrix}$$

where $v(u) = au^5 + bu^4 + cu^3 + du^2 + eu + f = 4u^5 - 8u^4 + 5u^3 - u^2$. The graphs of these functions appear as follows: