PIECEWISE LINEAR HOMOTOPIES AND AFFINE VARIATIONAL INEQUALITIES

By

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

(Computer Sciences)

at the

UNIVERSITY OF WISCONSIN – MADISON

1994

Abstract

PIECEWISE LINEAR HOMOTOPIES AND AFFINE VARIATIONAL INEQUALITIES

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The purpose of this thesis is to apply the theory of piecewise linear homotopies and the notion of a normal map in the construction and analysis of algorithms for affine variational inequalities.

An affine variational inequality can be expressed as a piecewise linear equation $A_C(x) = a$, where A is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, C is a polyhedral convex subset of $\mathbb{R}^n$, and A_C is the associated normal map. We introduce a path-following algorithm for solving the equation $A_C(x) = a$. When A_C is coherently oriented, we prove that the path following method terminates at the unique solution of $A_C(x) = a$. This generalizes the fact that Lemke's method terminates at the unique solution of LCP(q, M) with M being a P–matrix. In LCP study, termination of Lemke's method is established for two major classes of matrices, the class of L–matrices introduced by Eaves and the class of P_0–matrices studied by Cottle et al. We generalize the notion of L–matrices for polyhedral convex

sets in $\mathbb{R}^n$ and prove that, when A is a linear transformation associated with such matrices, our algorithm will find a solution for $A_C(x) = a$. unless the it is infeasible in a well specified sense.

Our approach to P_0 begins with the study of geometric characteristics of an LCP that contribute to the finite termination of Lemke's method. Given $K(M)$ as the set of solvable right hand sides for the matrix M and $\mathrm{SOL}(q, M)$ as the set of solutions for $\mathrm{LCP}(q, M)$, we prove that the convexity of $K(M)$ and the connectedness of $\mathrm{SOL}(q, M)$ for all $q \in \mathbb{R}^n$ guarantee finite termination of Lemke's method. We study those matrices such that $\mathrm{SOL}(q, M)$ is connected for all $q \in \mathbb{R}^n$ as a matrix class, denoted by P_c. We are interested in how P_c is related to P_0.

We also study variational inequalities from the perspective of maximal monotone multifunction theory. Our results are presented in the last two chapters.

Acknowledgements

I would like to thank members of my final exam committee: Professor Michael C. Ferris, Professor Stephen M. Robinson, Professor Olvi L. Mangasarian, Professor Robert R. Meyer, and Professor Hans R. Schneider. It has been a privilege to be associated with them in their rôles as both teacher and colleagues. Special thanks are due to my thesis advisor Professor Michael C. Ferris for his constant encouragement, patient advice and guidiance and for his meticulous reading and constructive comments on my work.

At the same time, I owe many thanks to my wife, Ruizhen Wu. Her support throughout my doctoral studies is enormous, I can only hope to match it with my gratitude.

This research was partially funded by grants from the National Science Foundation under Grant CCR–9157632 and the Air Force Office of Scientific Research under Grant AFOSR–89–0410.

Contents

Chapter 1

Introduction

The purpose of this thesis is to apply the theory of piecewise linear homotopies and normal maps in the construction and analysis of algorithms for affine variational inequalities.

Let F be a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and C be a non–empty closed convex set. The variational inequality problem $\mathrm{VI}(F, C)$ is to find $z \in C$ such that

$$\langle F(z), y - z \rangle \geq 0, \forall y \in C. \tag{VI}$$

This problem has appeared in the literature in several equivalent formulations, the most important of which is the generalized equation, that is

$$0 \in F(z) + \partial\psi(z \mid C), \tag{GE}$$

where $\psi(\cdot \mid C)$ is the indicator function of the set C defined by

$$\psi(z \mid C) := \begin{cases} 0 & \text{if } z \in C \\ \infty & \text{if } z \notin C \end{cases}$$

and $\partial\psi$ is the subdifferential of ψ (see [45]).

The variational inequality problem is a very fundamental problem in the theory and practice of optimization. This is mainly due to the fact that optimality conditions for various optimization problems when expressed in the form of minimum principle (see [32] and [41]) are variational inequality problems. Also, most complementarity problems can be equivalently formulated as variational inequalities. It is also an important tool for modeling various equilibrium problems ([26], [33], and [38]). For an up–to–date, comprehensive survey on formulation, theory, algorithms and applications of variational inequalities and complementarity problems, see [27] and [40].

The normal map relating to a function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ and a non–empty, closed, convex set C, is defined as

$$F_C(x) := F(\pi_C(x)) + x - \pi_C(x)$$

where $\pi_C(x)$ is the projection (with respect to the Euclidean norm) of x onto the set C. We call

$$F_C(x) = 0 \tag{NE}$$

a normal equation. Note that (NE) is equivalent to (GE), in the sense that if $F_C(x) = 0$, then $z := \pi_C(x)$ is a solution of (GE). Furthermore, if z is a solution of (GE), then $x := z - F(z)$ satisfies $F_C(x) = 0$.

A very familiar special case of (GE) is when $C \equiv K$ is a polyhedral convex cone. Then it is easy to show that (GE) is equivalent to the generalized complementarity problem [28]

$$z \in K, F(z) \in K^D, \langle F(z) - a, z\rangle = 0$$

where $K^D := \{z^* \mid \langle z^*, k\rangle \geq 0, \forall k \in K\}$ is the dual cone associated with K.

In this work, we focus on the special case where the map F is affine and C is polyhedral. In this case, the normal map is piecewise linear and the normal equation is a piecewise linear equation. The general theory of piecewise linear equations developed by Eaves in [15] and special properties of normal maps induced by linear transformation developed by Robinson [43] are used in constructing and analyzing algorithms for solving this type of normal equation. We generalize the notion of copositive–plus and L–matrices for polyhedral convex sets in $\mathbb{R}^n$ and then prove that our algorithm processes $A_C(x) = a$ when A is the linear transformation associated with such matrices. That is, when applied to such a problem, the algorithm will find a solution unless the problem is infeasible in a well specified sense.

Another important matrix class in the study of linear complementarity problem is P_0. Our approach to P_0 begins with the study of geometric characteristics of an LCP that contribute to the finite termination of Lemke's method. Given $K(M)$ as the set of solvable right hand sides for the matrix M and $\mathrm{SOL}(q, M)$ as the set of solutions for $\mathrm{LCP}(q, M)$, we prove that the convexity of $K(M)$ and the connectedness of $\mathrm{SOL}(q, M)$ for all $q \in \mathbb{R}^n$ guarantee finite termination of Lemke's method. We study those matrices such that $\mathrm{SOL}(q, M)$ is connected for all $q \in \mathbb{R}^n$ as a matrix class, which is denoted by P_c. This matrix class is not contained in P_0, but contains a substantial portion of P_0, e.g. all the column sufficient matrices. We are interested in knowing whether P_0 is a subclass of P_c.

Most of the existing algorithms for the mixed linear complementarity problem rely on a certain non–singularity property of the underlying matrix. Our study shows that copositive matrices have a special structural property which can be exploited in constructing algorithms that do not require any non–singularity assumptions.

In the final two chapters of this thesis, we also investigate variational inequalities from the perspective of maximal monotone multifunction theory.

The following is an introduction to our notation and some mathematical preliminaries.

1.1 Notation

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be the n–tuples of real numbers (n–vectors). The set of $m \times n$ matrices of real numbers is represented by $\mathbb{R}^{m\times n}$. A matrix in $\mathbb{R}^{m\times n}$ is usually represented by an upper case English letter and a vector in $\mathbb{R}^n$ is usually represented by a lower case English letter. Unless otherwise stated, the vector e represents the vector in $\mathbb{R}^n$ with all the components being 1, and the vector e_i represents the vector in $\mathbb{R}^n$ with all the components being 0 except the i–th, which is 1. For any vector or matrix, a superscript T indicates the transpose. Index sets are represented by lower case Greek letters. In particular, for the index set α, $|\alpha|$ denotes the cardinality of α. Given any vector v and index sets α, v_α denotes the set of components of v with index in α. Given any matrix M and index sets α and β, $M_{\alpha\cdot}$ denotes the submatrix formed by those rows of M with indices in α, $M_{\cdot\beta}$ denotes the submatrix formed by those columns of M with indices in β, and $M_{\alpha\beta}$ denotes the submatrix formed by those elements of M with row indices in α and column indices in β.

For any vectors x and y in $\mathbb{R}^n$, $\langle x, y\rangle$ or $x^T y$ denotes the inner product of x and y, and in this thesis, these two notations are freely interchangable. For any vector or matrix $\|\cdot\|_p$ denotes the p-norm, see [39]. B_p is used to denote the unit ball in $\mathbb{R}^n$ with respect to the norm $\|\cdot\|_p$ and B is used as a shorthand for B_2. Given a vector v, $\mathrm{diag}\{v\}$ is the diagonal matrix whose diagonal elements are the components of v.

Each $m \times n$ matrix A represents a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, the symbol A refers to either the matrix or the linear map as determined by the context. Given a linear map A from $\mathbb{R}^n$ to $\mathbb{R}^m$, for any $X \subset \mathbb{R}^n$, the set $A(X) := \{y \in \mathbb{R}^m \mid y = Ax, \text{ for some } x \in \mathbb{R}^n\}$ is called the image of X under A; for any set $Y \subset \mathbb{R}^m$, the set $A^{-1}(Y) := \{x \in \mathbb{R}^n \mid Ax \in Y\}$ is referred to as the inverse image of Y under A. In particular, the set $A^{-1}(\{0\})$ is called the kernel of A denoted as $\ker A$, and the set $A(\mathbb{R}^n)$ is called the image of A denoted as $\mathrm{im} A$.

Given any set $C \subset \mathbb{R}^n$ and the minimization problem

$$\min f(x) \qquad x \in C$$

the set of minimizers is denoted by $\arg\min \{f(x) \mid x \in C\}$. Similarly, for

$$\max f(x) \qquad x \in C$$

$\arg\max \{f(x) \mid x \in C\}$ denotes the set of maximizers.

1.2 Polyhedral Convex Sets in $\mathbb{R}^n$

A set C in $\mathbb{R}^n$ is said to be convex if for any two points $x, y \in C$ and $0 \le \lambda \le 1$ we have

$$\lambda x + (1 - \lambda) y \in C$$

It is a direct consequence of the definition that the intersection of any collection of convex sets is convex.

As examples of convex sets, we introduce sets of the form

$$\{x \in \mathbb{R}^n \mid \langle x, b\rangle \le \beta\}, \qquad \{x \in \mathbb{R}^n \mid \langle x, b\rangle \ge \beta\}$$

and

$$\{x \in \mathbb{R}^n \mid \langle x, b\rangle < \beta\}, \qquad \{x \in \mathbb{R}^n \mid \langle x, b\rangle > \beta\}$$

where $b \neq 0$ and $\beta \in \mathbb{R}$, and call them closed half–spaces and open half–spaces respectively. These sets are easily verified as convex.

A set C is called a polyhedral convex set if C is the intersection of finite number of closed half–spaces. Suppose I is an arbitrary finite index set, and $b_i \in \mathbb{R}^n$, $\beta_i \in \mathbb{R}$, for any $i \in I$. Then, a set of the form

$$\{x \in \mathbb{R}^n \mid \langle x, b_i \rangle \leq \beta_i, i \in I\}$$

is a polyhedral convex set.

A set M is called an affine set if for any two points $x, y \in M$ and $\lambda \in \mathbb{R}$ we have

$$\lambda x + (1 - \lambda) y \in M$$

$$\{x \in \mathbb{R}^n \mid \langle x, b_i \rangle = \beta_i, i \in I\}$$

is an affine set. The following theorem indicates a basic property of affine sets.

Theorem 1.1 ([45, Theorem 1.2]) *Each non–empty affine set M is parallel to a unique subspace L. This L is given by*

$$L = M - M = \{x - y \mid x \in M, y \in M\}$$

As a result, we can define the dimension of M, denoted as $\dim M$ to be the dimension of L.

For a non–empty, closed, convex set C, aff$C$, called the affine hull of C is the smallest affine set containing C. That is

$$\mathrm{aff} C = \bigcap_{C \subset S} S$$

where the sets S are affine sets. The dimension of C, denoted as $\dim C$, is defined to be the dimension of aff$C$.

The topological interior of C with respect to aff$C$ is call the relative interior of C, and is denoted as ri$C$. The closure of C, denoted as cl$C$ is defined as the topological closure of C. The set of points cl$C \setminus$ ri$C$ is called the relative boundary of C and is denoted as rbdry$C$.

For any closed convex set C, the set

$$\text{lin } C := \{d \in \mathbb{R}^n \mid x + \mu d \in C, \forall x \in C, \forall \mu \in \mathbb{R}\}$$

forms a linear subspace of $\mathbb{R}^n$ and is called the lineality space of C (see [45]).

A set K is called a cone if for any $x \in K$ we have $\lambda x \in K$ for any $\lambda > 0$. A cone K is called a convex cone if it is both a cone and a convex set.

Given a convex C, cone$C$ denotes the set

$$\{\lambda x \mid \lambda > 0, x \in C\}$$

We call this the cone generated by C. For any convex set C and $x \in C$, the set

$$N(x \mid C) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x \rangle \leq 0, \forall y \in C\}$$

is a convex cone. We call it the normal cone of C at x.

Let C be a non–empty convex set in $\mathbb{R}^n$, the set

$$\text{rec}C := \{d \in \mathbb{R}^n \mid x + \lambda d \in C, \forall x \in C, \forall \lambda \geq 0\}$$

is called the recession cone of C. A non–empty closed convex set C is bounded if and only if rec$C = \{0\}$ (see [45, Theorem 8.4]).

If C is a cone,

$$C^o = \left\{x \in \mathbb{R}^n \,\middle|\, x^T y \leq 0, \forall y \in C\right\}$$

is the polar cone of C, and

$$C^D = \left\{x \in \mathbb{R}^n \,\middle|\, x^T y \geq 0, \forall y \in C\right\}$$

is the dual cone of C. As a matter of fact $C^o = -C^D$.

Given a convex set C, a convex subset F of C is called a face of C if any line segment in C with a relative interior point in F has both of its endpoints in F. Furthermore if there exists a linear function f that is constant on F and such that $f(x) > f(y)$ for any $x \in F$ and $y \in C \setminus F$, F is called an exposed face of C. A face of dimension 0 is called an extreme point, and an exposed face of dimension 0 is called an exposed point. For a polyhedral convex set every face is an exposed face.

1.3 Piecewise Linear Manifolds

In the theory of piecewise linear manifolds, a polyhedral convex set is called a cell. A cell σ of dimension m is called an m–cell.

Let $\mathcal{M}$ be a finite or countable collection of m–cells in $\mathbb{R}^n$. Let $\mathcal{M}^i$, $i = 1, 2, \cdots, m$, be the set of i–faces of elements of $\mathcal{M}$, that is faces of a member of $\mathcal{M}$ of dimension i. We call members of $\mathcal{M}^i$, $i = 1, 2, \cdots, m$, and $\mathcal{M}^0$ cells and vertices of $\mathcal{M}$ respectively.

Let $M = \bigcup_{\sigma \in \mathcal{M}} \sigma$. We call $(M, \mathcal{M})$ a subdivided m–manifold if

1. any two m–cells of $\mathcal{M}$ are disjoint or meet in a common face.
2. each $(m-1)$–cell of $\mathcal{M}$ lies in at most two m–cells.
3. each point of M has a neighborhood meeting only finitely many m–cells of $\mathcal{M}$.

If $(M, \mathcal{M})$ is a subdivided m–manifold for some $\mathcal{M}$, we call M an m–manifold. Furthermore, if M is a connected set, we call M a connected m–manifold. Figure 1 shows an example of a 2–manifold.

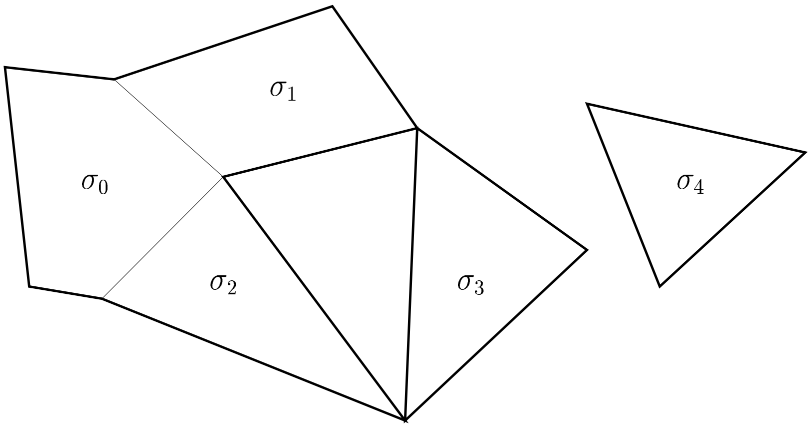

Figure 1: A 2–MANIFOLD

As another example, we show how an $(m+1)$–manifold can be constructed from an m–manifold by using Cartesian product with $\mathbb{R}_+$. Given an m-manifold $(M, \mathcal{M})$, we let

$$\begin{aligned} N &= M \times \mathbb{R}_+ \\ \mathcal{N} &= \{\tau \mid \tau = \sigma \times \mathbb{R}_+, \sigma \in \mathcal{M}\} \end{aligned}$$

Then, it is easy to verify that $(N, \mathcal{N})$ is an $(m+1)$–manifold.

The case of $m = 1$ is of particular interest in this work. This type of manifold can be characterized in a simple way. First, we refer to a convex subset of $\mathbb{R}$ that contains more than one point as an interval, and the set

$$\left\{x \in \mathbb{R}^2 \mid \langle x, x \rangle = 1\right\}$$

as a circle. We say that two sets are homeomorphic to each other if there is a bijective map from one to another and both the map and its inverse are continuous.

A connected 1–manifold is called a curve. The following lemma characterizes a curve.

Lemma 1.2 ([15, Lemma 5.1]) *A connected* 1*–manifold is homeomorphic to either a circle or an interval.*

We call a connected 1–manifold a loop if it is homeomorphic to a circle, and a route if it is homeomorphic to an interval. In general, we have the following characterization.

Lemma 1.3 ([15, Lemma 5.5]) *A* 1*–manifold is a disjoint collection of routes and loops.*

The intersection of a line and a cell σ of $\mathcal{M}$ is called a chord. A ray refers to a chord that is a half line. Considering an m–manifold $(M, \mathcal{M})$, a chord of $\mathcal{M}$ refers to a chord of an m–cell of $\mathcal{M}$. A ray of $\mathcal{M}$ is a chord that is a half line.

For an m–manifold M subdivided by $\mathcal{M}$, the boundary of M, denoted as ∂M, is the union of all $(m-1)$–cells of $\mathcal{M}$ which lie in exactly one m-cell of $\mathcal{M}$. As an example, in Figure 1, the boundary of the 2–manifold is indicated by the bold lines. For the case of 1–manifold, it is easy to see that a loop has empty boundary. However, the boundary of a route may contain 0, 1, or 2 points.

Basic properties of the boundary of a PL–manifold are summarized in the following two lemmas.

Lemma 1.4 ([15, Lemma 6.3]) *The boundary of a manifold is closed in the manifold.*

Lemma 1.5 ([15, Lemma 6.4]) *The boundary of a manifold is independent of the subdivision.*

Consider an m–manifold M and a 1–manifold W contained in M. If W is closed in M and $\partial W = W \cap \partial M$, then we say that W is neat in M. If M is subdivided by $\mathcal{M}$ and $\mathcal{W}$ is the set of 1–chords of $\mathcal{M}$ of the form $W \cap \sigma$ where σ is an m–cell of $\mathcal{M}$, we say that W is neat in $(M, \mathcal{M})$ if W is subdivided by $\mathcal{W}$.

1.4 Piecewise Linear Maps

Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be subdivided manifolds. Let $F : M \to N$ be a continuous map which is linear on each cell σ of $\mathcal{M}$, that is

$$F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y), \qquad \text{for all } x, y \in \sigma \text{ and } \lambda \in \mathbb{R}$$

and which carries each cell σ of $\mathcal{M}$ into a cell τ of $\mathcal{N}$. Then, we call such an F a piecewise linear map.

Given a cell $\sigma \in \mathcal{M}$ and $\tau \in \mathcal{N}$ with $F(\sigma) \subset \tau$, we define $F_\sigma : \text{aff}\sigma \to \text{aff}\tau$ to be the affine map which agrees with F on σ. Such an affine map can be represented as

$$F_\sigma(x) = A_\sigma x + a_\sigma$$

where A_σ and a_σ are matrix and vector of appropriate sizes.

In this work, we are particularly interested in the case of a piecewise linear map from an $(n+1)$–manifold to an n–manifold. Let M and N be an $(n+1)$ and an n–manifold respectively, a point x in M is said to be a degenerate (otherwise regular) point if x lies in a cell σ of $\mathcal{M}$ such that

$$\dim F(\sigma) < n$$

A value y in $F(M)$ is said to be a degenerate (otherwise regular) value if $F^{-1}(y)$ contains any degenerate points.

Suppose F is a piecewise linear map from an $(n+1)$–manifold $(M, \mathcal{M})$ to an n–manifold $(N, \mathcal{N})$ and F_σ is the affine map that agree with F on the cell $\sigma \in \mathcal{M}$. If y is a regular value, then, for each $(n+1)$–cell σ such that $\sigma \cap F^{-1}(y) \neq \emptyset$, the rank of the linear map A_σ is n. Hence $\sigma \cap F^{-1}(y)$ is a one chord. These chords actually form a 1–manifold.

Theorem 1.6 ([15, Theorem 9.1]) *If y is a regular value, then $F^{-1}(y)$ is a 1–manifold neat in $(M, \mathcal{M})$. In this case, $F^{-1}(y)$ is subdivided by sets of form $\sigma \cap F^{-1}(y) \neq \emptyset$ where $\sigma \in \mathcal{M}$.*

When $y = F(x)$ is a degenerate value, the structure of $F^{-1}(y)$ can be much more complicated. However, if x is a regular point or F is locally univalent on the boundary of M at x, structural properties similar to those in the proceeding theorem can also be derived. Note that F is locally univalent on the boundary of M at x if there exists a neighborhood U of x such that $F(x_1) = F(x_2)$ and $x_1, x_2 \in U \cap \partial M$ imply $x_1 = x_2$.

Theorem 1.7 ([15, Theorem 13.1]) *If F is locally univalent on the boundary of M at x, then $F^{-1}(F(x))$ contains a route W neat in $(M, \mathcal{M})$ with $x \in \partial W$. If x is a regular point, then $F^{-1}(F(x))$ contains a curve W neat in $(M, \mathcal{M})$ with $x \in W$. In either case W is subdivided by 1–chords of the form $\sigma \cap W$ with $\sigma \in \mathcal{M}$.*

In our analysis, we need to use regular values to approximate a given point in $F(M)$. Given $\epsilon > 0$, let $[\epsilon] = (\epsilon, \epsilon^2, \cdots, \epsilon^n)^T$. The following lemma identifies a situation in which a point in $F(M)$ can be approximated by a continuous path consisting of regular values.

Lemma 1.8 ([15, Lemma 14.2]) *Assume that $\mathcal{M}$ is finite, $y + Y[\epsilon] \in F(M)$ for all small positive ϵ, and the rank of Y is n. Then, $y + Y[\epsilon]$ is regular for all small positive ϵ.*

1.5 Piecewise Linear Equations

Let F is a piecewise linear map from an $(n+1)$–manifold M to an n–manifold N. Consider the piecewise linear equation

$$F(x) = y \qquad \text{where } y \in N$$

We are interested in algorithms for following paths in $F^{-1}(y)$. These are the basic tools that we use in approaching the piecewise linear equations derived from affine variational inequalities. We restrict our attention to the case where $\mathcal{M}$ and $\mathcal{N}$ are finite.

We first look at the case where y is a regular value and then a more general case. We need the following technical jargon. Given any point $x \in \sigma$, we say that a vector v points into σ from x if $x + \theta v \in \sigma$ for all $0 \leq \theta \leq \bar{\theta}$, where $\bar{\theta} > 0$.

Now, suppose y is a regular value, then $F^{-1}(y)$ forms a 1–manifold neat in $(M, \mathcal{M})$. Suppose we are given a point x_0 in $F^{-1}(y)$, an $(n+1)$–cell σ_0 of $\mathcal{M}$ containing x_0 and a vector v_0 such that $x_0 + \mu v_0 \in \sigma_0 \cap F^{-1}(y)$ for all $0 \leq \mu \leq \bar{\mu}$, where $\bar{\mu} > 0$. We describe an algorithm for moving along the curve of F^{-1} containing x_0 in the direction v_0 as follows.

Algorithm 1

1. **Initialization :**

   Given a triple (x_0, σ_0, v_0) where $F(x_0) = y$, $x_0 \in \sigma_0$, $A_{\sigma_0} v_0 = 0$, and v_0 points into σ_0 from x_0.

2. **Iteration :**

   Given the triple (x_k, σ_k, v_k) compute

   $$\theta_k = \sup \{ \theta \mid x_k + \theta v_k \in \sigma_k \}$$

   If $\theta_k = +\infty$, terminate with a ray.

   If $x_0 \in \sigma_k$ with $k \geq 2$, terminate with a loop.

   Otherwise, let

   $$x_{k+1} = x_k + \theta_k v_k$$

   If $x_{k+1} \in \partial M$, terminate at the boundary.

   Otherwise determine $\sigma_{k+1} \in \mathcal{M} \setminus \{\sigma_k\}$ which contains x_{k+1}. Compute $v_{k+1} \neq 0$ by solving the equation $A_{\sigma_{k+1}} v_{k+1} = 0$ and v_{k+1} points into σ_{k+1} from x_{k+1}. Proceed with the Iteration step with the triple $(x_{k+1}, \sigma_{k+1}, v_{k+1})$.

When y is not a regular value, we need to use the lexicographic rule to resolve degenerate pivots. A non–zero vector x is said to be lexicographically positive, denoted as $x \succ 0$ (negative, denoted as $x \prec 0$), if its first nonzero component is positive (negative). Let x and y be any vectors in $\mathbb{R}^n$, then x is lexicographically greater (less) than y if and only if $x - y \succ 0$ ($x - y \prec 0$). In this way, $\succ$ ($\prec$) defines a total ordering on $\mathbb{R}^n$. The following lemma put this in precise terms.

Lemma 1.9 *Given any x, y, $z \in \mathbb{R}^n$, the following are true*

1. *Either $x \succ y$, $x = y$, or $x \prec y$.*

2. *If $x \succ y$, then $y \prec x$.*

3. *If $x \succ y$, and $y \succ z$, then $x \succ z$.*

Proof By direct algebraic verification. **Q.E.D.**

The lexicographic order can be thought of as being induced by a perturbation term. For example, let $q \in \mathbb{R}^n$,

$$Q = \begin{pmatrix} q & I \end{pmatrix}$$

and

$$[\epsilon] = \begin{pmatrix} \epsilon & \epsilon^2 & \cdots & \epsilon^n \end{pmatrix}^T$$

Then $Q_{i\cdot} \succ Q_{j\cdots}$ if and only if $q_i + I \cdot [\epsilon] > q_j + I \cdot [\epsilon]$ for small positive ϵ.

Suppose x_0 is a regular point, or F is locally univalent at the boundary of M at x_0, let σ_0 be a cell containing x_0. Then, there exists a vector $v_0 \neq 0$ satisfying $A_{\sigma_0} v_0 = 0$. We can also find a set of vectors $x_{01}, x_{02}, \cdots, x_{0n}$, such that $x_0, x_0 + x_{01}, \cdots, x_0 + x_{0n}$ are in σ_0 and $v_0, x_{01}, \cdots x_{0n}$ are linearly independent. We can construct a matrix X_0 as follows

$$X_0 = \begin{pmatrix} x_{01} & x_{02} & \cdots & x_{0n} \end{pmatrix}$$

Then, $F(x_0 + X_0[\epsilon])$ is regular for all small $\epsilon > 0$, since $A_{\sigma_0} X_0$ has rank n. The algorithm starts with the triple $(x_0 + X_0[\epsilon], \sigma_0, v_0)$.

Algorithm 2

1. **Initialization :**

   Given a triple $(x_0+X_0[\epsilon], \sigma_0, v_0)$ where $F(x_0) = y$, $x_0 \in \sigma_0$, $A_{\sigma_0}v_0 = 0$, and v_0 points into σ_0 from x_0.

2. **Iteration :**

   Given the triple $(x_k + X_k[\epsilon], \sigma_k, v_k)$ compute

   $$\theta_k + \Theta_k[\epsilon] = \sup\{\theta \mid x_k + X_k[\epsilon] + \theta v_k \in \sigma_k\}$$

   for small $\epsilon > 0$.

   If $\theta_k = +\infty$, terminate with a ray.

   If $x_0 + X_0[\epsilon] \in \sigma_k$ with $k \geq 2$, terminate with loop.

   Otherwise, let

   $$x_{k+1} + X_{k+1}[\epsilon] = x_k + X_k[\epsilon] + (\theta_k + \Theta_k[\epsilon])v_k$$

   for small $\epsilon > 0$.

   If $x_{k+1} + X_{k+1}[\epsilon] \in \partial M$, terminate at the boundary.

   Otherwise determine $\sigma_{k+1} \in \mathcal{M} \setminus \{\sigma_k\}$ which contains $x_{k+1} + X_{k+1}[\epsilon]$ for small $\epsilon > 0$. Compute $v_{k+1} \neq 0$ by solving the equation $A_{\sigma_{k+1}}v_{k+1} = 0$ and v_{k+1} points into σ_{k+1} from $x_{k+1} + X_{k+1}[\epsilon]$ for small $\epsilon > 0$. Proceed with the Iteration step with the triple $(x_{k+1} + X_{k+1}[\epsilon], \sigma_{k+1}, v_{k+1})$.

This is the algorithm that we use for solving piecewise linear equation derived from affine variational inequalities. Later in Chapter 2, we will show how a starting point is chosen and how a pivot step is computed.

1.6 LCP and Matrix Classes

A special case of using the path following method on piecewise linear equations is Lemke's pivotal method for solving the linear complementarity problem

$$x \geq 0, \quad Mx + q \geq 0, \quad x^T(Mx+q) = 0 \tag{LCP}$$

where M is an $n \times n$ matrix and q is a vector in $\mathbb{R}^n$. The pair LCP(q, M) is used as a shorthand notation for (LCP). For LCP(q, M) the set

$$\mathrm{FEA}(q, M) = \{x \mid x \geq 0, \quad Mx + q \geq 0\}$$

is called the feasible set. An LCP is said to be feasible if its feasible set is non–empty. The set

$$\mathrm{SOL}(q, M) = \left\{x \in \mathrm{FEA}(q, M) \,\middle|\, x^T(Mx+q) = 0\right\}$$

is called the solution set. An LCP is said to be solvable if its solution set is non–empty. The set

$$K(M) = \{q \in \mathbb{R}^n \mid \mathrm{SOL}(q, M) \neq \emptyset\}$$

is the set of all right hand side vectors for which (LCP) is solvable.

The most extensively studied algorithm for solving LCP is Lemke's pivotal algorithm. Termination properties of this algorithm are well known on two classes of matrices, namely, copositive–plus and P (see, for example [7]). Generalizations are found throughout the LCP literature. For example, results concerning L and

L_* matrices, which are extensions of those for copositive–plus matrices, can be found in [12] and [13] (see also [37]). Termination results on P_0 matrices, which are extensions of those for P–matrices can be found in [1] and [9].

Matrix classes related to LCP are numerous (see [37] and [8]). Here is a brief survey of those that are closely related to this work. Most of these matrix classes are defined with respect to $\mathbb{R}^n_+$. Later in this thesis we generalize some of these, e.g. copositive–plus, L, P, column sufficient, etc. to arbitrary polyhedral convex sets in $\mathbb{R}^n$.

A matrix M is said to be copositive if

$$\langle x, Mx \rangle \geq 0, \forall x \geq 0$$

and M is said to be copositive–plus if M is copositive and

$$\langle x, Mx \rangle = 0, x \geq 0 \implies (M + M^T)x = 0$$

A matrix M is called an L matrix if the LCP

$$z \geq 0, \quad Mz + q \geq 0, \quad z^T(Mz + q) = 0$$

has a unique solution 0, and furthermore, for any $z \neq 0$ such that

$$z \geq 0, \quad Mz \geq 0, \quad z^T Mz = 0$$

there exist diagonal matrices $D \geq 0$ and $E \geq 0$ such that $Dz \neq 0$ and

$$(EM + M^T D)z = 0$$

A matrix M is called a P (P_0) matrix if all its principal minors are positive (non–negative).

The class of matrices such that

$$\mathrm{FEA}(q, M) \neq \emptyset \quad \Rightarrow \quad \mathrm{SOL}(q, M) \neq \emptyset$$

is referred to as Q_0 matrices. In another words a Q_0 matrix M is one such that LCP(q, M) is solvable whenever it is feasible. The class Q_0 can be characterized as follows.

Theorem 1.10 ([8, Proposition 3.2.1]) *For an $n \times n$ matrix M, the following are equivalent*

(a) $M \in Q_0$.

(b) $K(M)$ is convex.

(c) $K(M) = pos(I, -M)$.

Here $pos(I, -M)$ stands for the smallest polyhedral convex cone containing all the column vectors of the matrix $(I, -M)$ and the origin.

Definition 1.11 *A matrix M is said to be column sufficient if, given $z \in \mathbb{R}^n$*

$$z_i(Mz)_i \leq 0 \qquad \text{for all } i \qquad \Rightarrow \qquad z_i(Mz)_i = 0 \qquad \text{for all } i$$

M is row sufficient if its transpose is column sufficient. M is sufficient if it is both column and row sufficient.

An key property of row sufficient matrices is that a solution of LCP(q, M) can be obtained from a Karush–Kuhn–Tucker point (see [32]) of the following quadratic program

$$\begin{array}{rl} \min & x^T(Mx + q) \\ \text{subject to} & Mx + q \geq 0 \\ & x \geq 0 \end{array} \tag{1.1}$$

A Karush–Kuhn–Tucker point consists of primal and dual variables is also referred to as a Karush–Kuhn–Tucker pair.

Theorem 1.12 ([8, Proposition 3.5.4]) *Given an $n \times n$ matrix, the following are equivalent:*

(a) M is row sufficient.

(b) For each $q \in \mathbb{R}^n$, if (z, u) is a Karush–Kuhn–Tucker pair for the quadratic program (1.1) then z solves LCP(q, M).

A consequence of this theorem is that Q_0 contains the class of row sufficient matrices ([8, Corollary 3.5.5]).

Corollary 1.13 *Every row sufficient matrix is a Q_0 matrix.*

Proof If FEA$(q, M) \neq \emptyset$, then the quadratic program (1.1) is feasible, and hence has a solution by the Frank–Wolfe theorem (see [20]). Therefore a Karush–Kuhn–Tucker point (z^*, u^*) exists. Thus $z^* \in$ SOL(q, M) by the previous theorem.

Q.E.D.

Column sufficient matrices can be characterized by convexity of SOL(q, M) for all $q \in \mathbb{R}^n$. The following theorem put this in precise terms.

Theorem 1.14 ([8, Proposition 3.5.8]) *A matrix M is column sufficient if and only if for each $q \in \mathbb{R}^n$, the set SOL(q, M) is convex.*

As a result of Theorem 1.10, Corollary 1.13 and Theorem 1.14, we state the following theorem for the class of sufficient matrices.

Theorem 1.15 *If a matrix M is sufficient then*

(a) $M \in Q_0$.

(b) $K(M) = pos(I, -M)$ is a polyhedral convex cone.

(c) SOL(q, M) is a non–empty convex set for each $q \in K(M)$.

We conclude this section with a theorem on the connection between sufficient matrices and P or P_0 matrices.

Theorem 1.16 *Every P–matrix is sufficient, and every sufficient matrix is in P_0.*

Proof Every P–matrix M is column sufficient by part (b) of [8, Theorem 3.3.4]. M^T is a P–matrix provided M is a P–matrix. Hence M^T is column sufficient. Consequently, M is sufficient.

Suppose M is column sufficient. Let $0 \neq z \in \mathbb{R}^n$. Then, we have either $z_i(Mz)_i > 0$ for some i or $z_i(Mz)_i \leq 0$ for all i. In the latter case, $z_i(Mz)_i = 0$ for all i. In any event, there is an index i such that $z_i \neq 0$ and $z_i(Mz)_i \geq 0$. Hence, M is P_0 by part (b) of [8, Theorem 3.4.2]. So, every column sufficient matrix is a P_0–matrix. Therefore, every sufficient matrix is a P_0–matrix. **Q.E.D.**

A matrix M is said to be semi–monotone if for each non–zero vector x in $\mathbb{R}^n$ such that $x \geq 0$, there exists a index k such that

$$x_k > 0 \quad \text{and} \quad (Mx)_k \geq 0$$

The class of such matrices is denoted as E_0. The following theorem provides a characterization for semi–monotone matrices.

Theorem 1.17 *Given an $n \times n$ matrix, the following are equivalent:*

(a) M is semi–monotone.

(b) (LCP) has the unique solution 0 for all $q > 0$.

(c) For any index $\alpha \subset \{1, 2, \cdots, n\}$, the system

$$M_{\alpha\alpha} x_\alpha < 0, \qquad x_\alpha \geq 0$$

has no solution.

Proof See [8, Theorem 3.9.3]. **Q.E.D.**

It is clear from the definition that P matrices and column sufficient matrices are semi–monotone. Copositive–plus matrices and L–matrices are also semi–monotone as a result of part (b) of the proceeding theorem.

Chapter 2

Pivotal Method

In this chapter we are concerned with the affine variational inequality problem. The problem can be described as follows. Let $C \in \mathbb{R}^n$ be a polyhedral convex set and A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. We wish to find a point $z \in C$ such that

$$\langle A(z) - a, y - z \rangle \geq 0, \forall y \in C \tag{AVI}$$

The problem can be equivalently formulated as

$$0 \in A(z) - a + \partial\psi(z \mid C) \tag{GE}$$

where $\psi(\cdot \mid C)$ is the indicator function of the set C. It can be easily shown that $\partial\psi(z \mid C) = N(z \mid C)$, the normal cone to C at z, if $z \in C$ and is empty otherwise, and hence (AVI) is equivalent to (GE). The solutions of such problems arise for example in the determination of a Newton–type method for generalized equations.

The problem has also been termed the linear stationary problem and we refer the reader to the work of [51], [49], [11] and [10] for several methods for the solution of this problem either over a bounded polyhedron or a pointed convex polyhedron.

Our approach is to formulate (AVI) as a piecewise linear equation by using the normal map induced by the linear transformation A

$$A_C(x) := A(\pi_C(x)) + x - \pi_C(x)$$

and $\pi_C(x)$ is the projection (with respect to the Euclidean norm) of x onto the set C. We know that (AVI) is equivalent to

$$A_C(x) = a \tag{NE}$$

from Chapter 1.

In Section 2.1 we describe the theoretical algorithm and apply several results of Eaves and Robinson to establish its finite termination for coherently oriented normal maps. In Section 2.2 we carefully describe an implementation of such a method, under the assumption that C is given by

$$C := \{z \mid Bz \geq b, Hz = h\}.$$

In Section 2.3 we extend several well known results for linear complementarity problems to the affine variational inequality. In particular, we generalize the notions of copositive, copositive–plus and L–matrices from the complementarity literature and prove that our algorithm processes variational inequalities associated with such matrices. That is, when the algorithm is applied to such a problem, either a solution is found, or the problem is infeasible in a well specified sense. Our definition of L–matrices is new and enables the treatment of both coherently oriented normal maps and copositive–plus matrices within the same framework. Furthermore, this result (Theorem 2.11) includes many of the standard existence results for complementarity problems and variational inequalities as special cases.

2.1 Theoretical Algorithm

We describe briefly a theoretical algorithm that is guaranteed to find a solution in finitely many steps when the homeomorphism condition developed in [43] holds. This method is a realization of the general path–following algorithm described and justified in [15]. In what follows we use various terms and concepts that are explained in [15]. Related methods for finding stationary points of affine functions on polyhedral sets are given in [16, 17]. A more detailed description of an implementation of the method is given in the Section 2.2; here we deal with theoretical considerations underpinning the method. Other related work can be found in [5].

In order to formulate the algorithm, it is important to understand the underlying geometric structure of the problem. Our approach relies heavily on the normal manifold of the set C, [43], which we will now describe.

Theorem 2.1 *Let C be a nonempty polyhedral convex set in $I\!R^n$ and $\{F_i \mid i \in \mathcal{I}\}$ be the nonempty faces of C. For $i \in \mathcal{I}$, define N_{F_i} to be the common value of $N(\cdot \mid C)$ on $\mathrm{ri} F_i$ and let $\sigma_i := F_i + N_{F_i}$. The normal manifold $\mathcal{N}_C$ of C consists of the pair $(I\!R^n, \mathcal{S})$, where $\mathcal{S} := \{\sigma_i \mid i \in \mathcal{I}\}$. The faces of the σ_i having dimension $k \geq 0$ are called the k–cells of $\mathcal{N}_C$. $\mathcal{N}_C$ is a subdivided piecewise linear manifold of dimension n.*

It can be seen that the normal map A_C will agree in each n–cell of this manifold with an affine map, and therefore, with each such cell we can associate the determinant of the corresponding linear transformation. If each of these determinants has the same sign, we say that A_C is coherently oriented. The following is the central result from [43].

Theorem 2.2 *The normal map A_C is a Lipschitzian homeomorphism of $\mathbb{R}^n$ into $\mathbb{R}^n$ if and only if A_C is coherently oriented.*

We will assume first of all that A_C is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$, so that the same–sign condition holds and describe the algorithm within this framework. Later in the chapter, this condition will be weakened. The first step of the algorithm is to determine if C contains any lines. If it does, take orthonormal bases for lin C and its orthogonal complement according to the scheme explained in [43, Prop. 4.1]. The factoring procedure explained there shows how to reduce the problem to one (which we shall also write $A_C(x) = a$) in a possibly smaller space, in which the set C appearing in this problem contains no lines. In that case, as shown in [43], the determinants associated with A_C in the various cells of $\mathcal{N}_C$ must all have positive sign. Further, C will have an extreme point, say x_e, and as pointed out in [43, §5] the normal cone $N_C(x_e)$ must have an interior. Let e be any element of $\text{int}N_C(x_e)$. An implementation of the factoring procedure is given as stage one of the method described in Section 2.2. The construction of an extreme point and element in the interior of the normal cone corresponds to stage two of that method.

Now construct a piecewise-linear manifold $\mathcal{M}$ from $\mathcal{N}_C$ by forming the Cartesian product of each cell of $\mathcal{N}_C$ with $\mathbb{R}_+$, the non-negative half-line in $\mathbb{R}$. This $\mathcal{M}$ will be a PL $(n+1)$-manifold in $\mathbb{R}^{n+1}$, as can easily be verified (see [15, Example 4.3]). Define a PL function $F: \mathcal{M} \to \mathbb{R}^n$ (where $\mathbb{R}^n$ is regarded as a PL manifold of one cell) by:

$$F(x, \mu) = A_C(x) - (\mu e + a).$$

We shall consider solutions $x(\mu)$ of $F(x, \mu) = 0$; it is clear from (NE) that $x(0)$ will solve our problem. Note that since we have assumed A_C to be a homeomorphism, the function $x(\cdot)$ is single-valued and defined on all of $\mathbb{R}_+$, though this property is not essential to our argument.

Now define $w(\mu) = x_e + (a - Ax_e) + \mu e$. It is clear that since

$$w(\mu) = x_e + \mu[e + \mu^{-1}(a - Ax_e)] \tag{2.1}$$

for large positive μ, $w(\mu)$ lies interior to the cell $x_e + N_C(x_e)$ of $\mathcal{N}_C$. Therefore $(w(\mu), \mu)$ lies interior to the cell $[x_e + N_C(x_e)] \times R_+$ of $\mathcal{M}$, and so it is a regular point of $\mathcal{M}$. Further, for such μ we have $\pi_C(w(\mu)) = x_e$, so that

$$F(w(\mu), \mu) = Ax_e + (a - Ax_e) + \mu e - (\mu e + a) = 0,$$

and therefore for some $\mu_0 \geq 0$, $F^{-1}(0)$ contains the ray $\{\, (w(\mu), \mu) \mid \mu \geq \mu_0 \,\}$.

Now we apply the **Algorithm 2** from Chapter 1 to the PL equation $F(x, \mu) = 0$, using a ray start at $(w(\mu_1), \mu_1)$ for some $\mu_1 > \mu_0$ and proceeding in the direction $(-e, -1)$. As the manifold $\mathcal{M}$ is finite, according to [15, Th. 15.13] the algorithm generates, in finitely many steps, either a point (x_*, μ_*) in the boundary of $\mathcal{M}$ with $F(x_*, \mu_*) = 0$, or a ray in $F^{-1}(0)$ different from the starting ray. As the boundary of $\mathcal{M}$ is $\mathcal{N}_C \times \{0\}$, we see that in the first case $\mu_* = 0$ and, by our earlier remarks, x_* then satisfies $A_C(x_*) = a$. Therefore in order to justify the algorithm we need only show that it cannot produce a ray different from the starting ray.

The algorithm in question permits solving the perturbed system $F(x_\epsilon, \mu_\epsilon) = p(\epsilon)$, where $p(\epsilon)$ is of the form

$$p(\epsilon) = \sum_{i=1}^{n} p_i \epsilon^i$$

for appropriately chosen vectors p_i. It is shown in Chapter 1 that $p(\epsilon)$ is a regular value of F for each small positive ϵ, and it then follows by Theorem 1.6 that for such ϵ, $F^{-1}(p(\epsilon))$ is a connected 1-manifold $Y(\epsilon)$, whose boundary is equal to its intersection with the boundary of $\mathcal{M}$, and which is subdivided by the chords formed by its intersections with the cells of $\mathcal{M}$ that it meets. Finally, for an easily computed function

$$b(\epsilon) = \sum_{i=1}^{n} b_i \epsilon^i$$

we have $(w(\mu_1), \mu_1) + b(\epsilon) \in Y(\epsilon)$, and for small positive ϵ this point evidently lies on a ray in $F^{-1}(p(\epsilon))$. Because we start on this ray, $Y(\epsilon)$ cannot be homeomorphic to a circle, and therefore it is homeomorphic to an interval.

A simple computation at the starting point shows that the curve index [15, §12] at that point is -1. By [15, Lemma 12.1] this index will be constant along $Y(\epsilon)$. However, a computation similar to that in [15, Lemma 12.3] shows that in each cell of $\mathcal{M}$, if the direction of $Y(\epsilon)$ in that cell is (r, ρ) then

$$(\mathrm{sgn}\rho)(\mathrm{sgn}\det T) = -1$$

where T is the linear transformation associated with A_C in the corresponding cell of $\mathcal{N}_C$. Under our hypotheses, $\det T$ must be positive, and therefore ρ is negative everywhere along $Y(\epsilon)$. But this means that the parameter μ decreases strictly in each cell of linearity that $Y(\epsilon)$ enters, and it follows from the structure of $\mathcal{M}$ that after finitely many steps we must have $\mu = 0$, and therefore we have a point x_ϵ with $A_C(x_\epsilon) = a + p(\epsilon)$.

Now in practice the algorithm does not actually use a positive ϵ, but only maintains the information necessary to compute $Y(\epsilon)$ for all small positive ϵ, employing the lexicographic ordering to resolve possible ambiguities when $\epsilon = 0$. Therefore after finitely many steps it will actually have computed x_0 with $A_C(x_0) = a$.

Note that for linear complementarity problems, the above algorithm corresponds to Lemke's method [31]. It is well known that for linear complementarity problems associated with P–matrices, Lemke's method terminates at a solution. For variational inequalities, we have a similar result due to the analysis above.

Theorem 2.3 *Given the problem (NE), assume that A_C is coherently oriented; then the path following method given in this section terminates at a solution of (NE).*

2.2 Algorithm Implementation

The previous section described a method for solving the Affine Variational Inequality over a general polyhedral set and showed (under a lexicographical ordering) that a coherently oriented normal equation (NE) can be solved in a finite number of iterations by a path–following method. In this section, we describe the numerical implementation of such a method, giving emphasis to the numerical linear algebra required to perform the steps of the algorithm.

We shall specialize to the case where C is given as

$$C := \{z \mid Bz \geq b, Hz = h\} \tag{2.2}$$

and we shall assume that the linear transformation $A(z)$ is represented by the matrix A in our current coordinate system. We can describe our method to solve the normal equation in three stages. Note that by "solving", we mean producing a pair $(x, \pi(x))$, where x is a solution of (NE) and $\pi(x)$ is the projection of x onto the underlying set C.

In the first stage we remove lines from the set C, to form a reduced problem (over $\tilde{C}$) as outlined in the theory above. The lineality space of C as defined by

(2.2) is

$$\text{lin } C = \ker \begin{pmatrix} B \\ H \end{pmatrix}$$

We calculate bases for the lineality space and its orthogonal complement by performing a QR factorization (with column pivoting) of $\begin{pmatrix} B^T & H^T \end{pmatrix}$. If $\begin{pmatrix} W & V \end{pmatrix}$ represents these bases, the linear transformation A is represented by

$$A' = \begin{pmatrix} W^TAW & W^TAV \\ V^TAW & V^TAV \end{pmatrix}$$

and the vector a is represented by

$$a' = \begin{pmatrix} W^Ta \\ V^Ta \end{pmatrix}$$

under this basis. The reduced problem

$$\tilde{A}_{\tilde{C}}y = \tilde{a} \tag{2.3}$$

is constructed using the method outlined in [43, Proposition 4.1], which also appear in this work as Lemma 3.1. First of all, since V is a basis of $(\text{lin } C)^{\perp}$ and $\tilde{C} = C \cap (\text{lin } C)^{\perp}$, we have

$$\tilde{C} = \left\{ z \,\middle|\, \tilde{B}z \geq b, \tilde{H}z = h \right\}, \quad \tilde{B} = BV, \quad \tilde{H} = HV. \tag{2.4}$$

The matrix $\tilde{A}$ is the Shur complement of W^TAW in A' (see Lemma 3.1). That is

$$\tilde{A} = V^TAV - (V^TAW)(W^TAW)^{-1}(W^TAV)$$

Let

$$Z = W(W^TAW)^{-1}W^T, \qquad U = (I - ZA)V \tag{2.5}$$

Notice that Z satisfies $Z^TAZ = Z^T$ and by standard algebraic operations, we obtain

$$\tilde{A} = U^TAU \tag{2.6}$$

Similarly, by reference to Lemma 3.1, we have

$$\tilde{a} = V^Ta - (V^TAW)(W^TAW)^{-1}(W^Ta)$$

That is

$$\tilde{a} = V^T \left(\; I - AZ \; \right) a \tag{2.7}$$

In practice, $\tilde{A}$ and $\tilde{a}$ are calculated using one LU factorization of W^TAW. Furthermore, it follows from Lemma 3.1 that y solves (2.3) implies that

$$-W(W^TAW)^{-1}((W^TAV)y - W^Ta) + Vy = Z(a - AVy) + Vy$$

solves (NE). So, the solution pair $(x, \pi(x))$ of the original normal equation (NE) can be recovered from the solution pair $(y, \pi(y))$ of (2.3) using the identities

$$\begin{aligned} x_l &= Z(a - AVy) \\ x &= x_l + Vy \\ \pi(x) &= x_l + V\pi(y) \end{aligned}$$

Therefore, we can assume that the problem has the form (2.3), with $\tilde{C}$ given by (2.4) and that the matrix $\begin{pmatrix} \tilde{B} \\ \tilde{H} \end{pmatrix}$ has full column rank. We note that a similar construction is needed in [42, 44].

In the second stage, we determine an extreme point of the set $\tilde{C}$, and using this information reduce the problem further by forcing the iterates to lie in the affine space generated by the equality constraints. More precisely, we have the following result:

Lemma 2.4 *Suppose $y_e \in \tilde{C}$ and Y is an orthonormal basis for the kernel of $\tilde{H}$. Then $\bar{y}$ solves (2.3) if and only if $\bar{y} = y_e + Y\bar{x}$ where $\bar{x}$ solves*

$$\bar{A}_{\bar{C}} x = \bar{a} \tag{2.8}$$

Here $\bar{A} = Y^T \tilde{A} Y$, $\bar{a} = Y^T(\tilde{a} - \tilde{A} y_e)$ and $\bar{C} = \left\{ z \,\middle|\, \tilde{B} Y z \geq b - \tilde{B} y_e \right\}$. Furthermore, $\tilde{B}Y$ has full column rank if and only if $\begin{pmatrix} \tilde{B} \\ \tilde{H} \end{pmatrix}$ has full column rank.

Proof By definition, $y = y_e + Yx \in \tilde{C}$ if and only if $x \in \bar{C}$. Furthermore

$$\begin{aligned}
\pi_{\tilde{C}}(y) &= \arg\min \left\{ \|w - y\|_2 \,\middle|\, w \in \tilde{C} \right\} \\
&= \arg\min \left\{ \|(y_e + Yz) - (y_e + Yx)\|_2 \,\middle|\, w = y_e + Yz, z \in \bar{C} \right\} \\
&= \arg\min \left\{ \|Y(z - x)\|_2 \,\middle|\, w = y_e + Yz, z \in \bar{C} \right\} \\
&= \arg\min \left\{ \|(z - x)\|_2 \,\middle|\, w = y_e + Yz, z \in \bar{C} \right\}
\end{aligned}$$

Thus

$$\pi_{\tilde{C}}(y) = y_e + Y \pi_{\bar{C}}(x)$$

It follows that $\bar{y} = y_e + Y\bar{x}$ solves (2.3), that is

$$\tilde{A} \pi_{\tilde{C}}(\bar{y}) + \bar{y} - \pi_{\tilde{C}}(\bar{y}) = \tilde{a}$$

if and only if

$$\tilde{A}(y_e + Y \pi_{\bar{C}}(\bar{x})) + y_e + Y\bar{x} - (y_e + Y \pi_{\bar{C}}(x)) = \tilde{a}$$

or, equivalently

$$\tilde{A} Y \pi_{\bar{C}}(\bar{x})) + Y(\bar{x} - \pi_{\bar{C}}(x)) = \tilde{a} - \tilde{A} y_e$$

This is in turn, by orthonormality of V, equivalent to

$$Y^T \tilde{A} Y \pi_{\bar{C}}(\bar{x})) + \bar{x} - \pi_{\bar{C}}(x) = Y^T(\tilde{a} - \tilde{A} y_e)$$

Hence $\bar{y} = y_e + Y\bar{x}$ solves (2.3) exactly when $\bar{x}$ solves (2.8).

Suppose $\tilde{B}Yz = 0$ for some $z \neq 0$. Yz is nonzero since the columns form an orthonormal basis of $\ker \tilde{H}$. But then

$$\begin{pmatrix} \tilde{B} \\ \tilde{H} \end{pmatrix} Yz = \begin{pmatrix} \tilde{B}Yz \\ \tilde{H}Yz \end{pmatrix} = 0$$

Conversely if

$$\begin{pmatrix} \tilde{B} \\ \tilde{H} \end{pmatrix} w = 0$$

for some $w \neq 0$, then $w \in \ker \tilde{H}$. Hence $w = Yz$ for some $z \neq 0$. Also $\tilde{B}Yz = \tilde{B}w = 0$. **Q.E.D.**

Thus, to reduce our problem to one over an inequality constrained polyhedral set, it remains to show how we generate the point $y_e \in \tilde{C}$. In fact we show how to generate y_e as an extreme point of $\tilde{C}$ and further, how to project this extreme point into an extreme point of $\bar{C}$. The following result is a well known characterization of extreme points of polyhedral sets [36, §3.4].

Lemma 2.5 *Let u be partitioned into free and constrained variables $(u_{\mathcal{F}}, u_{\mathcal{C}})$. u is an extreme point of $\mathcal{D} = \{u = (u_{\mathcal{F}}, u_{\mathcal{C}}) \mid Du = d, u_{\mathcal{C}} \geq 0\}$ if and only if $u \in \mathcal{D}$ and $\{d_i \mid i \in \mathcal{B}\}$ are linearly independent, where $\mathcal{B} := \mathcal{F} \bigcup \{j \in \mathcal{C} \mid u_j > 0\}$.*

If we adopt the terminology of linear programming, then the variables corresponding to $\mathcal{B}$ are called basic variables; similarly, the columns of D corresponding to $\mathcal{B}$ are called basic columns; extreme points are called basic feasible solutions.

The extreme points of systems of inequalities and equalities are defined in an analogous manner. Note that extreme points of $\tilde{C}$ are (by definition) precisely the extreme points of

$$\begin{pmatrix} \tilde{B} & -I \\ \tilde{H} & 0 \end{pmatrix} \begin{pmatrix} z \\ s \end{pmatrix} = \begin{pmatrix} b \\ h \end{pmatrix}, \quad s \geq 0. \tag{2.9}$$

The slack variables s are implicitly defined by z, so without ambiguity we will refer to the above extreme point as z. For other systems of inequalities and equations a similar convention will be used. The following lemma outlines our method for constructing the relevant extreme points.

Lemma 2.6 *Suppose* $\begin{pmatrix} \tilde{B} \\ \tilde{H} \end{pmatrix}$ *has linearly independent columns,* Y *is a basis of the kernel of* $\tilde{H}$ *and* $\bar{B} = \tilde{B}Y$*. Then* y_e *is an extreme point of (2.9) if and only if* $y_e = y_* + Yz_*$*, for some* y_**,* z_* *where* $\tilde{H}y_* = h$ *and* z_* *is an extreme point of*

$$\begin{pmatrix} \bar{B} & -I \end{pmatrix} \begin{pmatrix} z \\ s \end{pmatrix} = b - \tilde{B}y_*, s \geq 0. \tag{2.10}$$

In our method we produce an extreme point of (2.9) as follows. Find orthonormal bases U and Y for $\mathrm{im}\tilde{H}^T$ and $\ker \tilde{H}$ respectively. This can be carried out by a singular value decomposition of $\tilde{H}$ or by QR factorizations of $\tilde{H}$ and $\tilde{H}^T$ (in fact, Y could be calculated as a by–product of stage 1 of the algorithm). In particular, if

$$H^T = \begin{pmatrix} Z & Y \end{pmatrix} \begin{pmatrix} R \\ 0 \end{pmatrix}$$

then Y is an orthonormal basis of $\ker \tilde{H}$ and we can let $y_* = ZR^{-T}h$, using this value of y_* in (2.10). If $b \notin \mathrm{im}\bar{B}$, then find an extreme point of (2.10) by solving the following auxiliary problem with the revised simplex method:

$$\begin{array}{ll} \text{minimize} & z_{aux} \\ \text{subject to} & \begin{pmatrix} \bar{B} & b - \tilde{B}y_* \end{pmatrix} \begin{pmatrix} z \\ z_{aux} \end{pmatrix} \geq b - \tilde{B}y_* \\ & z_{aux} \geq 0 \end{array}$$

Note that $z = 0$, $z_{aux} = 1$ is an initial feasible point for this problem, with basic variables (z, z_{aux}). In contrast to the usual square basis matrix (with corresponding LU factors), we use a QR factorization of the non–square basis matrix. The calculations of dual variables and incoming columns are performed in a least squares sense using the currently available QR factorization. This factorization is updated at each pivot step either by using a rank–one update to the factorization or by adding a column to the factorization (see [22]). In order to invoke Lemma 2.4, we let $y_e = y_* + Yz_*$ be the feasible point needed to define (2.8).

Note that in the well known method of Lemke, stages one and two are trivial since $C = \mathbb{R}^n_+$ has no lines and a single extreme point at 0. Furthermore, stage one is an exact implementation of the theory outlined in the previous section and stage two corresponds to determining an extreme point and treating the defining equalities of C in an effective computational manner.

It remains to describe stage three of our method. We are able to assume that our problem is given as

$$\bar{A}_{\bar{C}}x = \bar{a} \tag{2.11}$$

with $\bar{C} = \left\{ z \,\middle|\, \bar{B}z \geq \bar{b} \right\}$, where $\bar{B}$ has full column rank and x_e is an extreme point of $\bar{C}$ (easily determined from z_*). We also have available a basis matrix corresponding to this extreme point along with a QR factorization, courtesy of stage two.

The method that we use to solve this problem is precisely a realization of the general scheme for piecewise linear equations as described in **Algorithm 2**. The specific algorithm, which we label **Algorithm 2$'$**, is as follows:

Algorithm 2$'$

1. **Initialize :**

   Let L_{σ_k} denote the linear map representing F on the cell σ_k. Determine (x_1, σ_1, d_1) satisfying

   $$L_{\sigma_1} d_1 = 0, d_1 \text{ points into } \sigma_1 \text{ at } x_1.$$
   $$F(x_1) = v$$
   $$x_1 \in \sigma_1 \in \mathcal{M}, x_1 \in \operatorname{int}\{x - \theta d_1 \mid \theta \geq 0\} \subset F^{-1}v.$$

2. **Iteration :**

   Given (x_k, σ_k, d_k) let

   $$\theta_k := \sup\{\theta \mid x_k + \theta d_k \in \sigma_k\} \tag{2.12}$$

   if $\theta_k = +\infty$ then ray termination.

   if $x_{k+1} := x_k + \theta_k d_k \in \partial\mathcal{M}$ then boundary termination.

   Otherwise determine $(x_{k+1}, \sigma_{k+1}, d_{k+1})$, $d_{k+1} \neq 0$, satisfying

   $$L_{\sigma_{k+1}} d_{k+1} = 0, \text{and } d_{k+1} \text{ points into } \sigma_{k+1} \text{ from } x_{k+1}.$$
   $$\sigma_{k+1} \in \mathcal{M} \setminus \{\sigma_k\} \text{ with } x_{k+1} \in \sigma_{k+1}$$

   Set $k = k + 1$ and repeat iteration.

How does this relate to the description we gave in the previous section? The manifold we consider is

$$\mathcal{M} = \mathcal{N}_{\bar{C}} \times \mathbb{R}_+$$

and the corresponding cells $\sigma_{\mathcal{A}}$ are given by

$$(F_{\mathcal{A}} + N_{F_{\mathcal{A}}}) \times \mathbb{R}_+$$

where $F_{\mathcal{A}}$ are the faces of $\bar{C}$.

A face of $\bar{C}$ is described by the set of constraints from the system $\bar{B}z \geq \bar{b}$ which are active. Let $\mathcal{A}$ represent such a set so that

$$F_{\mathcal{A}} = \left\{ z \,\middle|\, \bar{B}_{\mathcal{A}}z = \bar{b}_{\mathcal{A}}, \bar{B}_{\mathcal{I}}z \geq \bar{b}_{\mathcal{I}} \right\}$$

where $\mathcal{I}$ is the complement of the set $\mathcal{A}$. The normal cone to the face (the normal cone to $\bar{C}$ at some point in the relative interior of $F_{\mathcal{A}}$) is given by

$$\left\{ \bar{B}^T u \mid u_{\mathcal{A}} \leq 0, u_{\mathcal{I}} = 0 \right\}$$

It now follows that an algebraic description of $(x, \mu) \in \sigma_{\mathcal{A}}$ is that there exist $(x, z, u_{\mathcal{A}}, s_{\mathcal{I}}, \mu)$ which satisfy

$$\begin{aligned} \bar{B}_{\mathcal{A}}z &= \bar{b}_{\mathcal{A}} \\ \bar{B}_{\mathcal{I}}z - s_{\mathcal{I}} &= \bar{b}_{\mathcal{I}}, s_{\mathcal{I}} \geq 0 \\ x &= z + \bar{B}_{\mathcal{A}}^T u_{\mathcal{A}}, u_{\mathcal{A}} \leq 0 \\ \mu &\geq 0 \end{aligned} \tag{2.13}$$

In particular, if x_e is the given extreme point, the corresponding face of the set $\bar{C}$ is used to define the initial cell σ_1. The piecewise linear system we solve is

$$F(x, \mu) := \bar{A}_{\bar{C}}(x) - (\mu e + \bar{a}) = 0$$

where e is a point in the interior of $N(x_e \mid \bar{C})$. An equivalent description of $N(x_e \mid \bar{C})$ is given by

$$\left\{ \bar{B}_{\mathcal{A}}^T u \mid u \leq 0 \right\}$$

from which it is clear that the interior of this set is nonempty if and only if $\bar{B}_{\mathcal{A}}$ has full column rank.

Lemma 2.7 *If x_e is an extreme point of $\left\{ z \,\middle|\, \bar{B}z \geq \bar{b} \right\}$ with active constraints $\mathcal{A}$, then $\bar{B}_{\mathcal{A}}$ has full column rank.*

Proof By definition,

$$G := \begin{pmatrix} \bar{B}_{\mathcal{A}} & 0 \\ \bar{B}_{\mathcal{I}} & -I \end{pmatrix} \tag{2.14}$$

has linearly independent columns. If $\bar{B}_{\mathcal{A}}$ does not have linearly independent columns, then $\bar{B}_{\mathcal{A}} w = 0$, for some $w \neq 0$, so that

$$G \begin{pmatrix} w \\ \bar{B}_{\mathcal{I}} w \end{pmatrix} = 0$$

with $(w, \bar{B}_{\mathcal{I}}) \neq 0$, a contradiction of (2.14). **Q.E.D.**

This is a simple proof (in this particular instance) of the comment from the previous section that the normal cone has interior at an extreme point. For consistency, we shall let e be any point in this interior $\left\{ \bar{B}_{\mathcal{A}}^T u \mid u < 0 \right\}$, and for concreteness we could take

$$e = -\bar{B}_{\mathcal{A}}^T \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Hence F is specified, $v = 0$ and the cells of $\sigma_{\mathcal{A}}$ are defined. By solving the perturbed system $F(x_\epsilon, \mu_\epsilon) = p(\epsilon)$ (as outlined in Section 2.1), we know that

$F^{-1}(p(\epsilon))$ is a connected 1–manifold whose boundary is equal to its intersection with the boundary of $\mathcal{M}$ and which is subdivided by the chords formed by its intersections with the cells of $\mathcal{M}$ that it meets. In practice, this means that (under the lexicographical ordering induced by $p(\epsilon)$) we may assume nondegeneracy. Thus, if ties ever occur in the description that follows, we will always choose the the lexicographical minimum from those which achieve the tie. Specific implementation techniques will be given later in this section.

Note that if $(x, \mu) \in \sigma_{\mathcal{A}}$ as defined in (2.13) then

$$F(x, \mu) = \bar{A}z + x - z - \mu e - \bar{a}$$

It follows that if $(x, \mu) \in \sigma_{\mathcal{A}} \bigcap F^{-1}(0)$ (i.e. (x, μ) is in one of the chords mentioned in the previous paragraph), then there exist $(x, z, u_{\mathcal{A}}, s_{\mathcal{I}}, \mu)$ satisfying

$$\begin{aligned} x - z &= -\bar{A}z + \mu e + \bar{a} \\ \bar{B}_{\mathcal{A}} z &= \bar{b}_{\mathcal{A}} \\ \bar{B}_{\mathcal{I}} z - s_{\mathcal{I}} &= \bar{b}_{\mathcal{I}}, s_{\mathcal{I}} \geq 0 \\ x - z &= \bar{B}_{\mathcal{A}}^T u_{\mathcal{A}}, u_{\mathcal{A}} \leq 0 \\ \mu &\geq 0 \end{aligned} \tag{2.15}$$

Furthermore, these equations determine the chord on the current cell of the manifold, or in the notation used to describe the algorithm of Eaves, the map $L_{\sigma_{\mathcal{A}}}$. The direction is determined from (2.12) by solving $L_{\sigma_{\mathcal{A}}} d = 0$, which can be calculated by solving

$$\begin{aligned} \Delta x - \Delta z &= -\bar{A}\Delta z + e\Delta\mu \\ \bar{B}_{\mathcal{A}} \Delta z &= 0 \\ \bar{B}_{\mathcal{I}} \Delta z - \Delta s_{\mathcal{I}} &= 0 \\ \Delta x - \Delta z &= \bar{B}_{\mathcal{A}}^T \Delta u_{\mathcal{A}} \end{aligned} \tag{2.16}$$

At the first iteration, $\bar{B}_{\mathcal{A}}$ has full column rank, so that $\Delta z = 0$, which also implies that $\Delta s_{\mathcal{I}} = 0$. The remaining system of equations is

$$\begin{array}{rcl} \Delta x & = & e\Delta\mu \\ \Delta x & = & \bar{B}_{\mathcal{A}}^T \Delta u_{\mathcal{A}} \end{array}$$

We choose $\Delta\mu = -1$ in order to force the direction to move into σ_1 (as required by (2.12)), and then it follows that $\Delta x = -e$ for the choice of e outlined above $\Delta u_{\mathcal{A}} = (1, \ldots, 1)^T$. The actual choice $x_1 = (w(\mu), \mu)$ given in the previous section ensures that (2.12) is satisfied.

We can now describe the general iteration and the resultant linear algebra that it entails. We are give a current point $(x, z, u_{\mathcal{A}}, s_{\mathcal{I}}, \mu)$ satisfying (2.15) for some cell $\sigma_{\mathcal{A}}$ and a direction $(\Delta x, \Delta z, \Delta u_{\mathcal{A}}, \Delta s_{\mathcal{I}}, \Delta\mu)$ satisfying (2.16). The value of θ_k to satisfy (2.12) can be calculated by the following ratio test; that is to find the largest θ such that

$$\begin{array}{rcl} u_{\mathcal{A}} + \theta\Delta u_{\mathcal{A}} & \leq & 0 \\ s_{\mathcal{I}} + \theta\Delta s_{\mathcal{I}} & \geq & 0 \\ \mu + \theta\Delta\mu & \geq & 0 \end{array} \tag{2.17}$$

Ray termination occurs if $\Delta u_{\mathcal{A}} \leq 0$, $\Delta s_{\mathcal{I}} \geq 0$ and $\Delta\mu \geq 0$. Obviously, if $\mu + \theta\Delta\mu = 0$, then we have a solution. Otherwise, at least one of the $\{u_i \mid i \in \mathcal{A}\}$ or $\{s_i \mid i \in \mathcal{I}\}$ hits a bound in (2.17). By the lexicographical ordering, which will be discussed more thoroughly in the next few paragraphs, we can determine the "leaving" variable from these uniquely. The set $\mathcal{A}$ is updated (corresponding to moving onto a new cell of the manifold) and a new direction is calculated as follows: if u_i, $i \in \mathcal{A}$ is the leaving variable, then $\mathcal{A} := \mathcal{A} \setminus \{i\}$, $\Delta s_i = 1$ and the new direction is found by solving (2.16); if s_i, $i \in \mathcal{I}$ is the leaving variable, then $\mathcal{A} := \mathcal{A} \bigcup \{i\}$, $\Delta u_i = -1$ and the new direction is found by solving (2.16). Note that in both cases, the choice of one component of the direction ensures movement into the new (uniquely specified) cell $\sigma_{\mathcal{A}}$ and forces a unique solution of (2.16).

The linear algebra needed for an implementation of the method is now clear. The actual steps used to carry out stage 3 are now described. First of all, x is eliminated from (2.15) to give

$$\begin{aligned} -\bar{A}z + \mu e + \bar{a} &= \bar{B}_{\mathcal{A}}^T u_{\mathcal{A}} + \bar{B}_{\mathcal{I}}^T u_{\mathcal{I}} \\ \bar{B}_{\mathcal{A}} z - s_{\mathcal{A}} &= \bar{b}_{\mathcal{A}} \\ \bar{B}_{\mathcal{I}} z - s_{\mathcal{I}} &= \bar{b}_{\mathcal{I}} \\ \mu \geq 0, u_{\mathcal{A}} \leq 0, u_{\mathcal{I}} &= 0, s_{\mathcal{I}} \geq 0, s_{\mathcal{A}} = 0 \end{aligned}$$

or, equivalently

$$\begin{aligned} \bar{B}_{\mathcal{A}}^T u_{\mathcal{A}} + \bar{B}_{\mathcal{I}}^T u_{\mathcal{I}} - \bar{A}z + \mu e + \bar{a} &= 0 \\ \bar{B}_{\mathcal{A}} z - s_{\mathcal{A}} &= \bar{b}_{\mathcal{A}} \\ \bar{B}_{\mathcal{I}} z - s_{\mathcal{I}} &= \bar{b}_{\mathcal{I}} \\ \mu \geq 0, u_{\mathcal{A}} \geq 0, u_{\mathcal{I}} &= 0, s_{\mathcal{I}} \geq 0, s_{\mathcal{A}} = 0 \end{aligned}$$

Note that we have added in the variables which are set to zero for completeness. The QR factorization corresponding to the given extreme point is used to eliminate the variables z. In fact, we take as our initial active set $\mathcal{A}$, the variables corresponding to $Q\hat{R}$, where $\hat{R}$ is the invertible submatrix of R. Thus

$$z = \bar{B}_{\mathcal{A}}^{-1}(s_{\mathcal{A}} + \bar{b}_{\mathcal{A}})$$

and substituting this into the above gives

$$\begin{aligned} \bar{B}_{\mathcal{A}}^T u_{\mathcal{A}} + \bar{B}_{\mathcal{I}}^T u_{\mathcal{I}} - \bar{A}\bar{B}_{\mathcal{A}}^{-1} s_{\mathcal{A}} + \mu e &= \bar{A}\bar{B}_{\mathcal{A}}^{-1}\bar{b}_{\mathcal{A}} - \bar{a} \\ -\bar{B}_{\mathcal{I}}\bar{B}_{\mathcal{A}}^{-1} s_{\mathcal{A}} + s_{\mathcal{I}} &= \bar{B}_{\mathcal{I}}\bar{B}_{\mathcal{A}}^{-1}\bar{b}_{\mathcal{A}} - \bar{b}_{\mathcal{I}} \\ \mu \geq 0, u_{\mathcal{A}} \leq 0, u_{\mathcal{I}} &= 0, s_{\mathcal{I}} \geq 0, s_{\mathcal{A}} = 0 \end{aligned} \tag{2.18}$$

Essentially we treat this system as in the method of Lemke. An initial basis is given by $(u_{\mathcal{A}}, s_{\mathcal{I}})$ and complementary pivots can then be executed (using the variables u and s as the complementary pair). Any basis updating technique or anti-cycling rule can be incorporated from the literature on linear programming and complementarity. In fact, by (2.18), we have

$$\begin{pmatrix} u_{\mathcal{A}} \\ s_{\mathcal{I}} \end{pmatrix} = \begin{pmatrix} -\bar{B}_{\mathcal{A}}^{-T} \bar{B}_{\mathcal{I}}^{T} & \bar{B}_{\mathcal{A}}^{-T} \bar{A} \bar{B}_{\mathcal{A}}^{-1} & \bar{B}_{\mathcal{A}}^{-T} e \\ 0 & \bar{B}_{\mathcal{I}} \bar{B}_{\mathcal{A}}^{-1} & 0 \end{pmatrix} \begin{pmatrix} u_{\mathcal{I}} \\ s_{\mathcal{A}} \\ \mu \end{pmatrix} + \begin{pmatrix} \bar{B}_{\mathcal{A}}^{-T} (\bar{A} \bar{B}_{\mathcal{A}}^{-1} \bar{b}_{\mathcal{A}} - \bar{a}) \\ \bar{B}_{\mathcal{I}} \bar{B}_{\mathcal{A}}^{-1} \bar{b}_{\mathcal{A}} - \bar{b}_{\mathcal{I}} \end{pmatrix} \tag{2.19}$$

$$\mu \geq 0, u_{\mathcal{A}} \leq 0, u_{\mathcal{I}} = 0, s_{\mathcal{I}} \geq 0, s_{\mathcal{A}} = 0$$

Lexicographic ordering can be achieved by introducing a perturbation

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \cdots \\ \epsilon^m \end{pmatrix}$$

to the vector

$$\begin{pmatrix} \bar{B}_{\mathcal{A}}^{-T} (\bar{A} \bar{B}_{\mathcal{A}}^{-1} \bar{b}_{\mathcal{A}} - \bar{a}) \\ \bar{B}_{\mathcal{I}} \bar{B}_{\mathcal{A}}^{-1} \bar{b}_{\mathcal{A}} - \bar{b}_{\mathcal{I}} \end{pmatrix}$$

Initially, lexicographic information is contained in the matrix

$$Q^0 = \begin{pmatrix} \bar{B}_{\mathcal{A}}^{-T} (\bar{A} \bar{B}_{\mathcal{A}}^{-1} \bar{b}_{\mathcal{A}} - \bar{a}) & I & 0 \\ \bar{B}_{\mathcal{I}} \bar{B}_{\mathcal{A}}^{-1} \bar{b}_{\mathcal{A}} - \bar{b}_{\mathcal{I}} & 0 & I \end{pmatrix} = \begin{pmatrix} u_{\mathcal{A}} & I & 0 \\ s_{\mathcal{I}} & 0 & I \end{pmatrix}$$

which has linearly independent and lexicographically positive rows (as defined in Section 1.5).

Suppose at iteration k, we have $(u_{\mathcal{A}}, s_{\mathcal{I}}) \geq 0$ and the matrix Q^0 is transformed to

$$Q^k = \begin{pmatrix} u_{\mathcal{A}} & W^k_{\mathcal{A}\mathcal{A}} & W^k_{\mathcal{A}\mathcal{I}} \\ s_{\mathcal{I}} & W^k_{\mathcal{I}\mathcal{A}} & W^k_{\mathcal{I}\mathcal{I}} \end{pmatrix}$$

and the linearly independence and lexicographic positivity of its rows are maintained. Let $(\Delta u_{\mathcal{A}}, \Delta s_{\mathcal{I}})$ be the direction determined by (2.16). Consider the set of vectors

$$\mathcal{Q} = \left\{ -\frac{1}{q_i} Q^k_{i\cdot} \mid q_i < 0 \right\}$$

where $q_i = \Delta u_i$ or Δs_i depending on whether $i \in \mathcal{A}$ or $\mathcal{I}$. If, there exist indices i and j such that

$$\frac{1}{q_i} Q^k_{i\cdot} = \frac{1}{q_j} Q^k_{j\cdot}$$

it would follow that the rows of Q^k are linearly dependent. Hence, no two such vectors are equal, and there is a unique index r such that $-(1/q_r)Q^k_{r\cdot}$ is the lexicographic minimum of $\mathcal{Q}$. The leaving variable is now uniquely determined by the index r. Furthermore, the updated matrix

$$Q^{k+1} = \begin{pmatrix} u_{\mathcal{A}} & W^{k+1}_{\mathcal{A}\mathcal{A}} & W^{k+1}_{\mathcal{A}\mathcal{I}} \\ s_{\mathcal{I}} & W^{k+1}_{\mathcal{I}\mathcal{A}} & W^{k+1}_{\mathcal{I}\mathcal{I}} \end{pmatrix}$$

will again have linearly independent, lexicographically positive rows (see [8, pp. 340 - 342] for a proof).

We showed in the previous section that if A_C was coherently oriented then following the above path gives a monotonic decrease in μ. However, the proof of the finite termination of the method (possibly ray termination) goes through without this assumption, and in the following section we will look at other conditions which guarantee that the method terminates either with a solution or a proof that no solution exists. The coherent orientation results are direct analogues of the P–matrix results for the linear complementarity problem – the results we shall give now generalize the notions of copositive plus and L–matrices.

2.3 Existence Results

The following definitions are generalizations of those found in the introduction.

Definition 2.8 *Let K be a given closed convex cone. A matrix A is said to be copositive with respect to the cone K if*

$$\langle x, Ax \rangle \geq 0, \forall x \in K$$

A matrix A is said to be copositive–plus with respect to the cone K if it is copositive with respect to K and

$$\langle x, Ax \rangle = 0, x \in K \implies (A + A^T)x = 0$$

Definition 2.9 *Let K be a given closed convex cone. A matrix A is said to be L–matrix with respect to K if both*

1. *For every $q \in ri(K^D)$, the solution set of the generalized complementarity problem*

   $$z \in K, \quad Az + q \in K^D, \quad z^T(Az + q) = 0 \tag{2.20}$$

   is contained in lin K.

2. *For any $z \notin$ lin K such that*

   $$z \in K, \quad Az \in K^D, \quad z^T Az = 0$$

   there exists $z' \notin$ lin K, such that z' is contained in every face of K containing z and $-A^T z'$ is contained in every face of K^D containing Az.

To see how these definitions relate to the standard ones given in the literature on linear complementarity problems (e.g. [37] and [8]), consider the case that

$C = \mathrm{IR}^n_+$ and $K = \mathrm{rec}C = \mathrm{IR}^n_+$. Condition a) says that LCP(q, A) has a unique solution 0 for all $q > 0$. Condition b) states that, if $z \neq 0$ is a solution of LCP$(0, A)$, then there exists $z' \neq 0$ such that z' is contained in every face of IR^n_+ containing z and $-A^T z'$ is contained in every face of IR^n_+ containing Az. In particular, $z' \in \{x \in \mathrm{IR}^n \mid x_i = 0\}$, for all $i \in \{i \mid z_i = 0\}$. Hence $z'_i = 0$ for each i such that $z_i = 0$. That is, supp$z' \subset$ supp$z$. In another words, there exists a diagonal matrix $D \geq 0$ such that $z' = Dz$. Similarly, there exists a diagonal matrix $E \geq 0$ such that $-A^T z' = EAz$. Hence

$$(EA + A^T D)z = 0$$

where $D, E \geq 0$ and $Dz \neq 0$. Thus the notion of L–matrix defined here is a natural extension of that presented in Section 1.6. The following lemma shows that the class of L–matrices contains the class of copositive–plus matrices.

Lemma 2.10 *If a matrix A is copositive–plus with respect to a closed convex cone K, then it is an L–matrix with respect to K.*

Proof Suppose that $q \in \mathrm{ri}(K^D)$ and $z \in K \setminus \mathrm{lin}\ K$, then $\pi_{(\mathrm{lin}\ K)^\perp}(z) \neq 0$. Furthermore, there exists an $\epsilon > 0$, such that $q - \epsilon\pi_{(\mathrm{lin}\ K)^\perp}(z) \in K^D$, since $\mathrm{aff}(K^D) = (\mathrm{lin}\ K)^\perp$ (cf. [45, Theorem 14.6]). It follows that

$$\begin{aligned} & \langle z, q\rangle - \epsilon \left\|\pi_{(\mathrm{lin}\ K)^\perp}(z)\right\|_2^2 \\ = \ & \langle z, q\rangle - \epsilon \left\langle z, \pi_{(\mathrm{lin}\ K)^\perp}(z)\right\rangle \\ = \ & \left\langle z, q - \epsilon\pi_{(\mathrm{lin}\ K)^\perp}(z)\right\rangle \\ \geq \ & 0 \end{aligned}$$

That is $\langle z, q\rangle \geq \epsilon \left\|\pi_{(\mathrm{lin}\ K)^\perp}(z)\right\|_2^2 > 0$. Also $z^T Az \geq 0$ since A is copositive with respect to K. Thus $z^T(Az + q) = z^T Az + z^T q \geq z^T q > 0$. This shows that the set $K \setminus \text{lin } K$ does not contain any solution of (2.20). Therefore the solution set of the problem (2.20) is contained in lin K.

To complete the proof, note that for any $z \in K$, such that $Az \in K^D$ and $z^T Az = 0$, we have $Az + A^T z = 0$, or $-A^T z = Az$, since A is copositive–plus. So the condition b) of Definition 2.9 is satisfied with $z' = z$. **Q.E.D.**

We now come to the main result of this section.

Theorem 2.11 *Suppose $C = \{z \mid Bz \geq b, Hz = h\}$. Suppose A is an L–matrix with respect to $recC$ and invertible on the lineality space of C. Then exactly one of the following occurs:*

- *The method given above solves (AVI)*
- *the following system has no solution*

$$Ax - a \in (recC)^D, \qquad x \in C \tag{2.21}$$

Proof We may assume that (AVI) is in the form (2.11) due to Lemma 2.17 and Lemma 2.18. The pivotal method fails to solve (AVI) only if, at some iterate x_k, it reaches an unbounded direction d_{k+1} in σ_{k+1}. We know that x_k satisfies (2.15), and the direction d_{k+1} which satisfies $L_{\sigma_{k+1}} d_{k+1} = 0$ can be found by solving (2.16). Suppose $(\Delta x, \Delta z, \Delta u_{\mathcal{A}}, \Delta s_{\mathcal{I}}, \Delta \mu)$ is a solution of (2.16), then

$$\Delta u_{\mathcal{A}} \leq 0, \quad \Delta s_{\mathcal{I}} \geq 0, \quad \Delta \mu \geq 0 \tag{2.22}$$

provided that $x_k + \theta d_{k+1}$ is an unbounded ray. By reference to (2.16), we have

$$\begin{aligned} \bar{B}_{\mathcal{A}}^T \Delta u_{\mathcal{A}} + \bar{A} \Delta z &= e \Delta \mu \\ \bar{B}_{\mathcal{A}} \Delta z &= 0 \\ \bar{B}_{\mathcal{I}} \Delta z &= \Delta s_{\mathcal{I}} \geq 0 \end{aligned} \tag{2.23}$$

That is, Δz satisfies

$$\begin{aligned}\Delta z &\in \mathrm{rec}\bar{C}\\ \bar{A}\Delta z - e\Delta\mu &= \bar{B}_{\mathcal{A}}^T(-\Delta u_{\mathcal{A}}) \in (\mathrm{rec}\bar{C})^D\\ \Delta z^T(\bar{A}\Delta z - e\Delta\mu) &= \Delta z^T\bar{B}_{\mathcal{A}}^T(-\Delta u_{\mathcal{A}}) = -(\bar{B}_{\mathcal{A}}\Delta z)^T\Delta u_{\mathcal{A}} = 0\end{aligned}$$

If $\Delta\mu > 0$, then $e\Delta\mu \in \mathrm{int}N(x_e \mid \bar{C})$, hence $-e\Delta\mu \in \mathrm{int}(\mathrm{rec}\bar{C})^D$. The above system has a unique solution $\Delta z = 0$ by the fact that $\bar{A}$ is an L–matrix with respect to $\mathrm{rec}\bar{C}$ and $\mathrm{lin}\,\bar{C} = \{0\}$. Therefore the terminating ray is the starting ray, a contradiction. Thus $\Delta\mu = 0$. It follows that $\Delta z \in \mathrm{rec}\bar{C}$, $\bar{A}\Delta z \in (\mathrm{rec}\bar{C})^D$, and $z^T\bar{A}z = 0$, therefore there exist $\tilde{z} \neq 0$, such that $\tilde{z}$ is contained in every face of $\mathrm{rec}\bar{C}$ containing Δz, and that $-\bar{A}^T\tilde{z}$ is contained in every face of $(\mathrm{rec}\bar{C})^D$ containing $\bar{A}\Delta z$. We observe that, since $x_k \in \sigma_k \cap \sigma_{k+1} \cap F^{-1}(0)$, there exist z_k, u_k, s_k, and μ_k such that (2.15) is satisfied. It is easy to verify that Δz is in the face

$$G_1 = \left\{ z \in \mathrm{rec}\bar{C} \,\middle|\, z^T(\bar{B}^T u_k) = 0 \right\}$$

of $\mathrm{rec}\bar{C}$, and $\bar{A}\Delta z$ is in the face

$$G_2 = \left\{ z \in (\mathrm{rec}\bar{C})^D \,\middle|\, z = \bar{B}^T u, u = (u_{\mathcal{A}}, 0) \geq 0 \right\}$$

of $(\mathrm{rec}\bar{C})^D$, and thus

$$-\bar{A}^T\tilde{z} = \bar{B}^T\tilde{u} \in G_2, \qquad \text{for some } \ \tilde{u} = (\tilde{u}_{\mathcal{A}}, 0) \geq 0 \tag{2.24}$$

Consequently, by (2.15) we have

$$\bar{a} = x_k - z_k + \bar{A}z_k - e\mu_k$$

$$\tilde{u}^T(\bar{B}z_k - \bar{b}) = (\tilde{u}_{\mathcal{A}}^T, 0)\begin{pmatrix} 0 \\ s_{\mathcal{I}} \end{pmatrix} = 0$$

and

$$\tilde{z}^T(x_k - z_k) = \tilde{z}^T \bar{B}^T u_k = 0$$

since $\tilde{z} \in G_1$. Therefore

$$\begin{aligned}
\tilde{u}^T\bar{b} + \tilde{z}^T\bar{a} &= \tilde{u}^T(\bar{b} - \bar{B}z_k) + \tilde{u}^T\bar{B}z_k + \tilde{z}^T(x_k - z_k + \bar{A}z_k - e\mu_k) \\
&= (\bar{B}^T\tilde{u} + \bar{A}^T\tilde{z})^T z_k - \mu_k e^T \tilde{z} \\
&= -\mu_k e^T \tilde{z} > 0
\end{aligned}$$

in which the last inequality is due to $\tilde{z} \in \mathrm{rec}\bar{C}$ and $e \in \mathrm{int}N(x_e \mid \bar{C}) \subset -\mathrm{int}(\mathrm{rec}\bar{C})^D$. We now claim that the the system

$$\bar{A}x - \bar{a} \in (\mathrm{rec}\bar{C})^D, \qquad x \in \bar{C} \tag{2.25}$$

has no solution. To see this, let $x \in \bar{C}$, then

$$\tilde{u}^T\bar{B}x + \tilde{z}^T\bar{A}x = 0$$

as a result of (2.24). Subtract from this the inequality

$$\tilde{u}^T\bar{b} + \tilde{z}^T\bar{a} > 0$$

which we have just proven, then

$$\tilde{u}^T(\bar{B}x - \bar{b}) + \tilde{z}^T(\bar{A}x - \bar{a}) < 0$$

But it is obvious that $\tilde{u}^T(\bar{B}x - \bar{b}) \geq 0$, hence

$$\tilde{z}^T(\bar{A}x - \bar{a}) < 0$$

But $\tilde{z} \in \mathrm{rec}\bar{C}$. Thus $\bar{A}x - \bar{a} \notin (\mathrm{rec}\bar{C})^D$.

The proof is complete by noting that (2.25) has a solution if and only if (2.21) has a solution. **Q.E.D.**

(AVI) is said to be feasible if (2.21) has a solution. x is said to be feasible for (AVI) if it satisfies (2.21). Notice that if x solves (AVI), then

$$0 \in Ax - a + N_C(x), \qquad x \in C$$

that is

$$-(Ax - a) \in N_C(x) \subset (\mathrm{rec}C)^o, \qquad x \in C$$

In another words

$$(Ax - a) \in (\mathrm{rec}C)^D, \qquad x \in C$$

So, every solution of (AVI) is feasible.

As a special case of Theorem 2.11, we have the following result for copositive–plus matrices.

Corollary 2.12 *Suppose $C = \{z \mid Bz \geq b, Hz = h\}$ and A is copositive–plus with respect to rec$C$ and invertible on the lineality space of C. Then exactly one of the following occurs:*

- *The method given above solves (AVI)*
- *the following system has no solution*

$$Ax - a \in (recC)^D, \qquad x \in C \tag{2.26}$$

Proof Obvious, in view of Lemma 2.10. **Q.E.D.**

We can also prove Theorem 2.3 as a special case of Theorem 2.11 by using the following lemma.

Lemma 2.13 *Suppose A_C is coherently oriented. Then*

a) $A_{\text{rec}C}$ is coherently oriented;

b) A is an L–matrix with respect to $recC$.

Proof a) This follows from the proof of [43, Theorem 4.3].

b) By the first part, $A_{\text{rec}C}$ is coherently oriented, so by [43, Theorem 4.3] it is a Lipschitzian homeomorphism, and hence $A_{\text{rec}C}(x) = q$ has a unique solution for all q. Therefore part 1 and 2 of the definition of L–matrix are trivially satisfied by the unique solution 0. **Q.E.D.**

2.4 Invariance Properties of L–matrices

In this section we show that the property of L–matrix with respect to a polyhedral convex cone is invariant under the two reductions presented in Section 2.2. We begin with the following technical lemmas.

Lemma 2.14 *Let C, $\tilde{C}$, and $\bar{C}$ be as in (AVI), (2.3) and (2.11); V and Y be as in (2.5) and Lemma 2.4. Then*

$$recC = V(rec\tilde{C}) \tag{2.27}$$

$$rec\tilde{C} = Y(rec\bar{C}) \tag{2.28}$$

and

$$V^T((recC)^D) = (rec\tilde{C})^D \tag{2.29}$$

$$Y^T((rec\tilde{C})^D) = (rec\bar{C})^D \tag{2.30}$$

Furthermore

$$V^T(ri((recC))^D) = ri(rec\tilde{C})^D \tag{2.31}$$

$$Y^T(ri(rec\tilde{C})^D) = ri(rec\bar{C})^D \tag{2.32}$$

Proof (2.27) and (2.28) are obvious from definition.

Based on these two equations and [45, Corollary 16.3.2], we have

$$\begin{aligned}(\mathrm{rec}C)^D &= -(\mathrm{rec}C)^o = -(V\mathrm{rec}\tilde{C})^o \\ &= -(V^T)^{-1}(\mathrm{rec}\tilde{C})^o = (V^T)^{-1}(\mathrm{rec}\tilde{C})^D\end{aligned}$$

where $K^o = -K^D$ is the polar cone of K and $(V^T)^{-1}$ is the inverse image of the linear map V^T (also see [45]). Similarly

$$(\mathrm{rec}\tilde{C})^D = (Y\mathrm{rec}\bar{C})^D = (Y^T)^{-1}(\mathrm{rec}\bar{C})^D$$

So we have proven (2.29) and (2.30).

(2.31) and (2.32) can be obtained from (2.29) and (2.30) by applying [45, Theorem 6.6]. **Q.E.D.**

Lemma 2.15 *For $z \in recC$, $\tilde{z} \in rec\tilde{C}$, and $\bar{z} \in rec\bar{C}$, define*

$$\begin{aligned}D(z) &:= \left\{d \in (recC)^D \mid \langle d, z\rangle = 0\right\} \\ \tilde{D}(\tilde{z}) &:= \left\{\tilde{d} \in (rec\tilde{C})^D \,\middle|\, \left\langle \tilde{d}, \tilde{z}\right\rangle = 0\right\} \\ \bar{D}(\bar{z}) &:= \left\{\bar{d} \in (rec\bar{C})^D \,\middle|\, \left\langle \bar{d}, \bar{z}\right\rangle = 0\right\}\end{aligned}$$

Then

$$\tilde{D}(\tilde{z}) = V^T D(V\tilde{z}) \tag{2.33}$$

$$\bar{D}(\bar{z}) = Y^T \tilde{D}(Y\bar{z}) \tag{2.34}$$

where V and Y are as in (2.5) and Lemma 2.4.

Proof

$$\begin{aligned}\tilde{D}(\tilde{z}) &= \left\{\tilde{d} \in (\mathrm{rec}\tilde{C})^D \,\middle|\, \left\langle \tilde{d}, \tilde{z}\right\rangle = 0\right\} = \left\{\tilde{d} \in V^T(\mathrm{rec}C)^D \,\middle|\, \left\langle \tilde{d}, \tilde{z}\right\rangle = 0\right\} \\ &= V^T\left\{d \in (\mathrm{rec}C)^D \,\middle|\, \left\langle d^T, V\tilde{z}\right\rangle = 0\right\} = V^T D(V\tilde{z})\end{aligned}$$

The other equation can be proven similarly. **Q.E.D.**

Actually, for $z \in \mathrm{rec}C$, $D(z)$ is the set of vectors defining faces of rec$C$ containing z, a vector z' is in every face of rec$C$ containing z if and only if $\langle d, z' \rangle = 0$ for all $d \in D(z)$. Similar observations can also be made for the set $\tilde{C}$ and $\bar{C}$.

Lemma 2.16 *For $w \in (recC)^D$, $\tilde{w} \in (rec\tilde{C})^D$, and $\bar{w} \in (rec\bar{C})^D$, define*

$$\begin{aligned} R(w) &:= \{r \in recC \mid \langle r, w \rangle = 0\} \\ \tilde{R}(\tilde{w}) &:= \left\{\tilde{r} \in rec\tilde{C} \mid \langle \tilde{r}, \tilde{w} \rangle = 0\right\} \\ \bar{R}(\bar{w}) &:= \left\{\bar{r} \in rec\bar{C} \mid \langle \bar{r}, \bar{w} \rangle = 0\right\} \end{aligned}$$

Then

$$V\tilde{R}(V^T w) = R(w) \tag{2.35}$$

$$Y\bar{R}(Y^T \tilde{w}) = \tilde{R}(\tilde{w}) \tag{2.36}$$

where V and Y are as in (2.5) and Lemma 2.4.

Proof

$$\begin{aligned} R(w) &= \{r \in \mathrm{rec}C \mid \langle r, w \rangle = 0\} = \left\{r \in V(\mathrm{rec}\tilde{C}) \mid \langle r, w \rangle = 0\right\} \\ &= V\left\{\tilde{r} \in \mathrm{rec}\tilde{C} \,\middle|\, \left\langle \tilde{r}, V^T w \right\rangle = 0\right\} = V\tilde{R}(V^T w) \end{aligned}$$

The other equation can be proven similarly. **Q.E.D.**

Similar to the case of Lemma 2.15, for $w \in (\mathrm{rec}C)^D$, $R(w)$ is the set of vectors defining faces of $(\mathrm{rec}C)^D$ containing w, a vector w' is in every face of $(\mathrm{rec}C)^D$ containing w if and only if $\langle r, w' \rangle = 0$ for all $r \in R(z)$. The situation is similar for the set $\tilde{C}$ and $\bar{C}$.

Now, we come to the invariance of the L–matrix property.

Lemma 2.17 *Given the problems (2.3) and (2.11), suppose $\tilde{A}$ is an L–matrix with respect to rec$\tilde{C}$. Then $\bar{A}$ is an L–matrix with respect to rec$\bar{C}$.*

Proof For $\bar{z} \in \mathrm{rec}\bar{C}$, $Y\bar{z} \in \mathrm{rec}\tilde{C}$. For any $\bar{q} \in \mathrm{ri}(\mathrm{rec}\bar{C})^D$, there exists $\tilde{q} \in \mathrm{ri}(\mathrm{rec}\tilde{C})^D$ such that $\bar{q} = Y^T\tilde{q}$ due to (2.32). If $\bar{A}\bar{z} + \bar{q} \in (\mathrm{rec}\bar{C})^D$ then

$$Y^T\tilde{A}Y\bar{z} + Y^T\tilde{q} \in (\mathrm{rec}\bar{C})^D$$

by definition of $\bar{A}$. Hence

$$\left\langle \tilde{A}Y\bar{z} + \tilde{q}, Y\bar{z} \right\rangle = \left\langle Y^T\tilde{A}Y\bar{z} + Y^T\tilde{q}, \bar{z} \right\rangle \geq 0, \qquad \forall \bar{z} \in \mathrm{rec}\bar{C}$$

It follows from (2.28) that

$$\left\langle \tilde{A}Y\bar{z} + \tilde{q}, \tilde{z} \right\rangle \geq 0, \qquad \forall \tilde{z} \in \mathrm{rec}\tilde{C}$$

Thus

$$\tilde{A}Y\bar{z} + \tilde{q} \in (\mathrm{rec}\tilde{C})^D$$

Therefore $\bar{z}$ satisfies

$$\bar{z} \in \mathrm{rec}\bar{C}, \quad \bar{A}\bar{z} + \bar{q} \in (\mathrm{rec}\bar{C})^D, \quad \text{and} \quad \bar{z}^T(\bar{A}\bar{z} + \bar{q}) = 0 \tag{2.37}$$

with $\bar{q} \in \mathrm{ri}(\mathrm{rec}\bar{C})^D$, implies $Y\bar{z}$ satisfies

$$Y\bar{z} \in \mathrm{rec}\tilde{C}, \quad \tilde{A}Y\bar{z} + \tilde{q} \in (\mathrm{rec}\tilde{C})^D, \quad \text{and} \quad (Y\bar{z})^T[\tilde{A}(Y\bar{z}) + \tilde{q}] = 0 \tag{2.38}$$

with $\tilde{q} \in \mathrm{ri}(\mathrm{rec}\tilde{C})^D$. Thus, the solution $Y\bar{z}$ of (2.38) is contained in lin $\tilde{C} = \{0\}$, which implies that $\bar{z} = 0$. Thus the solution set of (2.37) is $\{0\} \subset$ lin $\bar{C}$.

For any $0 \neq \bar{z} \in \mathrm{rec}\bar{C}$ such that

$$\bar{A}\bar{z} \in (\mathrm{rec}\bar{C})^D \quad \text{and} \quad \bar{z}^T\bar{A}\bar{z} = 0$$

we have, $0 \neq Y\bar{z} \in \mathrm{rec}\tilde{C}$, and

$$\tilde{A}Y\bar{z} \in (\mathrm{rec}\tilde{C})^D \quad \text{and} \quad (Y\bar{z})^T\tilde{A}(Y\bar{z}) = 0$$

So, there exists $0 \neq \tilde{z} \in \mathrm{rec}\tilde{C}$ such that $\tilde{z}$ is contained in every face of $\mathrm{rec}\tilde{C}$ containing $Y\bar{z}$, and $-\tilde{A}^T\tilde{z}$ is contained in every face of $(\mathrm{rec}\tilde{C})^D$ containing $\tilde{A}Y\bar{z}$. That is

$$\left\langle \tilde{d}, \tilde{z} \right\rangle = 0 \qquad \forall \tilde{d} \in \tilde{D}(Y\bar{z})$$
$$\left\langle \tilde{r}, -\tilde{A}^T\tilde{z} \right\rangle = 0 \qquad \forall \tilde{r} \in \tilde{R}(\tilde{A}Y\bar{z})$$

Consequently, there exists $0 \neq \bar{z}' \in \mathrm{rec}\bar{C}$ such that $\tilde{z} = Y\bar{z}'$. For any $\bar{d} \in \bar{D}(\bar{z})$, $\bar{d} = Y^T\tilde{d}$ for some $\tilde{d} \in \tilde{D}(Y\bar{z})$. Hence

$$\left\langle \bar{d}, \bar{z}' \right\rangle = \left\langle Y^T\tilde{d}, \bar{z}' \right\rangle = \left\langle \tilde{d}, Y\bar{z}' \right\rangle = 0$$

So, $\bar{z}'$ is contained every face of $\mathrm{rec}\bar{C}$ containing $\bar{z}$. Moreover, for any $\bar{r} \in \bar{R}(\bar{A}\bar{z})$

$$\left\langle \bar{r}, -\bar{A}^T\bar{z}' \right\rangle = \left\langle Y\bar{r}, -\tilde{A}^TY\bar{z}' \right\rangle = \left\langle Y\bar{r}, -\tilde{A}^T\tilde{z} \right\rangle = 0$$

since $Y\bar{z} \in \tilde{R}(\tilde{A}Y\bar{z})$. We see that $-\bar{A}^T\bar{z}'$ is contained in every face of $(\mathrm{rec}\bar{C})^D$ containing $\bar{A}\bar{z}$. Thus $\bar{A}$ is an L–matrix with respect to $\bar{C}$. **Q.E.D.**

Lemma 2.18 *Given the problems (NE) and(2.3), suppose A is an L–matrix with respect to rec$C$. Then $\tilde{A}$ is an L–matrix with respect to rec$\tilde{C}$.*

Proof For any $\tilde{z} \in \mathrm{rec}\tilde{C}$, $V\tilde{z} \in \mathrm{rec}C$ and

$$U\tilde{z} = (V - W(W^TAW)^{-1}W^TAV)\tilde{z} = V\tilde{z} - W(W^TAW)^{-1}W^TAV\tilde{z} \in \mathrm{rec}C$$

since $W(W^TAW)^{-1}W^TAV\tilde{z} \in \mathrm{lin}\ C$. For any $\tilde{q} \in \mathrm{ri}(\mathrm{rec}\tilde{C})^D$, there exists $q \in \mathrm{ri}(\mathrm{rec}C)^D$ such that $\tilde{q} = V^Tq$. If $\tilde{A}\tilde{z} + \tilde{q} \in (\mathrm{rec}\tilde{C})^D$ then

$$U^TAU\tilde{z} + V^Tq \in (\mathrm{rec}\tilde{C})^D, \qquad q \in (\mathrm{rec}C)^D$$

by definition of $\tilde{A}$. But

$$U^T A U = V^T A U - V^T A^T W (W^T A W)^{-T} W^T A U = V^T A U$$

since $W^T A U = 0$, as can be directly verified. Thus

$$V^T (AU\tilde{z} + q) = V^T AU\tilde{z} + V^T q \in (\mathrm{rec}\tilde{C})^D, \qquad q \in (\mathrm{rec}C)^D$$

which implies

$$\langle AU\tilde{z} + q, V\tilde{z}\rangle = \left\langle V^T(AU\tilde{z} + q), \tilde{z}\right\rangle \geq 0, \qquad \forall \tilde{z} \in \mathrm{rec}\tilde{C}$$

It follows from (2.27) that

$$\langle AU\tilde{z} + q, z\rangle \geq 0, \qquad \forall z \in \mathrm{rec}C$$

Thus

$$AU\tilde{z} + q \in (\mathrm{rec}C)^D$$

Also

$$(U\tilde{z})^T [A(U\tilde{z}) + q] = \tilde{z}^T \tilde{A}\tilde{z} = 0$$

Therefore $\tilde{z}$ satisfies

$$\tilde{z} \in \mathrm{rec}\tilde{C}, \quad \tilde{A}\tilde{z} + \tilde{q} \in (\mathrm{rec}\tilde{C})^D, \quad \text{and} \quad \tilde{z}^T(\tilde{A}\tilde{z} + \tilde{q}) = 0 \tag{2.39}$$

with $\bar{q} \in \mathrm{ri}(\mathrm{rec}\tilde{C})^D$ implies $U\tilde{z}$ satisfies

$$U\tilde{z} \in \mathrm{rec}C, \quad AU\tilde{z} + q \in (\mathrm{rec}C)^D, \quad \text{and} \quad (U\tilde{z})^T[A(U\tilde{z}) + q] = 0 \tag{2.40}$$

with $q \in \mathrm{ri}(\mathrm{rec}C)^D$. Hence the solution $U\tilde{z} \in \mathrm{lin}\ \mathrm{rec}C = \mathrm{lin}\ C$. But then

$$V\tilde{z} \in W(W^T AW)^{-1} A^T V\tilde{z} + \mathrm{lin}\ C \subset \mathrm{lin}\ C$$

which, by the definition of V, implies $\tilde{z} = 0$. This shows that the solution set of (2.39) is contained in lin $\tilde{C} = \{0\}$.

For any $0 \neq \tilde{z} \in \text{rec}\tilde{C}$ such that

$$\tilde{A}\tilde{z} \in (\text{rec}\tilde{C})^D \quad \text{and} \quad \tilde{z}^T\tilde{A}\tilde{z} = 0$$

we have $0 \neq U\tilde{z} \in \text{rec}C$, and

$$V^T AU\tilde{z} = U^T AU\tilde{z} = \tilde{A}\tilde{z} \in (\text{rec}\tilde{C})^D$$

which implies $A(U\tilde{z}) \in (\text{rec}C)^D$. We also have

$$(U\tilde{z})^T A(U\tilde{z}) = \tilde{z}^T\tilde{A}\tilde{z} = 0$$

So, there exists $0 \neq z' \in \text{rec}C$ such that z' is contained in every face of rec$C$ containing $U\tilde{z}$, and that $-A^T z'$ is contained in every face of $(\text{rec}C)^D$ containing $A(U\tilde{z})$. That is

$$\langle d, z' \rangle = 0 \qquad \forall d \in D(U\tilde{z})$$

$$\langle r, -Az' \rangle = 0 \qquad \forall r \in R(AU\tilde{z})$$

Consequently, there exists $0 \neq \tilde{z}' \in \text{rec}\tilde{C}$, such that $z' = V\tilde{z}'$, and for any $\tilde{d} \in \tilde{D}(\tilde{z})$, we have $\tilde{d} = V^T d$, for some $d \in D(V\tilde{z})$, but since $d \in (\text{rec}C)^D$, $W^T d = 0$, therefore $\langle d, V\tilde{z} \rangle = \langle d, U\tilde{z} \rangle$, so $d \in D(V\tilde{z})$ implies $d \in D(U\tilde{z})$, hence

$$\left\langle \tilde{d}, \tilde{z}' \right\rangle = \left\langle V^T d, \tilde{z}' \right\rangle = \langle d, V\tilde{z}' \rangle = \langle d, z' \rangle = 0$$

So, $\tilde{z}'$ is contained in every face of rec$\tilde{C}$ containing $\tilde{z}$. For any $\tilde{r} \in \tilde{R}(\tilde{A}\tilde{z})$

$$\begin{aligned} \left\langle \tilde{r}, -\tilde{A}^T\tilde{z}' \right\rangle &= \left\langle \tilde{r}, -U^T A^T U\tilde{z}' \right\rangle = \left\langle \tilde{r}, -U^T A^T V\tilde{z}' \right\rangle = \left\langle \tilde{r}, -U^T A^T z' \right\rangle \\ &= \left\langle \tilde{r}, -V^T A^T z' \right\rangle = \left\langle V\tilde{r}, -A^T z' \right\rangle = \left\langle r, -A^T z' \right\rangle = 0 \end{aligned}$$

since $r = V\tilde{r} \in R(AU\tilde{z})$ as a result of (2.36). This proved that $-\tilde{A}^T\tilde{z}'$ is contained in every face of $(\text{rec}\tilde{C})^D$ containing $\tilde{A}\tilde{z}$. **Q.E.D.**

2.5 P_c Matrices

Using a geometric approach, we generalize both the notion of P matrices for arbitrary polyhedral convex sets in $\mathbb{R}^n$, and the termination results for Lemke's pivotal method on P–matrices. These termination results can be generalized to a much broader class of matrices known as L as demonstrated by the work of Eaves (see [12] and [13]) and our work earlier in this chapter. Another approach for generalizing the termination results for Lemke's pivotal method on P–matrices is through the notion of P_0 matrices and the work of Cottle et.al. in [1] and [9].

In this section, we explore the possibility of generalizing the notion of P_0 for polyhedral convex sets. We begin with an analysis on the standard LCP. Our study focuses on geometric and topological properties of the sets $K(M)$ and $\mathrm{SOL}(q, M)$ that are crucial in analyzing termination behavior of Lemke's algorithm. We prove that the convexity of $K(M)$ and the connectedness of the set $\mathrm{SOL}(q, M)$ for all q are sufficient conditions for Lemke's algorithm to terminate at a solution of $\mathrm{LCP}(q, M)$. We study those matrices M for which $\mathrm{SOL}(q, M)$ is connected for all $q \in \mathbb{R}^n$ as a matrix class. We denote this matrix class as P_c. We show that P_c is a subclass of semi–monotone matrices. We also show that this class is not a subclass of P_0, but it contains at least a substantial portion of it, e.g. it contains all the column sufficient matrices. The question of whether P_0 is subclass of P_c is still unknown.

As we know, (LCP) is a special case of (NE). We have $F \equiv M$, a linear map, and $C = \mathbb{R}^n_+$. The normal equation is

$$M_{\mathbb{R}^n_+}(x) + q = 0$$

or equivalently

$$Mx_+ + x - x_+ + q = 0$$

Define a PL function $F:(N,\mathcal{N}) \to \mathbb{R}^n$ (where $\mathbb{R}^n$ is regarded as a PL manifold of one cell) by:

$$F(x,\mu) = M_{\mathbb{R}^n_+}(x) + (q + \mu e)$$

We shall consider solutions $x(\mu)$ of $F(x,\mu) = 0$; it is clear from (NE) that $x(0)$ will solve our problem.

We use the path following algorithm of Section 2.1 to find $x(0)$. In order to find a starting ray, consider $w(\mu) = -q - \mu e$. It is clear that since

$$w(\mu) = -\mu[e + \mu^{-1}q] \tag{2.41}$$

for large positive μ, $w(\mu)$ lies interior to the cell $\mathbb{R}^n_-$ of $\mathcal{N}_{\mathbb{R}^n_+}$. Therefore $(w(\mu),\mu)$ lies interior to the cell $\mathbb{R}^n_- \times R_+$ of $(N,\mathcal{N})$, and so it is a regular point of $(N,\mathcal{N})$. Further, for such μ we have $\pi_{\mathbb{R}^n_+}(w(\mu)) = 0$, so that

$$F(w(\mu),\mu) = -q - \mu e - (q + \mu e) = 0$$

Therefore for some $\mu_0 \geq 0$, $F^{-1}(0)$ contains the ray $\{\, (w(\mu),\mu) \mid \mu \geq \mu_0 \,\}$.

In analyzing the termination behavior of our algorithm, we assume that M is in Q_0, that is, LCP(q,M) is solvable whenever it is feasible (see Section 1.6). Our main result is summarized in the following theorem.

Theorem 2.19 *Suppose M is in $Q_0 \cap P_c$. Let*

$$q \in K(M) = pos(I, -M)$$

Then, the algorithm given in Section 2.1 terminates at a solution of LCP(q,M).

We first introduce a series of technical tools before proving the theorem.

Lemma 2.20 *Given a Q_0 matrix M and*

$$q \in K(M) = pos(I, -M)$$

there exists a set of n linearly independent vectors $\{y_1, y_2, \cdots, y_n\} \subset I\!R^n$ such that $q + y_i \in intK(M)$ for all $1 \leq i \leq n$.

Proof First we notice that $\mathrm{int}K(M) \supset \mathbb{R}^n_{++} \neq \emptyset$.

For any $q \in \mathrm{int}K(M)$, the lemma is trivially true by selecting the set of vectors

$$\{\delta e_i \,|\, 1 \leq i \leq n\}$$

for some $\delta > 0$ sufficiently small.

For any other vector $q \in K(M)$, we can first choose a vector $q_0 \in \mathrm{int}K(M)$ and a set of linearly independent vectors $\{y_1, y_2, \cdots, y_n\} \subset \mathbb{R}^n$ such that $q_0 + y_i \in \mathrm{int}K(M)$ for all $1 \leq i \leq n$. Let

$$z_i = y_i + q_0 - q \qquad\qquad 1 \leq i \leq n$$

then $z_1, z_2, \cdots, z_n$ are linearly independent and

$$q + z_i = y_i + q_0 \in \mathrm{int}K(M) \qquad\qquad 1 \leq i \leq n$$

Q.E.D.

Lemma 2.21 *Given a Q_0 matrix M and a vector $q \in K(M) = pos(I, -M)$, there exists a matrix Y of order n having linearly independent columns and $q + y_i \in K(M)$ for all $1 \leq i \leq n$, where y_i is the i–th column of Y. Let $[\epsilon]$ be the vector $(\epsilon, \epsilon^2, \cdots, \epsilon^n)^T$, then $q + Y[\epsilon] \in K(M)$ for $\epsilon > 0$ sufficiently small.*

Proof Choose a set of vectors $\{y_1, y_2, \cdots, y_n\}$ as specified by Lemma 2.20. Form the matrix Y by using y_i as the i–th column, for $1 \leq i \leq n$.

We observe that for any $\epsilon > 0$

$$\begin{aligned} q + Y[\epsilon] &= q + E\sum_{i=1}^{n} \frac{\epsilon^i}{E} y_i \\ &= E(\frac{q}{E} + \sum_{i=1}^{n} \frac{\epsilon^i}{E} y_i) \\ &= E\sum_{i=1}^{n} \frac{\epsilon^i}{E} \cdot \frac{q + Ey_i}{E} \end{aligned}$$

where $E = \sum_{i=1}^{n} \epsilon^i$.

We notice that $E < 1$ when ϵ is sufficiently small. Hence $q + Ey_i \in K(M)$ by convexity of $K(M)$ and $\frac{q+Ey_i}{E} \in K(M)$ by the fact that $K(M)$ is a cone. Therefore

$$\sum_{i=1}^{n} \frac{\epsilon^i}{E} \cdot \frac{q + Ey_i}{E} \in K(M)$$

as a convex combination of $\frac{q+Ey_i}{E}$'s, and

$$q + Y[\epsilon] = E\sum_{i=1}^{n} \frac{\epsilon^i}{E} \cdot \frac{q + Ey_i}{E} \in K(M)$$

since $K(M)$ is a cone. **Q.E.D.**

Now, we are ready to prove the main theorem.

Proof We wish to solve $F(x, \mu) = 0$. Unfortunately, 0 may not be a regular value of F. Thus we use **Algorithm 2** from Chapter 1 which permits solving the perturbed system

$$F(x, \mu) = Y[\epsilon]$$

We choose $-Y$ and ϵ according to Lemma 2.20 and Lemma 2.21 so that Y is of rank n and $q - Y[\epsilon] \in K(M)$ for all small non–negative ϵ. That is $Y[\epsilon]$ is in $F(N)$ for all small non–negative ϵ. Hence, by Theorem 1.6, $Y[\epsilon]$ is a regular value of F

for each small positive ϵ. It then follows by Theorem 1.6 that for such ϵ, $F^{-1}(Y[\epsilon])$ is a 1-manifold neat in $\mathcal{N}$. Furthermore, we have $(w(\mu) + Y[\epsilon], \mu) \in F^{-1}(Y[\epsilon])$, for sufficiently large μ.

Now, assume that the algorithm starts with the ray

$$\{(w(\mu) + Y[\epsilon], \mu) \mid \mu \text{ sufficiently large}\}$$

generates a sequence of points (x_1, μ_1), (x_2, μ_2), $\cdots$, (x_k, μ_k) and terminates at step k with a ray different from the starting one. Let $W(\epsilon)$ be the route formed by the set of chords traversed by the the algorithm. Then, due to the ray start, $W(\epsilon)$ cannot be homeomorphic to a circle, and therefore it is homeomorphic to an interval.

Upon ray termination, μ is non–decreasing on the terminating ray. Thus, the set

$$\Xi = \{\mu \mid (x, \mu) \in W(\epsilon)\}$$

admits a minimum $0 < \bar{\mu} = \inf\{\mu \in \Xi\}$ which is achieved on (x_j, μ_j) for some $1 \leq j \leq k$. Let

$$S = \{x \mid (x, \bar{\mu}) \in W(\epsilon)\}$$

then $F(x, \bar{\mu}) = Y[\epsilon]$ for $x \in S$. Hence

$$S \subset \mathrm{SOL}(q - Y[\epsilon] + \bar{\mu}e, M)$$

But $\mathrm{SOL}(q - Y[\epsilon] + \bar{\mu}e, M)$ cannot contain any other point z_1 such that $(z_1, \bar{\mu}) \notin W(\epsilon)$, otherwise, by our hypothesis on the connectedness of the solution set, there is a continuous path map $z : [0, 1] \to \mathrm{SOL}(q - Y[\epsilon] + \bar{\mu}e, M)$ with $z(1) = z_1$ and $z(0) = z_0$ for any $z_0 \in S$. Thus

$$\{(z(t), \bar{\mu}) \mid 0 \leq t \leq 1\} \subset F^{-1}(Y[\epsilon])$$

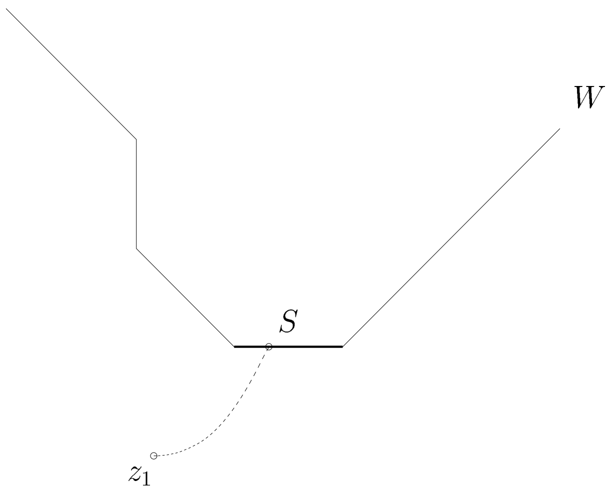

Figure 2: THE PATH CONNECTING z_1 TO S FORMS A BRANCH OF W

But this contradicts the fact that $F^{-1}(Y[\epsilon])$ is a 1–manifold, since $(z_0, \bar{\mu})$ contains a neighborhood not homeomorphic to an interval (see Figure 2).

Thus $S = \mathrm{SOL}(q - Y[\epsilon] + \bar{\mu}e, M)$ is a connected set. It is either a single point, or the union of finite number of consecutive chords in $W(\epsilon)$. In particular, S is closed.

Considering that $K(M)$ is convex and that $\mathrm{SOL}(q - Y[\epsilon] + \mu e, M) \neq \emptyset$ for $\mu = \bar{\mu}$ and for $\mu = 0$

$$\mathrm{SOL}(q - Y[\epsilon] + \mu e, M) \neq \emptyset$$

for all $0 \leq \mu < \bar{\mu}$. Consider a strictly increasing sequence $\{\mu_j \mid j = 1, 2, \cdots\}$ with $\mu_1 < \bar{\mu}$ and $\lim_{j \to \infty} \mu_j = \bar{\mu}$. Assume that $x(\mu_j) \in \mathrm{SOL}(q - Y[\epsilon] + \mu e, M)$. Then, $(x(\mu_j), \mu_j) \in F^{-1}(Y[\epsilon])$, hence each $(x(\mu_j), \mu_j) \in F^{-1}(Y[\epsilon])$ is contained in a 1–chord of $F^{-1}(Y[\epsilon])$. Since the 1–manifold $F^{-1}(Y[\epsilon])$ is finite, there exists a chord d such that $(x(\mu_j), \mu_j) \in d$ for infinitely many j, and without loss of generality we

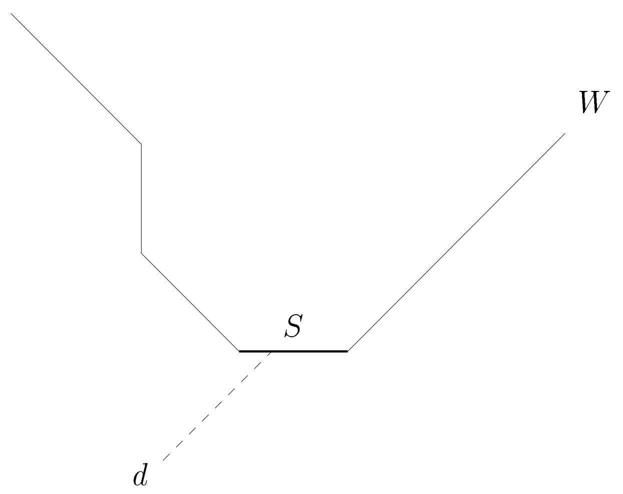

Figure 3: The chord d forms a branch of W

can assume that $(x(\mu_j), \mu_j) \in d$ for all j. Therefore d contains the set

$$\left\{(x(\mu), \mu) \in F^{-1}(Y[\epsilon]) \mid \bar{\mu} - \delta \leq \mu < \bar{\mu}\right\}$$

for some $\delta > 0$. Thus d contains a point $(w(\bar{\mu}), \bar{\mu})$ with $w(\bar{\mu}) \in S$. On the other hand, by definition of $\bar{\mu}$

$$(w(\mu), \mu) \notin W(\epsilon)$$

for any $\mu < \bar{\mu}$. Hence d is not a subset of $W(\epsilon)$, and d forms a branch from $S \times \{\bar{\mu}\}$ (see Figure 3). This is in contradiction to the fact that $F^{-1}(Y[\epsilon])$ is a 1–manifold.

So the algorithm terminates at a point at the boundary, that is a solution of $F(x, 0) = Y[\epsilon]$.

Now in practice the algorithm does not actually use a positive ϵ, but only maintains the information necessary to compute $W(\epsilon)$ for all small positive ϵ, employing the lexicographic ordering to resolve possible ambiguities when $\epsilon =$

0. Therefore after finitely many steps it will actually have computed x_0 with $M_{\mathbb{R}^n_+}(x_0) + q = 0$. **Q.E.D.**

By reference to Theorem 1.15, we obtain the termination property of the pivotal algorithm for sufficient matrices.

Corollary 2.22 *Suppose M is a column sufficient matrix and M is in Q_0. Let*

$$q \in K(M) = pos(I, -M)$$

Then, the algorithm given in Section 2.1 terminates at a solution of LCP(q, M).

Proof Since M is column sufficient, SOL(q, M) is convex, and is hence connected for all q. The corollary now follows from Theorem 2.19. **Q.E.D.**

Now that we know our new matrix class contains a substantial portion of P_0, e.g. column sufficient matrices, we will be interested to find out how is it related to P_0 itself. The following example indicates this new matrix class is not a subclass of P_0. Note that this example also shows that a matrix M being in P_c does not guarantee that M is Q_0.

Let

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then, M is not a P_0 matrix. But for any $q \in \mathbb{R}^n$, we have

$$\mathrm{SOL}(q, M) = \begin{cases} \{(0,0)\} & \text{if} \quad q_1 > 0, q_2 > 0 \\ \emptyset & \text{if} \quad q_1 > 0, q_2 < 0 \\ \emptyset & \text{if} \quad q_1 < 0, q_2 > 0 \\ \{(-q_2, -q_1)\} & \text{if} \quad q_1 < 0, q_2 < 0 \\ \{(0,y) \mid y \geq 0\} & \text{if} \quad q_1 > 0, q_2 = 0 \\ \{(x,0) \mid x \geq 0\} & \text{if} \quad q_1 = 0, q_2 > 0 \\ \{(0,y) \mid y \geq -q_1\} \cup \{(x, -q_1) \mid x \geq 0\} & \text{if} \quad q_1 < 0, q_2 = 0 \\ \{(x,0) \mid x \geq -q_2\} \cup \{(-q_2, y) \mid y \geq 0\} & \text{if} \quad q_1 = 0, q_2 < 0 \end{cases}$$

We see that $\mathrm{SOL}(q, M)$ is connected for all q. Now that P_0 does not contain P_c, does P_c contains P_0? According to a result in [23, Theorem 2], originally due to Cottle and Guu, $\mathrm{SOL}(q, M)$ contains either 0, 1, or infinitely many points, given that M is a P_0 matrix. Hence, $\mathrm{SOL}(q, M)$ is connected when it has finitely may elements. The question whether $\mathrm{SOL}(q, M)$ is connected when it has infinitely many elements remains open.

2.6 An Implementation in MATLAB

The algorithm described in this chapter has been implemented in `MATLAB` [34]. Copies of the code and the testing script files are available.

The algorithm `NEPOLY` is implemented as three function files in `MATLAB` . The development of the code is exactly as outlined in Section 2.2. The first function removes the lineality of the set C, then calls the second routine which proceeds to determine an extreme point and factor out the equality constraints. Having accomplished this, the third routine then executes the pivot steps. We note in particular, that Lemke's original pivot algorithm can be carried out just using the third routine, since the defining set $C = \mathbb{R}^n_+$ has no lines, no equality constraints and a single extreme point 0.

We now present two tables of our results of applying this algorithm to some small quadratic programs. In Table 1 we present a comparison of `NEPOLY` to the standard QP solver that is available as part of the optimization tool box of `MATLAB` . This QP solver is an active set method, similar to that described in [21]. Further details are available in [34].

The problems that we generate are of the form

$$\min \quad \frac{1}{2}x^TQx + c^Tx + \frac{1}{2}y^Ty \tag{2.42}$$

$$\text{subject to} \quad Ax + By = b, \;\; x \geq 0 \tag{2.43}$$

where $Q \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{p\times n}$ and $B \in \mathbb{R}^{p\times m}$. The minimum principle generates an affine variational inequality which under convexity is equivalent to (2.42). In general, the variational inequality represents necessary optimality conditions for (2.42).

We generate Q as a random sparse symmetric matrix. Unfortunately, the `MATLAB` QP solver did not solve (2.42) unless Q was positive semi-definite, so in Table 1, Q was generated positive semi-definite. The matrices A and B were generated using the `MATLAB` random generator, the feasible region was guaranteed to be non–empty by randomly generating a feasible point (x_0, y_0) and setting $b = Ax_0 + By_0$.

`MATLAB` 4.0 was used with dedicated access to a Hewlett Packard 9000/705 workstation. The times reported are elapsed times in seconds using the built–in stopwatch timer of `MATLAB` . The ordering of entries in the table is by total problem size. Since the problems are convex, both codes always found the solution of (2.42). The constraint error was always less than 10^{-14}. All `MATLAB` codes reported here do not use the sparse matrix facility of MATLAB .

m	n	p	NEPOLY time	MATLAB QP time
10	10	10	0.3	0.8
20	10	10	0.2	0.2
30	20	10	0.3	0.3
10	40	10	3.4	10.5
10	10	50	0.6	5.7
20	20	30	1.2	4.6
10	60	20	5.8	45.1
70	10	30	0.8	0.9
40	40	40	4.6	14.3
100	10	10	0.5	0.6
10	10	100	3.1	9.8
10	100	10	28.0	121.1
50	30	40	7.9	6.8
40	100	60	32.4	208.5
80	40	100	10.2	37.4
60	60	100	13.3	114.5

Table 1: NEPOLY and MATLAB QP

Notice that NEPOLY solves all but one of these instances more quickly than the MATLAB code. On the bigger problems, NEPOLY is much quicker that QP. These results are averaged over 10 randomly generated problems of the given size. The times vary slightly for different random problems of the same dimension, but the main conclusion is that NEPOLY outperforms MATLAB QP.

In Table 2, we present similar results comparing NEPOLY with a standard Lemke code. As outlined above, NEPOLY is easily adapted to generate the Lemke path as a special case. In order to carry out this comparison, we reformulate (2.42) as the following quadratic program:

$$\begin{array}{ll} \min & \frac{1}{2}x^TQx + c^Tx + \frac{1}{2}(z-e\xi)^T(z-e\xi) \\ & Ax + B(z-e\xi) \geq b, \\ \text{subject to} & e^T(Ax + B(z-e\xi)) \leq e^Tb, \\ & x, z, \xi \geq 0. \end{array}$$

The necessary optimality conditions for this problem give rise to a standard form LCP to which Lemke's method can then be applied. Table 2 reports the iteration count and elapsed time for problems of various sizes. In all cases, the problems were solved to high accuracy (constraint errors less than 10^{-14}).

Notice on some of the problems, one or other of the codes failed (denoted by F in the table). This is because for these experiments, Q was generated sparse and symmetric but not positive definite. The convergence theory does not guarantee finding a solution in these case, but note that the number of failures are small for NEPOLY . The number of failures can be made large by testing problems with large n since the failures are entirely due to the indefiniteness of Q. However, it is easy to infer that NEPOLY is significantly quicker than the standard Lemke code.

			NEPOLY		Lemke	
m	n	p	iter	time	iter	time
10	10	10	8	0.3	46	2.6
10	10	10	9	0.3	69	3.5
20	10	5	0	0.1	64	4.0
10	14	24	9	0.5	75	7.7
13	26	10	37	2.4	80	11.3
13	26	10	29	2.3	114	16.5
13	26	10	18	2.1	F	
20	40	20	32	4.6	126	46.5
20	40	20	23	2.8	173	62.6
10	50	30	F		F	
30	30	30	20	2.1	168	60.8
50	30	40	10	8.1	196	109.6
10	50	70	F		F	
40	70	50	40	13.5	298	471.2
40	100	60	55	33.8	323	1199.5
80	40	100	29	21.9	349	860.7

Table 2: NEPOLY and Lemke code

Chapter 3

Lineality Space

The pivotal method for solving the normal equation

$$A_C(x) = a$$

described in the last chapter depends on a non–singularity property of A with respect to the lineality of C. In this chapter, we prove that if A is copositive–plus, we can remove the lineality space in the absence of such non–singularity assumption. For convenience of terminology, we refer to $A_C(x) = a$ as a copositive–plus normal equation when the matrix A is copositive–plus.

Recall that the property of being copositive–plus is defined with respect to a cone. The bigger the cone, the stronger is the assumption of being copositive–plus. For example, a matrix is positive semi–definite when it is copositive–plus with respect to $\mathbb{R}^n$, on the other hand, any matrix in $\mathbb{R}^{n\times n}$ is copositive–plus with respect to $\{0\}$. The analysis of this chapter requires that the matrix A be copositive–plus with respect to a cone K with non–empty interior. In the context of a normal equation $A_C(x) = a$, we assume that $K \supset \text{rec}C$. When $\text{int rec}C \neq \emptyset$, the assumption that A is copositive–plus on $\text{rec}C$ will suffice.

3.1 Basic Techniques

We introduce a standard form of the normal equation $A_C(x) = a$ by using a reduction procedure similar to the one described in the paper [43, Proposition 4.1]. For easy reference, we summarize the relevant results from [43, Proposition 4.1] as follows.

Lemma 3.1 *Let C be a nonempty polyhedral convex set in $\mathbb{R}^n$ and A be a linear map. Let $\bar{C} = C \cap (lin\ C)^{\perp}$ so that $C = lin\ C + \bar{C}$. Let $\{e_1, e_2, \cdots, e_j\}$ be a basis of lin C, and $\{e_{j+1}, e_{j+2}, \cdots, e_n\}$ be a basis of $(lin\ C)^{\perp}$, and let A be the matrix that represent the linear map A with respect to this basis. Let $\alpha = \{1, 2, \cdots, j\}$, $\beta = \{j+1, j+2, \cdots, n\}$ and*

$$A = \begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{pmatrix}$$

Assume that $A_{\alpha\alpha}$ is non–singular. Let $A/A_{\alpha\alpha}$ be the Schur complement of $A_{\alpha\alpha}$ in A, i.e.

$$A/A_{\alpha\alpha} = A_{\beta\beta} - A_{\beta\alpha}A_{\alpha\alpha}^{-1}A_{\alpha\beta}$$

and let

$$\bar{a} = a_{\beta} - A_{\beta\alpha}A_{\alpha\alpha}^{-1}a_{\alpha}$$

Then the normal equation

$$A_C(x) = a$$

is equivalent to

$$(A/A_{\alpha\alpha})_{\bar{C}}(x) = \bar{a}$$

in the sense that for any $\bar{x} = (\bar{x}_{\alpha}, \bar{x}_{\beta})$ satisfying the former, $\bar{x}_{\beta}$ satisfies the latter, and for any $\bar{x}_{\beta}$ that solves the latter, there exists an $\bar{x}_{\alpha}$ such that $(\bar{x}_{\alpha}, \bar{x}_{\beta})$ solves the former.

Proof By selection of basis, we have

$$C = \mathbb{R}^j \times \bar{C}$$

If $\bar{x} = (\bar{x}_\alpha, \bar{x}_\beta)$ solves $A_C(x) = a$, then $\bar{x}_\beta \in \bar{C}$ and for any $y_\beta \in \bar{C}$,

$$\begin{array}{rcl} A_{\alpha\alpha}\bar{x}_\alpha + A_{\alpha\beta}\bar{x}_\beta - a_\alpha & = & 0 \\ \langle A_{\beta\alpha}\bar{x}_\alpha + A_{\beta\beta}\bar{x}_\beta - a_\beta, y_\beta - \bar{x}_\beta \rangle & \geq & 0 \end{array}$$

Therefore we have, from the first equation,

$$\bar{x}_\alpha = -A_{\alpha\alpha}^{-1}(A_{\alpha\beta}\bar{x}_\beta - a_\alpha)$$

and, applying this to the second equation

$$\langle (A/A_{\alpha\alpha})\bar{x}_\beta - \bar{a}, y_\beta - \bar{x}_\beta \rangle \geq 0$$

for any $y_\beta \in \bar{C}$. It follows that x_β solves $(A/A_{\alpha\alpha})_{\bar{C}}(x) = \bar{a}$.

Conversely, if $\bar{x}_\beta$ solves $(A/A_{\alpha\alpha})_{\bar{C}}(x) = \bar{a}$, then

$$\bar{x} = (-A_{\alpha\alpha}^{-1}(A_{\alpha\beta}\bar{x}_\beta - a_\alpha), \bar{x}_\beta) \in \mathbb{R}^j \times \bar{C} = C$$

it is now easy to verify that x solves $A_C(x) = a$. **Q.E.D.**

This type of reduction can be carried out with respect to any principal submatrix of $A_{\alpha\alpha}$. In particular, we have the following corollary.

Corollary 3.2 *Let C be a nonempty polyhedral convex set in $\mathbb{R}^n$ in the form of $C = \mathbb{R}^j \times \bar{C}$ with lin $\bar{C} = \{0\}$. Let A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, and A_C be the corresponding normal map. Let $\kappa \subset \{1, 2, \cdots, j\}$ and $A_{\kappa\kappa}$ non–singular and $C(\kappa) = C \cap \{e_l\}_{l\in\kappa}^\perp$. Denote*

$$\bar{\kappa} \;=\; \{1, 2, \cdots, n\} \setminus \kappa$$

let

$$(A/A_{\kappa\kappa}) = A_{\bar{\kappa}\bar{\kappa}} - A_{\bar{\kappa}\kappa}A_{\kappa\kappa}^{-1}A_{\kappa\bar{\kappa}}$$

and

$$a(\kappa) = a_{\bar{\kappa}} - A_{\bar{\kappa}\kappa}a_{\kappa}$$

Then

$$A_C(x) = a$$

is equivalent to

$$(A/A_{\kappa\kappa})_{C(\kappa)}(x) = a(\kappa)$$

Proof Similar to that of Lemma 3.1. **Q.E.D.**

It is a crucial fact that the property of copositive–plus is invariant under such reductions. In fact, we have the following theorem.

Theorem 3.3 *Let C be a nonempty polyhedral convex set in $\mathbb{R}^n$ and A a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume that the matrix A is copositive–plus with respect to rec$C$, where $C = \mathbb{R}^j \times \bar{C}$ with lin $\bar{C} = \{0\}$. Let $\kappa \subset \{1, 2, \cdots, j\}$ and $A_{\kappa\kappa}$ be non–singular and let $C(\kappa) = C \cap \{e_l\}_{l\in\kappa}^{\perp}$,*

$$\bar{\kappa} \quad = \quad \{1, 2, \cdots, n\} \setminus \kappa$$

Then

$$(A/A_{\kappa\kappa}) = A_{\bar{\kappa}\bar{\kappa}} - A_{\bar{\kappa}\kappa}A_{\kappa\kappa}^{-1}A_{\kappa\bar{\kappa}}$$

is copositive–plus over rec$C(\kappa)$.

Furthermore, if K is any cone containing rec$C$, $K(\kappa) = K \cap \{e_l\}_{l\in\kappa}^{\perp}$, and A is copositive–plus with respect to K, then $A/A_{\kappa\kappa}$ is copositive–plus with respect to $K(\kappa)$.

Proof For any $z \in \mathrm{rec}C(\kappa)$

$$\begin{aligned}
& z^T(A/A_{\kappa\kappa})z \\
= \;& z^T(A_{\bar\kappa\bar\kappa} - A_{\bar\kappa\kappa}A_{\kappa\kappa}^{-1}A_{\kappa\bar\kappa})z \\
= \;& \begin{pmatrix} w^T & z^T \end{pmatrix} \begin{pmatrix} A_{\kappa\kappa} & A_{\kappa\bar\kappa} \\ A_{\bar\kappa\kappa} & A_{\bar\kappa\bar\kappa} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}
\end{aligned}$$

where $w = -A_{\kappa\kappa}^{-1}A_{\kappa\bar\kappa}z$. By our assumption on the structure of C, we have

$$C = \mathbb{R}^{|\kappa|} \times C(\kappa)$$

It follows from $z \in \mathrm{rec}C(\kappa)$ that $(w, z) \in \mathrm{rec}C$. Therefore, by assuming A copositive–plus with respect to $\mathrm{rec}C$, we have

$$z^T(A/A_{\kappa\kappa})z = \begin{pmatrix} w^T & z^T \end{pmatrix} \begin{pmatrix} A_{\kappa\kappa} & A_{\kappa\bar\kappa} \\ A_{\bar\kappa\kappa} & A_{\bar\kappa\bar\kappa} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \geq 0$$

For any $z \in \mathrm{rec}C(\kappa)$ such that

$$z^T(A/A_{\kappa\kappa})z = 0$$

we have

$$\begin{pmatrix} w^T & z^T \end{pmatrix} \begin{pmatrix} A_{\kappa\kappa} & A_{\kappa\bar\kappa} \\ A_{\bar\kappa\kappa} & A_{\bar\kappa\bar\kappa} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = 0$$

where $w = -A_{\kappa\kappa}^{-1}A_{\kappa\bar\kappa}z$. Hence

$$\begin{pmatrix} A_{\kappa\kappa} & A_{\kappa\bar\kappa} \\ A_{\bar\kappa\kappa} & A_{\bar\kappa\bar\kappa} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} A_{\kappa\kappa} & A_{\kappa\bar\kappa} \\ A_{\bar\kappa\kappa} & A_{\bar\kappa\bar\kappa} \end{pmatrix}^T \begin{pmatrix} w \\ z \end{pmatrix} = 0 \tag{3.1}$$

due to A is copositive–plus with respect to $\mathrm{rec}C$. In particular

$$\begin{aligned}
A_{\kappa\kappa}w + A_{\kappa\bar\kappa}z &= 0 \\
A_{\kappa\kappa}^T w + A_{\bar\kappa\kappa}^T z &= 0 \\
A_{\bar\kappa\kappa}w + A_{\bar\kappa\bar\kappa}z + A_{\kappa\bar\kappa}^T w + A_{\bar\kappa\bar\kappa}^T z &= 0
\end{aligned}$$

where the first equation is due to the definition of w, the second equation follows from the first and (3.1). By using the first two equations on the third

$$(A_{\bar{\kappa}\bar{\kappa}} - A_{\bar{\kappa}\kappa}A_{\kappa\kappa}^{-1}A_{\kappa\bar{\kappa}})z + (A_{\bar{\kappa}\bar{\kappa}} - A_{\bar{\kappa}\kappa}A_{\kappa\kappa}^{-1}A_{\kappa\bar{\kappa}})^T z = 0$$

That is

$$(A/A_{\kappa\kappa})z + (A/A_{\kappa\kappa})^T z = 0$$

Thus $(A/A_{\kappa\kappa})$ is copositive–plus with respect to rec$C(\kappa)$.

The conclusion regarding K and $K(\kappa)$ can be proven in a similar way. **Q.E.D.**

3.2 Copositive–plus Normal Equations

In this section we show that the invertibility assumption over the lineality space is unnecessary in the case that A is copositive–plus with respect to a cone $K \supset \text{rec}C$ with int$K \neq \emptyset$. The proof of this result requires two separate reductions which we give as Lemma 3.6 and Lemma 3.7, which lead to the results in Theorem 3.8. The main theorem follows as Theorem 3.9. We first state some technical results.

Lemma 3.4 ([37, Result 1.6]) *Let M be a positive semi-definite matrix, and assume*

$$M = \begin{pmatrix} 0 & u^T \\ 0 & M' \end{pmatrix}$$

then $u = 0$.

Consequently, we have the following corollary.

Corollary 3.5 *Let M be an $n \times n$ positive semi-definite matrix, and let*

$$\gamma \subset \{1, 2, \cdots, n\}$$

Assume $M_{\cdot\gamma} = 0$, then $M_{\gamma\cdot} = 0$.

Proof Apply the previous Lemma to each index of γ. **Q.E.D.**

Lemma 3.6 describes the first of our reductions. Essentially we make a change of variables over the lin C which transforms the submatrix $A_{\alpha\alpha}$, which corresponds to the lineality space, into a matrix of the form

$$\begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix}$$

where D_0 is a positive definite matrix. This form will be exploited in Lemma 3.7.

Lemma 3.6 *Given a normal equation $A_C(x) = a$, where A and C are as in Lemma 3.1. Then, there exists a linear transformation*

$$x = Uy$$

such that the restriction of U to $L^\perp = (lin\, C)^\perp$ is the identity. This transformation maps C onto itself. Let $\bar{A}_C(y) = \bar{a}$ be the representation of $A_C(x) = a$ in the variable y. Then, we can choose U such that $\bar{A}$ is in the form

$$\bar{A} = \begin{pmatrix} D & \bar{A}_{\alpha\beta} \\ \bar{A}^T_{\beta\alpha} & \bar{A}_{\beta\beta} \end{pmatrix}, \qquad \bar{a} = U^T a \tag{3.2}$$

where D is given by

$$D = \begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.3}$$

with D_0 being a positive definite matrix. Furthermore, if A is copositive–plus with respect to a cone K containing rec$C$, then $\bar{A}$ is copositive–plus with respect to K.

Proof Since A is copositive with respect to rec$C = \mathbb{R}^{|\alpha|} \times \bar{C}$

$$x_\alpha^T A_{\alpha\alpha} x_\alpha = \begin{pmatrix} x_\alpha^T & 0 \end{pmatrix} \begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{pmatrix} \begin{pmatrix} x_\alpha^T \\ 0 \end{pmatrix} \geq 0$$

for all $x_\alpha \in \mathbb{R}^{|\alpha|}$. That is, $A_{\alpha\alpha}$ is positive semi–definite. Consider a QR factorization of $A_{\alpha\alpha}$

$$A_{\alpha\alpha} = Q_{\alpha\alpha} R$$

where

$$R = \begin{pmatrix} R_0 \\ 0 \end{pmatrix}$$

Here, R_0 is an upper triangular matrix whose row rank equals the rank of $A_{\alpha\alpha}$. By orthonormality of $Q_{\alpha\alpha}$,

$$Q_{\alpha\alpha}^T A_{\alpha\alpha} Q_{\alpha\alpha} = D$$

where $D = RQ_{\alpha\alpha}$. Furthermore D is of the form

$$D = \begin{pmatrix} D' \\ 0 \end{pmatrix}$$

and D is positive semi–definite, and

$$\text{rank} D' = \text{rank} R = \text{rank} R_0$$

Thus, D is a matrix in the form of (3.3) due to Corollary 3.5.

Let

$$U = \begin{pmatrix} Q_{\alpha\alpha} & \\ & I \end{pmatrix}$$

then U is orthonormal and the transformation

$$x = Uy$$

maps C onto itself. The linear transformation A is represented by $U^T AU$ with respect to the variable y, and therefore the normal map will be $\bar{A}_C(y)$ as claimed.

The verification that $\bar{A}$ is copositive–plus with respect to K is straight forward.

Q.E.D.

In the following lemma, we reduce the problem resulting from Lemma 3.6 by eliminating the variables associated with the positive definite matrix D_0. The statement of the result is somewhat technical, but this reduction is crucial step for establishing our main result in Theorem 3.9. The proof of Lemma 3.7 relies heavily on Corollary 3.2 and Theorem 3.3.

Lemma 3.7 *Given a normal equation $A_C(x) = a$, where A is as in (3.2) and C is as in Lemma 3.1 and suppose that A is copositive–plus with respect to a cone $K \supset recC$, $intK \neq \emptyset$. Let $\kappa = \{1, 2, \cdots, k\}$ be the set of indices for the submatrix D_0 in (3.3), and $C(\kappa)$ be as in Theorem 3.3. Then, the given normal equation is equivalent to $\bar{A}_{C(\kappa)}(x) = \bar{a}$ where*

$$\bar{A} = \begin{pmatrix} 0 & \bar{A}_{\alpha'\beta} \\ \bar{A}_{\beta\alpha'} & \bar{A}_{\beta\beta} \end{pmatrix} = \begin{pmatrix} 0 & \bar{A}_{\alpha'\beta} \\ -\bar{A}^T_{\alpha'\beta} & \bar{A}_{\beta\beta} \end{pmatrix} \tag{3.4}$$

where $\alpha' = \alpha \setminus \kappa$. Furthermore, $\bar{A}$ is copositive–plus with respect to $K(\kappa)$.

Proof By Corollary 3.2, $A_C(x) = a$ is equivalent to $\bar{A}_{C(\kappa)}(x) = \bar{a}$, where

$$\bar{A} = \begin{pmatrix} 0 & \bar{A}_{\alpha'\beta} \\ \bar{A}_{\beta\alpha'} & \bar{A}_{\beta\beta} \end{pmatrix}$$

Furthermore, it is easy to see that $K(\kappa) \supset \mathrm{rec}C(\kappa)$. For any $x = (x_{\alpha'}, x_\beta) \in K(\kappa)$

$$\begin{pmatrix} x^T_{\alpha'} & x^T_\beta \end{pmatrix} \begin{pmatrix} 0 & \bar{A}_{\alpha'\beta} \\ \bar{A}_{\beta\alpha'} & \bar{A}_{\beta\beta} \end{pmatrix} \begin{pmatrix} x_{\alpha'} & x_\beta \end{pmatrix} \geq 0$$

That is

$$x^T_\beta(\bar{A}^T_{\alpha'\beta} + \bar{A}_{\beta\alpha'})x_{\alpha'} + x^T_\beta \bar{A}_{\beta\beta} x^T_\beta \geq 0 \tag{3.5}$$

for all $x_{\alpha'} \in \mathrm{IR}^{|\alpha'|}$.

If

$$\bar{A}^T_{\alpha'\beta} + \bar{A}_{\beta\alpha'} \neq 0$$

then $\dim\ker(\bar{A}^T_{\alpha'\beta} + \bar{A}_{\beta\alpha'}) < |\beta| = \dim K(\kappa)$. Hence there exists an $\bar{x}_\beta \in K(\kappa)$ such that

$$(\bar{A}^T_{\alpha'\beta} + \bar{A}_{\beta\alpha'})\bar{x}_\beta \neq 0$$

Let

$$x_{\alpha'} = -\lambda(\bar{A}^T_{\alpha'\beta} + \bar{A}_{\beta\alpha'})\bar{x}_\beta$$

then

$$\bar{x}^T_\beta(\bar{A}^T_{\alpha'\beta} + \bar{A}_{\beta\alpha'})x_{\alpha'} + \bar{x}^T_\beta \bar{A}_{\beta\beta}\bar{x}^T_\beta < 0$$

for sufficiently large λ, a contradiction to (3.5). So we have

$$\bar{A}^T_{\alpha'\beta} + \bar{A}_{\beta\alpha'} = 0$$

The last statement follows easily from Theorem 3.3. **Q.E.D.**

The following theorem summarize the outcome of the two reduction steps described in Lemma 3.6 and Lemma 3.7 which lead to a standard form for copositive–plus normal equations.

Theorem 3.8 *Given a normal equation $A_C(x) = a$, where A is copositive–plus with respect to a polyhedral convex cone $K \supset recC$ such that $intK \neq \emptyset$, there is an equivalent normal equation $\bar{A}_{\bar{C}}(x) = \bar{a}$, where $\bar{A}$ is copositive–plus with respect to $rec\bar{C}$. Furthermore*

$$\bar{C} = \left\{ x \,\middle|\, \begin{pmatrix} 0 & \bar{B} \end{pmatrix} \begin{pmatrix} x_\alpha \\ x_\beta \end{pmatrix} \geq \bar{b} \right\}$$

and

$$\bar{A} = \begin{pmatrix} 0 & \bar{A}_{\alpha\beta} \\ -\bar{A}^T_{\alpha\beta} & \bar{A}_{\beta\beta} \end{pmatrix}$$

Proof We can first perform a transformation as given in Lemma 2.4, so that C is in the form

$$C = \left\{ x \,\middle|\, \begin{pmatrix} 0 & B \end{pmatrix} \begin{pmatrix} x_\alpha \\ x_\beta \end{pmatrix} \geq b \right\}$$

The theorem now follows by applying Lemma 3.6 and Lemma 3.7. **Q.E.D.**

Given a normal equation in standard form, we are able to reduce it to one whose feasible set has zero lineality. This is the subject of our main result of this chapter.

Theorem 3.9 *Consider a normal equation $A_C(x) = a$, where A and C is given by*

$$A = \begin{pmatrix} 0 & A_{\alpha\beta} \\ -A_{\alpha\beta}^T & A_{\beta\beta} \end{pmatrix}$$

and

$$C = \left\{ x \,\middle|\, \begin{pmatrix} 0 & B \end{pmatrix} \begin{pmatrix} x_\alpha \\ x_\beta \end{pmatrix} \geq b \right\}$$

Suppose A is copositive–plus with respect to $recC$, let

$$\bar{A} = A_{\beta\beta} \quad \bar{a} = a_\beta$$

and

$$\tilde{C} = \{x_\beta \mid Bx_\beta \geq b, A_{\alpha\beta}x_\beta = a_\alpha\}$$

Then, $A_C(x) = a$ is equivalent to $\bar{A}_{\tilde{C}}(x) = \bar{a}$, in the sense that for any $\bar{x} = (\bar{x}_\alpha, \bar{x}_\beta)$ satisfying $A_C(x) = a$, $\bar{x}_\beta$ satisfies $\bar{A}_{\tilde{C}}(z) = \bar{a}$, and for any $\bar{x}_\beta$ satisfying $\bar{A}_{\tilde{C}}(z) = \bar{a}$, there exists an $\bar{x}_\alpha$ such that $(\bar{x}_\alpha, \bar{x}_\beta)$ satisfies $A_C(x) = a$. Moreover, $\bar{A}$ is copositive–plus with respect to $rec\tilde{C}$.

Proof Let $\bar{C} = \{x_\beta \mid Bx_\beta \geq b\}$. Notice that $\bar{x}_\beta \in \tilde{C}$ satisfies $\bar{A}_{\tilde{C}}(z) = \bar{a}$ if and only if

$$A_{\beta\beta}\bar{x}_\beta - a_\beta \in N_{\tilde{C}}(\bar{x}_\beta)$$

Notice that $\tilde{C} = \bar{C} \cap \{z \mid A_{\alpha\beta}z = b\}$ and by reference to [45, Corollary 23.8.1], we have

$$A_{\beta\beta}\bar{x}_\beta - a_\beta \in N_{\bar{C}}(\bar{x}_\beta) + \mathrm{im}A_{\alpha\beta}^T$$

or

$$A_{\beta\beta}\bar{x}_\beta - a_\beta - A_{\alpha\beta}^T\bar{x}_\alpha \in N_{\bar{C}}(\bar{x}_\beta) \tag{3.6}$$

for some $\bar{x}_\alpha$. Hence $\bar{x}_\beta$, together with $\bar{x}_\alpha$, satisfies

$$\begin{aligned}
&\bar{x}_\beta \in \{x_\beta \mid Bx_\beta \geq b\} \\
&A_{\alpha\beta}\bar{x}_\beta - a_\alpha = 0 \\
&A_{\beta\beta}\bar{x}_\beta - a_\beta - A_{\alpha\beta}^T\bar{x}_\alpha \in N_{\bar{C}}(\bar{x}_\beta)
\end{aligned}$$

that is

$$\begin{aligned}
&(\bar{x}_\alpha, \bar{x}_\beta) \in C \\
&\begin{pmatrix} 0 & A_{\alpha\beta} \\ -A_{\alpha\beta}^T & A_{\beta\beta} \end{pmatrix} \begin{pmatrix} \bar{x}_\alpha \\ \bar{x}_\beta \end{pmatrix} - \begin{pmatrix} a_\alpha \\ a_\beta \end{pmatrix} \in N_{\mathbb{R}^{|\alpha|} \times \bar{C}}(\bar{x}_\alpha, \bar{x}_\beta)
\end{aligned}$$

or

$$A\bar{x} - a \in N_C(x)$$

Therefore $x = (x_\alpha, x_\beta)$ solve $A_C(x) = a$.

It is obvious that $\bar{A}$ is copositive–plus with respect to $\bar{C}$, and $\tilde{C} \subset \bar{C}$. Hence, $\bar{A}$ is copositive–plus with respect to $\tilde{C}$. **Q.E.D.**

Notice that β can be determined easily from a single QR factorization (see Section 2.2). Thus $\bar{A}$ and $\tilde{C}$ can be easily formed. Furthermore, the path following algorithm of Chapter 2 can be used to solve this problem, starting at stage 2. The fact that $\bar{A}$ is copositive–plus with respect to $\tilde{C}$ guarantees that the algorithm will process the normal equation. Given a solution $\bar{x}_\beta$ of $\bar{A}_{\tilde{C}} = \bar{a}$, a solution of $A_C(x) = a$ can be constructed from (3.6), which is equivalent to

$$A_{\beta\beta}\bar{x}_\beta + a_\beta - A_{\alpha\beta}^T\bar{x}_\alpha = B_{\mathcal{A}}^T u, \qquad u \leq 0$$

that is

$$\begin{aligned} A_{\alpha\beta}^T\bar{x}_\alpha + B_{\mathcal{A}}^T u &= A_{\beta\beta}\bar{x}_\beta - a_\beta \\ u &\leq 0 \end{aligned}$$

So, x can be constructed from x_β by solving a linear program.

Theorem 3.9 is actually a variant of the results regarding augmented LCP discussed by Eaves in [14], also see [24].

Chapter 4

Monotonicity and Interior Point Methods

Given a variational inequality $\mathrm{VI}(F, C)$ where F is a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, and C be a non–empty closed convex set in $\mathbb{R}^n$. We say that $\mathrm{VI}(F, C)$ is monotone if F is a monotone mapping, that is

$$\langle F(x_2) - F(x_1), x_2 - x_1 \rangle \geq 0$$

for any $x_1, x_2 \in \mathbb{R}^n$. In particular, the affine variational inequality $\mathrm{AVI}(q, M, X)$, where M is an $n \times n$ matrix and X is a polyhedral set in $\mathbb{R}^n$, is monotone if M is positive semi-definite. In this chapter, we investigate monotone affine variational inequalities from the perspective of maximal monotone multifunction theory.

We begin with a few basic concepts from the theory of monotone multifunctions. A multifunction from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a subset of $\mathbb{R}^n \times \mathbb{R}^m$. For any $T \subset \mathbb{R}^n \times \mathbb{R}^m$ and $x \in \mathbb{R}^n$ we define

$$T(x) := \{y \in \mathbb{R}^m | (x, y) \in T\}$$

and

$$T^{-1}(y) := \{x \in \mathbb{R}^n | (x, y) \in T\}$$

In particular, $T^{-1}(0)$ is called the zero set of T.

A multifunction, T, from $\mathbb{R}^n$ to $\mathbb{R}^n$ is said to be monotone if for each pair $(x_1, y_1), (x_2, y_2)$ of points in T,

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$$

T is said to be maximal if it is not properly contained in any other monotone multifunction. Also, T is said to be affine if T is an affine subset of $\mathbb{R}^n$ to $\mathbb{R}^n$.

For each monotone multifunction T and $\lambda > 0$ define

$$J_\lambda = (I + \lambda T)^{-1}$$

and J_λ is called the resolvent of T. The following theorem due to Minty characterize a maximal monotone multifunction in terms of its resolvent J_λ.

Theorem 4.1 ([35]) *Suppose that T is a monotone multifunction from $\mathbb{R}^n$ to $\mathbb{R}^n$. T is maximal monotone if and only if $dom J_\lambda = \mathbb{R}^n$.*

We refer readers to [2] for a comprehensive treatment of the theory of maximal monotone multifunctions.

Given a monotone multifunction T we can define a complementarity problem of finding (x, y) such that

$$(x, y) \in T, \qquad (x, y) \geq 0 \quad \text{and} \quad x^T y = 0 \tag{4.1}$$

We call it the generalized linear complementarity problem when T is affine and maximal monotone (see [25]).

In Section 4.1, we will show how AVI(q, M, X) is related to a generalized linear complementarity problem and how interior point methods can be used for solving monotone affine variational inequalities. In Section 4.2, we try to address some related computational issues.

4.1 Generalized Linear Complementarity Problem

It is well known (see [27]) that AVI(q, M, X) is equivalent to the following complementarity problem

$$
\begin{array}{l}
(s, x, u) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^m_+ \\
H(s, x, u) = \begin{pmatrix} 0 & -B & 0 \\ B^T & M & A^T \\ 0 & -A & 0 \end{pmatrix} \begin{pmatrix} s \\ x \\ u \end{pmatrix} + \begin{pmatrix} d \\ q \\ b \end{pmatrix} \in \{0\} \times \{0\} \times \mathbb{R}^m_+ \qquad \text{(CP)} \\
(s, x, u)^T H(s, x, u) = 0
\end{array}
$$

Furthermore, this complementarity problem can be put into the framework of (4.1).

Theorem 4.2 *Suppose that M is positive semidefinite. Then, the problem (CP) is a generalized linear complementarity problem with*

$$T = \left\{ (u, v) \in I\!R^m \times I\!R^m \,\middle|\, v = b - Ax, Bx = d, Mx + A^T u + B^T s + q = 0 \right\} \quad (4.2)$$

Proof Obviously T is affine. It suffice to show that T is maximal monotone.

For any $(u_i, v_i) \in T$, $i = 1, 2$,

$$\Delta u = u_2 - u_1, \text{ and } \Delta v = v_2 - v_1$$

and some appropriate Δs and Δx satisfy the following homogeneous equation

$$\begin{pmatrix} B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} = 0 \tag{4.3}$$

Therefore

$$\Delta u^T \Delta v = \Delta x^T M^T \Delta x$$

It follows from the positive semi-definiteness of M that

$$\Delta u^T \Delta v \geq 0 \tag{4.4}$$

which implies that T is a monotone multifunction.

In showing that T is maximal, we may assume without loss of generality that $b = 0$, $d = 0$, and $q = 0$. By Minty's Theorem, it suffices to show that the range of $I + T$ is $\mathbb{R}^m$. Let $z \in \mathbb{R}^m$ be arbitrary. We show the existence of $(u, v) \in T$ such that $z = u + v$. It follows from (4.2) that this is equivalent to the solvability of the system

$$\begin{aligned} u + v &= z \\ Bx &= 0 \\ Ax + v &= 0 \\ Mx + A^T u + B^T s &= 0 \end{aligned}$$

Equivalently the system

$$Bx = 0, \qquad Mx + A^T(Ax + z) + B^T s = 0 \tag{4.5}$$

must be solvable for (x, s). Let $C \in \mathbb{R}^{n \times n-p}$ be a matrix such that $\ker B = \mathrm{im} C$. Letting $x = Ct$, (4.5) reduces to the system

$$(M + A^T A)Ct + A^T z \in \mathrm{im} B^T = \ker C^T$$

or

$$C^T(M + A^T A)Ct + C^T A^T z = 0$$

Since z is arbitrary, we must show

$$\mathrm{im} C^T A^T \subset \mathrm{im} C^T(M + A^T A)C$$

which is in turn equivalent to the statement

$$\ker(C^T(M^T + A^T A)C) \subset \ker(AC)$$

To prove the last statement, assume that $C^T(M^T + A^T A)Cw = 0$. Then,

$$w^T C^T(M^T + A^T A)Cw = 0$$

or

$$w^T C^T(M^T)Cw + \|ACw\|_2^2 = 0$$

But M is positive semi–definite, hence $w^T C^T(M^T)Cw = 0$ and $ACw = 0$. The claim is proved. **Q.E.D.**

Now that we know that T is maximal monotone, the following result ([25, Corollary 2.1]) illustrates the connection between (CP) and a class of horizontal LCP as defined in [48].

Theorem 4.3 *Let T be an affine multifunction on $\mathbb{R}^m$, T is maximal monotone if and only if there exist matrices H_1, $H_2 \in \mathbb{R}^{m \times m}$ and $a \in \mathbb{R}^m$ such that the pair*

H_1 and H_2 is column monotone, i.e. $H_1 + H_2 = I$, $H_1^T H_2$ is positive semi–definite, and

$$T = \{(u, v) \mid H_1 u - H_2 v = a\}$$

With T represented as in Theorem 4.3, (CP) is equivalent to the following horizontal LCP (see [48])

$$H_1 u - H_2 v = a, \quad u, v \geq 0, \quad u^T v = 0 \tag{4.6}$$

The pair H_1 and H_2 is column monotone due to the maximality of T, therefore following theorem ([48, Theorem 7]) applies.

Theorem 4.4 *Given H_1 and H_2 column monotone, then, for any $a \in I\!R^m$ (4.6) is equivalent to LCP($C^{-1}D, C^{-1}a$), where C and D are column representatives (see [48]) of H_1 and H_2 and $C^{-1}D$ is positive semi–definite.*

Since (CP) is equivalent to a standard monotone LCP, interior point algorithms, e.g., the path following algorithm in [29], the potential reduction algorithm in [30], and the infeasible path following algorithm in [50] and [3], can be applied to provide polynomial algorithms for (CP) and hence for (AVI). In the next section, we show that the path following and the potential reduction algorithms can be carried out without specifically reducing (CP) to a monotone LCP.

4.2 Interior Point Algorithms

Section 4.1 shows that (CP) is equivalent to a standard LCP. However, directly reducing (CP) to a standard LCP using the method outlined in the last section will not provide a practical algorithm. We now show how to exploit the structure of the problem (CP) in applying the path following and potential reduction algorithms.

We assume that all elements of the matrix

$$Q = \begin{pmatrix} M & q \\ A & b \\ B & d \end{pmatrix}$$

are integers. The size of the problem (AVI) is defined by

$$L = 1 + \log(m+n+p)^2 + \lfloor \sum_{i=1}^{m+n+p} \sum_{j=1}^{n+1} \log(1 + |q_{ij}|) \rfloor$$

where q_{ij}'s are element of the matrix Q.

To solve (CP) using path following method, we begin with an initial point (s^0, u^0, v^0, x^0) which is close to the central path, that is, a point in the set

$$S^\alpha := \left\{ (s,u,v,x) \in S \;\middle|\; u, v > 0, \|UVe - \zeta e\|_2 \leq \alpha\zeta, \text{ where } \zeta = \frac{1}{m} u^T v \right\} \quad (4.7)$$

At each step, Newton's step for the nonlinear equation

$$\begin{aligned} & F(s,u,v,x,\mu) && (4.8) \\ = \; & (UV - \mu e, Mx + q + B^T s + A^T u, v + Ax - b, Bx - d) && (4.9) \\ = \; & 0 && (4.10) \end{aligned}$$

is used to compute a new point in S^α such that ζ is reduced from the previous value by a constant factor. The algorithm terminates when ζ is sufficiently small.

Given a point $(s^0, u^0, v^0, x^0) \in S^\alpha$, here is the algorithm:

Algorithm 3

1. Choose $0 < \alpha \leq \frac{1}{10}$, let $\delta = \frac{\alpha}{1-\alpha}$, and let $k = 0$.

2. If $u^{kT} v^k < 2^{-4L}$, then stop.

3. Let

$$\begin{aligned} \zeta &= u^{kT} v^k / m \\ \mu &= (1 - \delta / m^{\frac{1}{2}}) \zeta \\ (s, u, v, x) &= (s^k, u^k, v^k, x^k) \end{aligned}$$

4. Compute $(\Delta s, \Delta u, \Delta v, \Delta x)$ by constructing a Newton step for the nonlinear equation (4.8), that is, solving

$$\begin{pmatrix} 0 & V & U & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} = \begin{pmatrix} UVe - \mu e \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.11}$$

   and set

$$(s^{k+1}, u^{k+1}, v^{k+1}, x^{k+1}) = (s, u, v, x) - (\Delta s, \Delta u, \Delta v, \Delta x)$$

5. Set $k = k + 1$, and go to Step 2.

There are two crucial issues concerning the validity of the algorithm, one is the solvability of (4.11), and the other is the justification that each new iterate stays in S^α and that ζ is reduced. In fact, by the analysis from the last section, (4.11) is equivalent to

$$H_1\Delta u - H_2\Delta v = 0, \qquad V\Delta u + U\Delta v = UVe - \mu e \tag{4.12}$$

Hence, $(\Delta u, \Delta v)$ is uniquely solvable from (4.11) by the maximality of T (see [25, Theorem 2.1]). Furthermore, in view of Theorem 4.4, the step computed from (4.11) is the same as the interior step used by Kojima et.al. in [29] for LCP$(C^{-1}D, C^{-1}a)$. Therefore we have the following theorem.

Theorem 4.5 *Let $(s, u, v, x) \in S$ with $u,\ v > 0$ satisfy*

$$\|UVe - \zeta e\|_2 \leq \alpha\zeta \quad \textit{with} \quad \zeta = \frac{1}{m}u^T v$$

for $\alpha \in (0, \frac{1}{10})$. Let

$$\mu \ = \ (1 - \delta/m^{\frac{1}{2}})\zeta$$

Suppose $(\Delta s, \Delta u, \Delta v, \Delta x)$ is a solution of (4.11), and

$$(\bar{s}, \bar{u}, \bar{v}, \bar{x}) = (s, u, v, x) - (\Delta s, \Delta u, \Delta v, \Delta x)$$

Then, $(\bar{u}, \bar{v}) > 0$, and

$$\begin{aligned} \left\|\bar{U}\bar{V}e - \bar{\zeta}e\right\|_2 \ &\leq \ \alpha\bar{\zeta} \\ \bar{\zeta} = \frac{1}{m}\bar{u}^T\bar{v} \ &\leq \ (1 - \frac{\delta}{6m^{\frac{1}{2}}})\zeta \end{aligned}$$

As a result of this theorem, **Algorithm 3** stops in $O(m^{\frac{1}{2}}L)$ iterations, each of which requires $O((m+n+p)^3)$ operations to compute a new point. Therefore, the number of arithmetic operations needed for finding a point $\{(s^k, u^k, v^k, x^k)\}$ such that $u^{kT}v^k < 2^{-4L}$ is no more than $O(m^{\frac{1}{2}}(m+n+p)^3L)$. Furthermore, an exact solution of AVI(q, M, X) can be constructed from such a point in no more than $O((m+n+p)^3)$ arithmetic operations by using a technique similar to that of [29].

Potential reduction algorithms start with a point in

$$S^0 := \{(s, u, v, x) \in S \mid u, v > 0\}$$

such that $f(u, v)$ does not exceed $O(m^{\frac{1}{2}}L)$, where the potential function f is defined by

$$f(u, v) = \sqrt{m} \log u^T v - \sum_{i=1}^{m} log(u_i v_i) - m \log m \quad \text{for } (s, u, v, x) \in S^0 \quad (4.13)$$

The algorithm is as follows.

Algorithm 4

1. Choose $(s^0, u^0, v^0, x^0) \in S^0$, such that $f(u, v)$ does not exceed $O(m^{\frac{1}{2}}L)$, and let $k = 0$.

2. Let $(s, u, v, x) = (s^k, u^k, v^k, x^k)$, if $f(u^k, v^k) < -4m^{\frac{1}{2}}L$, then stop.

3. Let

$$\begin{aligned} w &= (\sqrt{u_1 v_1}, \sqrt{u_2 v_2}, \cdots, \sqrt{u_m v_m}) \\ W &= \operatorname{diag}\{w\} \\ z &= W^{-1}e - ((n + \sqrt{n})/ \|w\|_2^2)w \end{aligned}$$

4. Compute $(\Delta s, \Delta u, \Delta v, \Delta x)$ by constructing a Newton step for the function F, that is, solving

$$\begin{aligned} \begin{pmatrix} B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} &= 0 \\ W^{-1}(U\Delta v + V\Delta u) &= \frac{z}{\|z\|_2} \end{aligned} \tag{4.14}$$

   and set

$$(s^{k+1}, u^{k+1}, v^{k+1}, x^{k+1}) = (s, u, v, x) - (\Delta s, \Delta u, \Delta v, \Delta x)$$

5. Set $k = k + 1$, and go to Step 2.

We notice that the system (4.14) is equivalent to

$$H_1 \Delta u - H_2 \Delta v = 0, \qquad U \Delta v + V \Delta u = W \frac{z}{\|z\|_2} \tag{4.15}$$

A reference to Theorem 4.4 and [30, Theorem 2.2] leads to the following result.

Theorem 4.6 $(\Delta u, \Delta v)$ *is uniquely determined by (4.14) and at each iteration we have*

$$f(u^{k+1}, v^{k+1}) < f(u^k, v^k) - 0.2$$

Similar to case of **Algorithm 3**, Theorem 4.6 guarantees that the number of arithmetic operations needed by the potential reduction algorithm for finding a solution of AVI(q, M, X) is bounded by $O(m^{\frac{1}{2}}(m+n+p)^3 L)$.

4.3 An Implementation Issue

Although $(\Delta u, \Delta v)$ can be uniquely determined from the system (4.12) or (4.15), in practice we are dealing with (4.11) or (4.14). The task of of computing $(\Delta u, \Delta v)$ can be significantly simplified if solution to each of these systems is unique. Our next lemma shows that the following assumption

$$\operatorname{rank} \begin{pmatrix} 0 & -B \\ B^T & M \\ 0 & -A \end{pmatrix} = n + p \tag{4.16}$$

guarantees the uniqueness of solution for (4.11) and (4.14). The general case will be dealt with in the rest of this section.

Lemma 4.7 *Suppose that the condition (4.16) holds. Then, for any positive diagonal matrices D_1, D_2, and $r \in \mathbb{R}^m$, the equation*

$$\begin{pmatrix} 0 & D_1 & D_2 & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has a unique solution.

Proof It suffices to show that the homogeneous system

$$\begin{pmatrix} 0 & D_1 & D_2 & 0 \\ B^T & A^T & 0 & M \\ 0 & 0 & I & A \\ 0 & 0 & 0 & B \end{pmatrix} \begin{pmatrix} \Delta s \\ \Delta u \\ \Delta v \\ \Delta x \end{pmatrix} = 0 \tag{4.17}$$

has a unique solution.

Suppose $(\Delta s, \Delta u, \Delta v, \Delta x)$ is a solution, then

$$D_1 \Delta u + D_2 \Delta v = 0$$

hence

$$D \Delta u + D^{-1} \Delta v = 0$$

where $D = (D_1 D_2^{-1})^{\frac{1}{2}}$. Therefore

$$\|D\Delta u\|_2^2 + 2(D\Delta u)^T (D^{-1}\Delta v) + \left\|D^{-1}\Delta v\right\|_2^2 = 0$$

Notice that $(D\Delta u)^T(D^{-1}\Delta v) = \Delta u^T \Delta v \geq 0$ as a result of (4.4), so we have

$$\|D\Delta u\|_2 = 0, \quad \|D^{-1}\Delta v\|_2 = 0$$

It follows that

$$\Delta u = 0, \quad \Delta v = 0$$

Consequently, $\Delta s = 0$ and $\Delta x = 0$ since rank $\begin{pmatrix} 0 & -B \\ B^T & M \\ 0 & -A \end{pmatrix} = n + p.$ **Q.E.D.**

In general, a problem in the form of (CP) can be reduced to a smaller problem satisfying (4.16). Define the feasible set of (CP) by

$$S := \left\{ (u, v) \,\middle|\, u, v \geq 0, v = Ax - b, Bx - d = 0, Mx + A^T u + B^T s + q = 0 \right\} \tag{4.18}$$

Then the lineality space (see [45]) of S is

$$L(S) = \left\{ (s, 0, 0, x) \,\middle|\, B^T s + Mx = 0, -Ax = 0, -Bx = 0 \right\}$$

So, $L(S) = \{0\}$ if and only if (4.16) holds.

For convenience of notation, define

$$Q = \begin{pmatrix} 0 & -B \\ B^T & M \end{pmatrix} \qquad C = \begin{pmatrix} 0 & A \end{pmatrix}$$

(CP) can be reformulated as

$$\begin{aligned} & (z, u) \in \mathbb{R}^{p+n} \times \mathbb{R}^m_+ \\ & H(z, u) = \begin{pmatrix} Q & C^T \\ -C & 0 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix} + \begin{pmatrix} q' \\ b \end{pmatrix} \in \{0\} \times \mathbb{R}^m_+ \\ & (z, u)^T H(z, u) = 0 \end{aligned} \tag{CP$'$}$$

where $z = \begin{pmatrix} s \\ x \end{pmatrix}$ and $q' = \begin{pmatrix} d \\ q \end{pmatrix}$.

Suppose $L(S) \neq \{0\}$, then the columns of the matrix $\begin{pmatrix} Q \\ -C \end{pmatrix}$ are linearly dependent. There exist index sets α and β such that

$$\begin{pmatrix} Q \\ -C \end{pmatrix} = \begin{pmatrix} Q_{\cdot\alpha} & Q_{\cdot\beta} \\ -C_{\cdot\alpha} & -C_{\cdot\beta} \end{pmatrix}$$

and $\begin{pmatrix} Q_{\cdot\alpha} \\ -C_{\cdot\alpha} \end{pmatrix}$ is a maximum subset of linearly independent columns of the matrix $\begin{pmatrix} Q \\ -C \end{pmatrix}$. Thus

$$\begin{pmatrix} Q_{\cdot\beta} \\ -C_{\cdot\beta} \end{pmatrix} = \begin{pmatrix} Q_{\cdot\alpha} \\ -C_{\cdot\alpha} \end{pmatrix} P \tag{4.19}$$

for some $|\alpha| \times |\beta|$ matrix P.

The following lemma will be useful as a technical tool.

Lemma 4.8 *Let M be an $n \times n$ positive semi-definite matrix, and γ, α, β be a partition of $\{1, 2, \cdots, n\}$, so that*

$$M = \begin{pmatrix} M_{\cdot\gamma} & M_{\cdot\alpha} & M_{\cdot\beta} \end{pmatrix}$$

Assume that

$$M_{\cdot\gamma} = M_{\cdot\alpha} P$$

for some $|\alpha| \times |\gamma|$ matrix P, then

$$M_{\gamma\cdot} = P^T M_{\alpha\cdot}$$

Proof

$$\begin{pmatrix} I & -P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} M \begin{pmatrix} I & 0 & 0 \\ -P & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & -P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & M_{\cdot\alpha} & M_{\cdot\beta} \end{pmatrix}$$

$$= \begin{pmatrix} I & -P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & M_{\gamma\alpha} & M_{\gamma\beta} \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & * & * \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix}$$

where the last equality follows from Lemma 3.4. It now follows that

$$\begin{aligned} M &= \begin{pmatrix} I & P^T & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\ 0 & M_{\beta\alpha} & M_{\beta\beta} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ P & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} P^T M_{\alpha\alpha} P & P^T M_{\alpha\alpha} & P^T M_{\alpha\beta} \\ M_{\alpha\alpha} P & M_{\alpha\alpha} & M_{\alpha\beta} \\ M_{\alpha\beta} P & M_{\alpha\beta} & M_{\beta\beta} \end{pmatrix} \end{aligned}$$

therefore

$$\begin{aligned} M_{\gamma\cdot} &= \begin{pmatrix} P^T M_{\alpha\alpha} P & P^T M_{\alpha\alpha} & P^T M_{\alpha\beta} \end{pmatrix} \\ &= P^T \begin{pmatrix} M_{\alpha\alpha} P & M_{\alpha\alpha} & M_{\alpha\beta} \end{pmatrix} \\ &= P^T M_{\alpha\cdot} \end{aligned}$$

Q.E.D.

The method of reducing (CP) to a smaller problem satisfying (4.16) is derived from the following two lemmas.

Lemma 4.9 *Let α, β and P be as in (4.19), $\beta \neq \emptyset$. If (CP$'$) is solvable, then there exists a solution $(\bar{z}, \bar{u})$ such that $\bar{z}_\beta = 0$.*

Proof Let $(\bar{z}, \bar{u}) = (\bar{z}_\alpha, \bar{z}_\beta, \bar{u})$ be a solution of (CP$'$), then it is clear that $(\bar{z}_\alpha + P\bar{z}_\beta, 0, \bar{u})$ is the desired solution. **Q.E.D.**

Lemma 4.10 *Define (CP$''$) by*

$$(w, u) \in I\!R^{p+n-|\beta|} \times I\!R^m_+$$

$$\tilde{H}(w, u) = \begin{pmatrix} Q_{\alpha\alpha} & (C^T)_{\alpha\cdot} \\ -C_{\cdot\alpha} & 0 \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} + \begin{pmatrix} {q'}_\alpha \\ b \end{pmatrix} \in \{0\} \times I\!R^m_+ \qquad (CP'')$$

$$(w, u)^T \tilde{H}(w, u) = 0$$

Then (z, u) is a solution of (CP$'$) with $z_\beta = 0$ if and only if (z_α, u) is a solution of (CP$''$).

Proof If (z, u) is a solution of (CP$'$) with $z_\beta = 0$, then it is easily verified that (z_α, u) is a solution of (CP$''$).

If (z_α, u) is a solution of (CP$''$), then

$$Q_{\alpha\alpha} z_\alpha + (C^T)_{\alpha\cdot} u + {q'}_\alpha \;=\; 0 \qquad (4.20)$$

$$-\, C_{\cdot\alpha} z_\alpha + b \;\in\; \mathbb{R}^m_+ \qquad (4.21)$$

and

$$u^T(-C_{\cdot\alpha} z_\alpha + b) = 0 \qquad (4.22)$$

Moreover, since the matrix $\begin{pmatrix} Q & C^T \\ -C & 0 \end{pmatrix}$ is positive semi-definite, we can apply Lemma 4.8 to (4.19) resulting in

$$\begin{pmatrix} Q_{\beta\alpha} & Q_{\beta\beta} & (C^T)_{\beta\cdot} \end{pmatrix} = P^T \begin{pmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & (C^T)_{\alpha\cdot} \end{pmatrix}$$

Also, taking into account (4.20), we have

$$\begin{aligned} & \begin{pmatrix} Q_{\beta\alpha} & Q_{\beta\beta} & (C^T)_{\beta\cdot} \end{pmatrix} \begin{pmatrix} z_\alpha \\ 0 \\ u \end{pmatrix} + {q'}_\beta \\ = \ & P^T \begin{pmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & (C^T)_{\alpha\cdot} \end{pmatrix} \begin{pmatrix} z_\alpha \\ 0 \\ u \end{pmatrix} + {q'}_\beta \\ = \ & {q'}_\beta - P^T {q'}_\alpha \end{aligned}$$

If ${q'}_\beta - P^T {q'}_\alpha \neq 0$, then the system

$$\begin{pmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & (C^T)_{\alpha\cdot} \\ Q_{\beta\alpha} & Q_{\beta\beta} & (C^T)_{\beta\cdot} \end{pmatrix} \begin{pmatrix} z_\alpha \\ 0 \\ u \end{pmatrix} + \begin{pmatrix} {q'}_\alpha \\ {q'}_\beta \end{pmatrix} = 0$$

is inconsistent, a contradiction to the solvability of (CP$'$) and Lemma 4.9. Hence

$${q'}_\beta - P^T {q'}_\alpha = 0$$

Let $z_0 = (z_\alpha, 0)$, then

$$H(z_0, u) \ = \ \begin{pmatrix} 0 \\ 0 \\ -C_{\cdot\alpha} z_\alpha + b \end{pmatrix} \in \{0\} \times \mathbb{R}^m_+$$

follows from (4.20), (4.21). We also have $(z_0, u^T)H(z_0, u) = 0$ by reference to (4.22). **Q.E.D.**

By definition, we can write

$$Q_{\alpha\alpha} = \begin{pmatrix} 0 & -\bar{B} \\ \bar{B}^T & \bar{M} \end{pmatrix} \qquad C_{\cdot\alpha} = \begin{pmatrix} 0 & \bar{A} \end{pmatrix}$$

for appropriate submatrices $\bar{A}$, $\bar{B}$, and $\bar{M}$ of A, B, and M respectively. Note that $\bar{M}$ is positive semi-definite and the matrix

$$\begin{pmatrix} 0 & -\bar{B} \\ \bar{B}^T & \bar{M} \\ 0 & -\bar{A} \end{pmatrix}$$

has full column rank. Therefore (CP$''$) is equivalent to AVI$(\bar{q}, \bar{M}, \bar{X})$ where

$$\bar{X} = \left\{ y \,\middle|\, \bar{A}y \le \bar{b}, \bar{B}y = \bar{d} \right\}$$

and $\bar{q}$, $\bar{b}$, and $\bar{d}$ are vectors which consist of appropriate components of q, b and d respectively.

The procedure of reducing AVI(q, M, X) to AVI$(\bar{q}, \bar{M}, \bar{X})$ can be carried out by using Gaussian elimination and deleting rows and columns from a matrix. A solution of AVI$(\bar{q}, \bar{M}, \bar{X})$ is found by solving (CP$''$). A solution of (CP), and hence a solution of AVI(q, M, X), can then be constructed from that of (CP$''$) by applying Lemma 4.10. Therefore these operations will not increase the order of complexity.

Chapter 5

Monotone Variational Inequalities

The proximal point algorithm is an iterative method for solving the generalized equation

$$0 \in T(x) \tag{5.1}$$

for $x \in \mathbb{R}^n$, where T is a maximal monotone multifunction on $\mathbb{R}^n$. The iterates are constructed by

$$x_{k+1} = J_{\lambda_k}(x_k) = (I + \lambda_k T)^{-1}(x_k) \tag{5.2}$$

where $J_{\lambda_k} = (I + \lambda_k T)^{-1}$ is called the resolvent of T. The well known Minty's theorem (see Theorem 4.1) guarantees that such a sequence $\{x_k\}$ is well defined given any starting point x_0. Basic convergence results are summarized in the following theorem (cf. [46]).

Theorem 5.1 *Suppose that T is a maximal monotone operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $0 \in imT$. Let $\{x_k\}$ be generated by (PP) using a sequence of positive numbers $\{\lambda_k\}$ such that $\sum_{n=0}^{\infty} \lambda_k^2 = \infty$. Then $\{x_k\}$ converges to a point $\bar{x}$ such that $0 \in T(\bar{x})$.*

In the case of convex programming, the minimization problem

$$\min_{x \in C} f(x) \tag{5.3}$$

where f is a proper closed convex function and C is a closed convex set in $\mathbb{R}^n$, is equivalent to (GE) with

$$T = \partial f + N(\cdot \mid C)$$

The finite termination of (PP) is studied in [18] and [4] in conjunction with the notions of weak sharp minima and minimum principle sufficiency (see [19] and [6]). We proceed to extend some of these results to the case of general maximal monotone multifunction and show how these results can be applied to monotone variational inequalities.

5.1 Finite Termination

We begin with the definition of a sharp zero set for a maximal monotone multifunction. The concept of sharp zero set is crucial in generalizing the finite termination results in [18] to a generalized equation in the form of (5.1).

Definition 5.2 *Given a maximal monotone multifunction T. The set $Z = T^{-1}(0)$ is called a sharp zero of T if $Z \neq \emptyset$ and there exists a $\delta > 0$ such that*

$$\delta B \cap N(z \mid Z) \subset T(z) \quad \forall z \in Z \tag{5.4}$$

The constant δ is called the modulus of sharpness.

As an example, consider the case where $T = \partial f + N(\cdot \mid C)$ [18, Theorem 2] shows that a sharp zero set of $T = \partial f + N(\cdot \mid C)$ is a set of weak sharp minima for the convex program (5.3).

T is assumed to have a sharp zero set throughout the rest of this chapter. Our first step is to establish a result similar to that of [18, Lemma 4] for (5.1).

Lemma 5.3 *Let δ be the modulus of sharpness for T and $0 < \epsilon < \delta$. Suppose*

$$y = \epsilon \frac{z - \pi_Z(z)}{\|z - \pi_Z(z)\|}$$

and $y \in T(w)$ for some $w \in Z$, then $y \in T(\pi_Z(z))$.

Proof By the definition $z - \pi_Z(z) \in N(\pi_Z(z) \mid Z)$, hence $y \in \delta B \cap N(\pi_Z(z) \mid Z)$ since $\epsilon < \delta$. Therefore $y \in T(\pi_Z(z))$ by (5.4). **Q.E.D.**

The next lemma is similar to [18, Lemma 5].

Lemma 5.4 *Let ϵ and δ be as in Lemma 5.3. If $w \in T(z)$ and $\|w\| < \epsilon$, then $z \in Z$.*

Proof If $z \neq \pi_Z(z)$, then

$$y = \epsilon \frac{z - \pi_Z(z)}{\|z - \pi_Z(z)\|} \in T(\pi_Z(z))$$

by Lemma 5.3. It follows from monotonicity of T that

$$0 \leq \langle z - \pi_Z(z), w - y \rangle$$

By definition of y

$$\begin{aligned} & \frac{\epsilon}{\|z - \pi_Z(z)\|} \langle z - \pi_Z(z), z - \pi_Z(z) \rangle \\ = \; & \langle z - \pi_Z(z), y \rangle \\ \leq \; & \langle z - \pi_Z(z), w \rangle \\ \leq \; & \|z - \pi_Z(z)\| \, \|w\| \end{aligned}$$

Consequently, $\|w\| \geq \epsilon$, a contradiction. **Q.E.D.**

Now, we are ready to present the main theorem of this section.

Theorem 5.5 *Suppose Z is sharp zero set of T with modulus δ. Let $\{\lambda_k\}$ be any sequence of positive number which is bounded below and let $x_0 \in I\!R^n$. The (PP) terminates in a finite number of iterations.*

Proof For any $0 < \epsilon < \delta$, we have

$$\epsilon \mathbf{B} \cap N(z \mid Z) \subset T(z)$$

for all $z \in Z$.

Let $\lambda_k \geq \lambda > 0$ for the given sequence. Then, for any $z \in Z$ we know that the sequence $\{\|x_k - z\|\}$ is bounded and hence converges (see [47]). Furthermore

$$\|x_{K+1} - z\|^2 + \sum_{k=0}^{K} \lambda_k^2 \|v_k\|^2 \leq \|x_0 - z\|^2$$

where $v_k \in T(x_k)$ and $x_k + \lambda_{k-1} v_k = x_{k-1}$. Since $\{\|x_k - z\|\}$ is bounded, it follows that

$$\sum_{k=0}^{K} \lambda_k^2 \|v_k\|^2 \leq M$$

for some constant M. Hence

$$\lambda_k^2 \|v_{K+1}\|^2 (K+1) \leq M$$

Therefore, there exists a sufficiently large K such that

$$\|v_{K+1}\|^2 \leq \frac{M}{\lambda^2 (K+1)} < \epsilon^2$$

by the non-increasing property of $\|v_{K+1}\|$ (see [47]). It follows from Lemma 5.4 that x_{K+1} is in the solution set. **Q.E.D.**

Corollary 5.6 *Suppose Z is a sharp zero set of T with modulus δ. Let $\{\lambda_k\}$ be any sequence of positive number which is bounded below by $\lambda > 0$. Then for any given $x_0 \in \mathbb{R}^n$, (PP) terminates in one iteration for a sufficiently large choice of λ.*

Proof See the proof of [18, Theorem 8]. **Q.E.D.**

5.2 An Equivalence Relation

In the studying the connection between weak sharp minima and finite termination of (PP), the equivalence between

$$\alpha\mathbf{B} \cap N(x \mid Z) \subset T(x)$$

and

$$\alpha\mathbf{B} \cap \bigcup_{x \in Z} N(x \mid Z) \subset \bigcup_{x \in Z} T(x)$$

where $T = \partial f + N(\cdot \mid C)$ and $Z = T^{-1}(0)$, plays an important role. We extend it to the case where T is a maximal monotone multifunction with the further assumption that Z is polyhedral. Such a generalization turns out to be useful in establishing the connection between finite termination of (PP) and minimum principle sufficiency for monotone variational inequalities.

Since T is maximal monotone, the set $Z = T^{-1}(0)$ is closed convex. The following result is a direct consequence of the monotonicity of T.

Lemma 5.7 *Suppose T is maximal monotone, $Z = T^{-1}(0)$, then*

$$T(x) \subset N(x \mid Z) \qquad \textit{for all } x \in Z \tag{5.5}$$

Proof Let x be any point in Z, $u \in T(x)$. For all $z \in Z$, we have, $0 \in T(z)$. So, by monotonicity, $\langle 0 - u, z - x \rangle \geq 0$, i.e. $\langle u, z - x \rangle \leq 0$, which implies that $u \in N(x \mid Z)$. **Q.E.D.**

For those points not in Z, their images under T also possess interesting properties.

Lemma 5.8 *For any points $x, y \in I\!R^n$, $0 \leq \lambda \leq 1$, we have*

$$T(x) \cap T(y) \subset T(\lambda x + (1 - \lambda) y) \tag{5.6}$$

In general, for any $x_i \in I\!R^n$, $\lambda_i \geq 0$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \lambda_i = 1$

$$\bigcap_{i=1}^{k} T(x_i) \subset T(\sum_{i=1}^{k} \lambda_i x_i) \tag{5.7}$$

Assume that $Z \neq \emptyset$. Then, for $x \in I\!R^n$, $d \in recZ$, $\lambda > 0$

$$T(x) \cap \{d\}^{\perp} \subset T(x + \lambda d) \tag{5.8}$$

Proof Let $u \in T(x) \cap T(y)$, then for any $z \in \mathbb{R}^n$ and $w \in T(z)$

$$\begin{aligned} & \langle u - w, \lambda x + (1 - \lambda) y - z \rangle \\ = \; & \lambda \langle u - w, x - z \rangle + (1 - \lambda) \langle u - w, y - z \rangle \\ \geq \; & 0 \end{aligned}$$

So $u \in T(\lambda x + (1 - \lambda) y)$ by maximality of T.

(5.7) can be proven by using induction on (5.6).

To prove (5.8), let $z \in \mathbb{R}^n$, $\bar{x} \in Z$, $v \in T(z)$, then

$$\langle d, v \rangle = - \lim_{\mu \to +\infty} \frac{1}{\mu} \langle z - (\bar{x} + \mu d), v - 0 \rangle \leq 0 \tag{5.9}$$

since T is monotone, $\bar{x} + \mu d \in Z$, $0 \in T(\bar{x} + \mu d)$, and $v \in T(z)$.

If $u \in T(x) \cap \{d\}^{\perp}$, $z \in \mathbb{R}^n, v \in T(z)$ then $\langle u - v, x - z \rangle \geq 0$ by monotonicity of T. Knowing that $u \in \{d\}^{\perp}$ and $\langle d, v \rangle \leq 0$, it follows that

$$\langle u - v, x + \lambda d - z \rangle \geq 0$$

Hence, $u \in T(x + \lambda d)$ by maximality of T. **Q.E.D.**

We further claim that the inequalities of Lemma 5.7 hold as equalities if $T \equiv N(\cdot \mid Z)$.

Lemma 5.9 *For any points $x, y \in Z$, $0 < \lambda < 1$, we have*

$$N(x \mid Z) \cap N(y \mid Z) = N(\lambda x + (1 - \lambda)y \mid Z) \tag{5.10}$$

For $x_i \in \mathbb{R}^n$, $\lambda_i > 0$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} \lambda_i = 1$

$$\bigcap_{i=1}^{k} N(x_i \mid Z) = N(\sum_{i=1}^{k} \lambda_i x_i \mid Z) \tag{5.11}$$

For any $\bar{x} \in Z$, $d \in recZ$, and $\lambda > 0$

$$N(\bar{x} \mid Z) \cap \{d\}^{\perp} = N(\bar{x} + \lambda d \mid Z) \tag{5.12}$$

Proof In view of Lemma 5.7, the only thing needed to prove (5.10) is

$$N(x \mid Z) \cap N(y \mid Z) \supset N(\lambda x + (1 - \lambda)y \mid Z)$$

Let $u \in N(\lambda x + (1 - \lambda)y \mid Z)$, then for any $z \in Z$

$$\begin{aligned}
\langle u, z - x \rangle &= \frac{1}{\lambda} \langle u, \lambda z - \lambda x \rangle \\
&= \frac{1}{\lambda} \langle u, \lambda z + (1 - \lambda)y - (\lambda x + (1 - \lambda)y) \rangle \\
&\leq 0
\end{aligned}$$

since $u \in N(\lambda x + (1-\lambda)y \mid Z)$ and $\lambda z + (1-\lambda)y \in Z$. Thus, $u \in N(x \mid Z)$. Interchanging the roles of x and y, we see that $u \in N(y \mid Z)$.

(5.11) follows from induction on (5.10).

To prove (5.12), it is sufficient to show that

$$N(\bar{x} \mid Z) \cap \{d\}^{\perp} \supset N(\bar{x} + \lambda d \mid Z)$$

Let $u \in N(\bar{x} + \lambda d \mid Z)$, then $\langle u, z - (\bar{x} + \lambda d)\rangle \leq 0$ for any $z \in Z$. By taking $z = \bar{x}$ and $z = \bar{x} + 2\lambda d$, it follows that $\langle u, d\rangle = 0$. Hence

$$\langle u, z - \bar{x}\rangle = \langle u, z - (\bar{x} + \lambda d)\rangle \leq 0$$

We now have $u \in N(\bar{x} \mid Z) \cap \{d\}^{\perp}$. **Q.E.D.**

Lemma 5.10 *Suppose $x, y \in Z$, then*

$$N(x \mid Z) \cap N(y \mid Z) = \begin{cases} N(x \mid Z) \\ \text{a subset of } rbdry N(x \mid Z) \end{cases} \tag{5.13}$$

Proof Suppose that $N(x \mid Z) \cap N(y \mid Z)$ does not equal $N(x \mid Z)$, so that we can find $u \in N(x \mid Z) \setminus N(y \mid Z)$. For any point $v \in N(x \mid Z) \cap N(y \mid Z)$, we have

$$\langle v, x - y\rangle = 0$$

since $v \in N(x \mid Z)$ implies

$$\langle v, y - x\rangle \leq 0$$

and $v \in N(y \mid Z)$ implies

$$\langle v, x - y\rangle \leq 0$$

Knowing that $u \in N(x \mid Z)$, we have $\langle u, x - y\rangle \geq 0$. We further claim that $\langle u, x - y\rangle > 0$, since if $\langle u, x - y\rangle = 0$, then for all $z \in Z$

$$\langle u, z - y\rangle = \langle u, z - x + x - y\rangle \tag{5.14}$$

$$= \langle u, z - x\rangle \tag{5.15}$$

$$\leq 0 \tag{5.16}$$

and hence $u \in N(y \mid Z)$, a contradiction to $u \notin N(y \mid Z)$. It follows that

$$\langle v + \epsilon(v - u), y - x\rangle = \epsilon \langle u, x - y\rangle > 0$$

for all $\epsilon > 0$. Hence $v + \epsilon(v - u) \notin N(x \mid Z)$, which implies $v \notin \mathrm{ri}N(x \mid Z)$. Therefore $v \in \mathrm{rbdry}N(x \mid Z)$. **Q.E.D.**

With these four lemma as technical tools, we are now ready to establish our main result. Note that the converse statement is obvious, even without polyhedrality of Z.

Theorem 5.11 *Let T be maximal monotone and $Z = T^{-1}(0)$ be polyhedral. If*

$$\alpha \boldsymbol{B} \cap \bigcup_{x \in Z} N(x \mid Z) \subset \bigcup_{x \in Z} T(x) \tag{5.17}$$

then Z is a sharp zero set of T with modulus α. That is

$$\alpha \boldsymbol{B} \cap N(x \mid Z) \subset T(x) \tag{5.18}$$

for all $x \in Z$.

Proof i) Assume that Z contains no lines. We first show (5.18) for an extreme point of Z, then show it for a convex combination of extreme points, and finally show it for an arbitrary point in Z.

Suppose that z is an extreme point of Z, then according to [45, Theorem 18.6 and Corollary 19.1.1], z is also an exposed point, i.e. there exists $c \in \mathbb{R}^n$ such that

$$\langle c, w - z\rangle < 0$$

for all $w \in Z \setminus \{z\}$. So, $c \in N(z \mid Z)$, but $c \notin N(w \mid Z)$. Hence

$$N(z \mid Z) \cap N(w \mid Z) \neq N(z \mid Z)$$

It follows, by Lemma 5.10 , that

$$N(z \mid Z) \cap N(w \mid Z) \subset \mathrm{rbdry} N(z \mid Z) \tag{5.19}$$

for all $w \in Z \setminus \{z\}$.

If there exists $v \in (\alpha\mathbf{B} \cap \mathrm{ri} N(z \mid Z)) \setminus T(z)$. Then, because of (5.17), there exists $w \in Z$ such that

$$v \in T(w) \subset N(w \mid Z)$$

and then

$$v \in N(z \mid Z) \cap N(w \mid Z)$$

But since we have shown (5.19), there must be

$$v \in \mathrm{rbdry} N(z \mid Z)$$

a contradiction to $v \in \mathrm{ri} N(z \mid Z)$. So

$$(\alpha\mathbf{B} \cap \mathrm{ri} N(z \mid Z)) \setminus T(z) = \emptyset$$

Therefore

$$T(z) \supset \alpha\mathbf{B} \cap \mathrm{ri} N(z \mid Z)$$

But since T is maximum monotone, $T(z)$ is closed; consequently $T(z) \supset \alpha\mathbf{B} \cap N(z \mid Z)$.

Now let $x = \sum_{i=1}^{k} \lambda_i x_i$, where x_i's are extreme points of Z, $\lambda_i > 0$ for $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} \lambda_i = 1$, we have, according to previous lemma

$$\alpha\mathbf{B} \cap N(x \mid Z) \quad = \quad \alpha\mathbf{B} \cap N(\sum_{i=1}^{n} \lambda_i x_i \mid Z)$$

$$
\begin{aligned}
&= \bigcap_{i=1}^{n} (\alpha \mathbf{B} \cap N(x_i \mid Z)) \\
&\subset \bigcap_{i=1}^{n} T(x_i) \\
&\subset T(\sum_{i=1}^{n} \lambda_i x_i) \\
&= T(x)
\end{aligned}
$$

According to [45, Theorem 18.5, pp. 166], any polyhedral convex set containing no lines equals the convex hull of all its extreme points and extreme directions. Therefore, for any $y \in Z$, we can write $y = x + \lambda d$, for some x as a convex combination of extreme points, $d \in \mathrm{rec} Z$ and $\lambda > 0$. It follows that

$$
\begin{aligned}
\alpha \mathbf{B} \cap N(y \mid Z) &= \alpha \mathbf{B} \cap N(x + \lambda d \mid Z) \\
&= \alpha \mathbf{B} \cap N(x \mid Z) \cap \{d\}^{\perp} \\
&\subset T(x) \cap \{d\}^{\perp} \\
&\subset T(x + \lambda d) \\
&= T(y)
\end{aligned}
$$

ii) Let L be the lineality space of Z, then Z can be decomposed as

$$Z = L + (Z \cap L^{\perp})$$

with no lines in $Z \cap L^{\perp}$. Furthermore, L is perpendicular to $T(z)$ for any $z \in \mathbb{R}^n$. We can see this by looking at any $d \in L$, we know that $\pm d \in \mathrm{rec} Z$, and therefore $d \perp T(z)$ by (5.9).

Since $T(z) \subset L^{\perp}$, for all $z \in \mathbb{R}^n$

$$T \subset \mathbb{R}^n \times L^{\perp}$$

By restricting the domain of T to $L^\perp$, we obtain a multifunction

$$\begin{aligned} T_{|L^\perp} &:= T \cap (L^\perp \times \mathbb{R}^n) \\ &\subset (\mathbb{R}^n \times L^\perp) \cap (L^\perp \times \mathbb{R}^n) \\ &= L^\perp \times L^\perp \end{aligned}$$

which is a multifunction from $L^\perp$ to $L^\perp$. Moreover, it is monotone due to monotonicity of T.

Let $z \in \mathbb{R}^n$, $d \in L$, and $\lambda > 0$, then since $\pm d \in \text{rec}Z$, and $T(z) \subset L^\perp$, it follows from (5.8) that

$$\begin{aligned} T(z) &= T(z) \cap L^\perp \\ &\subset T(z) \cap \{d\}^\perp \\ &\subset T(z + \lambda d) \end{aligned}$$

and

$$\begin{aligned} T(z + \lambda d) &\subset T(z + \lambda d) \cap \{-d\}^\perp \\ &\subset T(z + \lambda d + \lambda(-d)) \\ &= T(z) \end{aligned}$$

Thus

$$T(z) = T(z + \lambda d)$$

We see that T is constant on any direction $d \in L$. Hence

$$T = T_{|L^\perp} \circ \pi_{L^\perp}(\cdot) \tag{5.20}$$

where $\pi_{L^\perp}(\cdot)$ is the linear projector onto $L^\perp$.

We also claim that $T_{|L^\perp}$ is maximal. Otherwise, there will be some monotone multifunction T_m from $L^\perp$ to $L^\perp$ properly containing $T_{|L^\perp}$. Thus T will be properly contained in the multifunction $T_m \circ \pi_{L^\perp}(\cdot)$. Furthermore, for any $x,\ y \in \mathbb{R}^n$, we have

$$x = \pi_L(x) + \pi_{L^\perp}(x)$$

and

$$y = \pi_L(y) + \pi_{L^\perp}(y)$$

Hence

$$\begin{aligned} & \langle T_m \circ \pi_{L^\perp}(x) - T_m \circ \pi_{L^\perp}(y), x - y \rangle \\ = & \ \langle T_m \circ \pi_{L^\perp}(x) - T_m \circ \pi_{L^\perp}(y), \pi_L(x) - \pi_L(y) \rangle \\ + & \ \langle T_m \circ \pi_{L^\perp}(x) - T_m \circ \pi_{L^\perp}(y), \pi_{L^\perp}(x) - \pi_{L^\perp}(y) \rangle \end{aligned}$$

But the second term is non–negative due to the monotonicity of T_m and we have

$$\langle T_m \circ \pi_{L^\perp}(x) - T_m \circ \pi_{L^\perp}(y), x - y \rangle \ \geq \ \langle T_m \circ \pi_{L^\perp}(x) - T_m \circ \pi_{L^\perp}(y), \pi_L(x) - \pi_L(y) \rangle$$

The right hand side is 0 since $\mathrm{im} T_m \subset L^\perp$, $\pi_L(x) - \pi_L(y) \in L$ and L is perpendicular to $L^\perp$. Thus

$$\langle T_m \circ \pi_{L^\perp}(x) - T_m \circ \pi_{L^\perp}(y), x - y \rangle \ \geq \ 0$$

for all $x,\ y \in \mathbb{R}^n$. In another words, $T_m \circ \pi_{L^\perp}(\cdot)$ is monotone. But the proper inclusion of T in $T_m \circ \pi_{L^\perp}(\cdot)$ contradicts the maximality of T. So, we know that $T_{|L^\perp}$ is a maximal monotone.

We also observed that the zero set of $T_{|L^\perp}$ is $Z \cap L^\perp$, and for any $x \in Z \cap L^\perp$

$$N(x \mid Z \cap L^\perp) = N(x \mid Z) \tag{5.21}$$

with the first normal cone taken with respect to $L^\perp$.

Now, by using (5.20), we can reduce (5.17) into

$$\alpha \mathbf{B} \cap \bigcup_{x \in Z \cap L^\perp} N(x \mid Z) \subset \bigcup_{x \in Z \cap L^\perp} T(x)$$

which is, by (5.21), equivalent to

$$\alpha\mathbf{B} \cap \bigcup_{x \in Z \cap L^{\perp}} N(x \mid Z \cap L^{\perp}) \subset \bigcup_{x \in Z \cap L^{\perp}} T_{|L^{\perp}}(x)$$

By applying the result of i) on the multifunction $T_{|L^{\perp}}$, we have

$$\alpha\mathbf{B} \cap N(x \mid Z) = \alpha\mathbf{B} \cap N(x \mid Z \cap L^{\perp}) \subset T_{|L^{\perp}}(x) = T(x) \tag{5.22}$$

for all $x \in Z \cap L^{\perp}$.

We further conclude that (5.18) is true for all $x \in Z$ based on (5.20), (5.21) and (5.22). In fact, for any $x \in Z$

$$\begin{aligned} T(x) &= T_{|L^{\perp}}(\pi_{L^{\perp}}(x)) \\ &\supset \alpha\mathbf{B} \cap N(\pi_{L^{\perp}}(x) \mid Z \cap L^{\perp}) \\ &= \alpha\mathbf{B} \cap N(\pi_{L^{\perp}}(x) \mid Z) \\ &= \alpha\mathbf{B} \cap N(x \mid Z) \end{aligned}$$

where the inclusion on the second line follows from (5.22) due to $\pi_{L^{\perp}}(x) \in Z \cap L^{\perp}$ and the last equality is true because $N(\cdot \mid Z)$ is constant along any direction of L just like T. **Q.E.D.**

The method used in part i) of the proof can be used on another class of maximal monotone multifunctions to obtain a result similar to that of the proceeding theorem. A set $C \subset \mathbb{R}^n$ is called strictly convex, if for any $x, y \in C$ and $0 < \lambda < 1$, $\lambda x + (1 - \lambda)y \in \text{ri}C$. The following Corollary is a consequence of Lemma 5.8, Lemma 5.9 and the method mentioned above.

Corollary 5.12 *Let T be maximal monotone, and $Z = T^{-1}(0)$. Assume that Z is strictly convex, and*

$$\alpha\boldsymbol{B} \cap \bigcup_{x \in Z} N(x \mid Z) \subset \bigcup_{x \in Z} T(x) \tag{5.23}$$

then

$$\alpha \boldsymbol{B} \cap N(x \mid Z) \subset T(x) \tag{5.24}$$

for all $x \in Z$.

5.3 An Application to Monotone Variational Inequalities

Consider a monotone variational inequality

$$0 \in f(x) + N(x \mid X) \tag{5.25}$$

where f is a monotone, continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$, X is a polyhedral set. The solution set Z is the zero set of the maximal monotone multifunction

$$T = f(\cdot) + N(\cdot \mid X)$$

We prove that a necessary condition for T to be a sharp zero set of T is that Z is an exposed face of X.

Before presenting our main results, we need the following lemmas.

Lemma 5.13 *If* $x_0 \in riF$ *for some face* F *of a polyhedral convex set* C, *then*

$$N(x_0 \mid C) = N(x_0 \mid C \cap F^{\perp}) \cap F^{\perp} \tag{5.26}$$

Proof We first observe that

$$N(x_0 \mid C) \subset F^{\perp}$$

due to $x_0 \in \mathrm{ri}F$. We also know

$$N(x_0 \mid C) \subset N(x_0 \mid C \cap F^{\perp})$$

from $C \cap F^\perp \subset C$. Therefore

$$N(x_0 \mid C) \subset N(x_0 \mid C \cap F^\perp) \cap F^\perp \tag{5.27}$$

If there exists $v \in N(x_0 \mid C \cap F^\perp) \cap F^\perp \setminus N(x_0 \mid C)$, then we can find $c \in F^\perp$ such that c strongly separates v from $N(x_0 \mid C)$. That is

$$\langle c, v\rangle > 0 \quad and \quad \langle c, y\rangle \leq 0 \tag{5.28}$$

for all $y \in N(x_0 \mid C)$. That is to say

$$c \in (N(x_0 \mid C))^o = \text{cone}C - x_0 \tag{5.29}$$

Furthermore, because $c \in F^\perp$

$$\begin{aligned} c &\in \text{cone}C - x_0 \cap F^\perp \\ &= \text{cone}(C \cap F^\perp) - x_0 \\ &= N(x_0 \mid C \cap F^\perp) \cap F^\perp)^o \end{aligned}$$

The last equality is due to the fact $N(x_0 \mid C \cap F^\perp) \cap F^\perp$ is the normal cone of $C \cap F^\perp$ at x_0 with respect to $F^\perp$ as opposed to $\mathbb{R}^n$, which is the case in (5.29). But we assumed $v \in N(x_0 \mid C \cap F^\perp) \cap F^\perp$, therefore

$$\langle c, v\rangle \leq 0$$

contradicting (5.28). So we see that

$$N(x_0 \mid C \cap F^\perp) \cap F^\perp \setminus N(x_0 \mid C) = \emptyset$$

which, together with (5.27), leads to

$$N(x_0 \mid C) = N(x_0 \mid C \cap F^\perp) \cap F^\perp$$

Q.E.D.

Lemma 5.14 *Let* $Z = T^{-1}(0)$*, then*

$$\langle f(x_1), x_2 - x_1 \rangle = 0 \tag{5.30}$$

for any $x_1, x_2 \in Z$*.*

If Z *is a sharp zero set, then*

$$\mathit{aff} Z \cap X = Z \tag{5.31}$$

Proof Let $x_1, x_2 \in Z$, we have

$$\langle f(x_1), x_2 - x_1 \rangle \geq 0$$

If $\langle f(x_1), x_2 - x_1 \rangle > 0$, then

$$\langle f(x_2), x_1 - x_2 \rangle < -\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \leq 0$$

contradicting the fact that $x_2 \in Z$. Hence

$$\langle f(x_1), x_2 - x_1 \rangle = 0$$

for all $x_1, x_2 \in Z$.

Assume $Z \neq \emptyset$ satisfies (5.18). It is obvious that

$$Z \subset \mathrm{aff} Z \cap X \subset \mathrm{aff} Z$$

because Z is contained in both aff$Z$ and X. Thus ri$Z \subset$ ri(aff$Z \cap X$).

Suppose $x \in ((\mathrm{aff} Z) \cap X) \setminus Z$ and $\bar{x} \in \mathrm{ri} Z$. Let

$$\mu = \sup\{0 \leq \lambda | \lambda x + (1 - \lambda)\bar{x} \in Z\}$$

then, because Z is closed

$$x_\mu = \mu x + (1 - \mu)\bar{x} \in Z$$

hence $\mu < 1$. Also notice that $\bar{x} \in \mathrm{ri}Z \subset \mathrm{ri}(\mathrm{aff}Z \cap X)$, we have

$$x_\mu \in \mathrm{ri}(\mathrm{aff}Z \cap X)$$

Therefore $N(x_\mu \mid X) \subset (\mathrm{aff}Z \cap X)^\perp \subset Z^\perp = (\mathrm{aff}Z)^\perp$, which then implies

$$\dim N(x_\mu \mid X) \leq n - \dim(\mathrm{aff}Z) \tag{5.32}$$

where dim denotes the dimensionality of a set.

On the other hand, for any $\epsilon > 0$

$$\begin{aligned} x_\mu - \epsilon(\bar{x} - x_\mu) &= (1+\epsilon)x_\mu - \epsilon\bar{x} \\ &= (1+\epsilon)\mu x + (1 - (1+\epsilon)\mu)\bar{x} \notin Z \end{aligned}$$

by the definition of μ. Thus $x_\mu \notin \mathrm{ri}Z$. So x_μ is in $\mathrm{ri}F$ for some face F of Z (see [45, Theorem 18.2]) with

$$\dim F < \dim Z = \dim(\mathrm{aff}Z)$$

We now show that

$$\dim N(x_\mu \mid Z) = n - \dim F$$

In fact, x_μ is an extreme point of $Z \cap F^\perp$. Otherwise

$$x_\mu = \lambda x_1 + (1-\lambda)x_2$$

where $0 < \lambda < 1$, $x_1, x_2 \in Z \cap F^\perp$. But given that $x_1, x_2 \in Z$, $x_\mu \in \mathrm{ri}[x_1, x_2]$, and $x_\mu \in F$, we conclude that $x_1, x_2 \in F$ by the fact that F is a face of Z. Consequently, $x_1, x_2 \in F \cap F^\perp = \{x_\mu\}$, a contradiction. Now that x_μ is an extreme point of $Z \cap F^\perp$

$$\dim(x_\mu | Z \cap F^\perp) = \dim F^\perp$$

where the normal cone is taken in $F^\perp$, or equivalently

$$\dim(N(x_\mu \mid Z \cap F^\perp) \cap F^\perp) = \dim F^\perp$$

with the normal cone taken in $\mathbb{R}^n$. This in turn gives

$$\begin{aligned} \dim N(x_\mu \mid Z) &= \dim(N(x_\mu \mid Z \cap F^\perp) \cap F^\perp) \\ &= \dim F^\perp \\ &= n - \dim F \end{aligned}$$

by virtue of (5.26). Combining this with (5.32)

$$\begin{aligned} \dim N(x_\mu \mid Z) &= n - \dim F \\ &> n - \dim(\mathrm{aff} Z) \\ &= \dim N(x_\mu \mid X) \end{aligned}$$

which makes

$$\alpha \mathbf{B} \cap N(x_\mu \mid Z) \subset T(x_\mu) = f(x_\mu) + N(x_\mu \mid X)$$

impossible and hence contradicts (5.18). We therefore conclude that

$$((\mathrm{aff} Z) \cap X) \setminus Z = \emptyset$$

or equivalently

$$\mathrm{aff} Z \cap X = Z$$

Q.E.D.

Theorem 5.15 *Let Z be a sharp zero set of T, then Z is an exposed face of X, and furthermore*

$$Z = \{x | x \in X, \langle f(\bar{x}), x - \bar{x} \rangle = 0\} \tag{5.33}$$

for $\bar{x} \in riZ$.

Proof We first prove (5.33). Note that if $Z = X$, then (5.33) holds trivially. So we assume $Z \neq X$. Let S be the right hand side of (5.33), then $Z \subset S$ is a direct consequence of Lemma 5.14 (see (5.30)).

To prove $S \subset Z$, we notice that since $\bar{x} \in \mathrm{ri}Z$,

$$N(\bar{x} \mid Z) = \{\mathrm{aff}Z\}^{\perp}$$

By the fact that Z is a sharp zero set

$$\alpha\mathbf{B} \cap \{\mathrm{aff}Z\}^{\perp} \subset T(\bar{x}) = f(\bar{x}) + N(\bar{x} \mid X) \tag{5.34}$$

for some $\alpha > 0$. Considering that $N(\bar{x} \mid X) \subset N(\bar{x} \mid Z) = \{\mathrm{aff}Z\}^{\perp}$

$$-f(\bar{x}) + \alpha\mathbf{B} \cap \{\mathrm{aff}Z\}^{\perp} \subset N(\bar{x} \mid X) \subset \{\mathrm{aff}Z\}^{\perp}$$

It follows that $\mathrm{aff}N(\bar{x} \mid X) = \{\mathrm{aff}Z\}^{\perp}$ and

$$-f(\bar{x}) \in \mathrm{ri}N(\bar{x} \mid X)$$

For any $x \in X$, we can write

$$x - \bar{x} = \pi_{\mathrm{aff}Z}(x - \bar{x}) + \pi_{\{\mathrm{aff}Z\}^{\perp}}(x - \bar{x})$$

where $\pi_L(\cdot)$ denotes the linear projection onto a subspace L.

Knowing that $-f(\bar{x}) \in \mathrm{ri}N(\bar{x} \mid X) \subset N(\bar{x} \mid Z) \subset \{\mathrm{aff}Z\}^{\perp}$

$$\left\langle -f(\bar{x}), \pi_{\{\mathrm{aff}Z\}^{\perp}}(x - \bar{x}) \right\rangle = \langle -f(\bar{x}), x - \bar{x} \rangle \leq 0$$

If $x \in X \setminus Z$, then $x \notin \mathrm{aff}Z$ by (5.31), hence we have $\pi_{\{\mathrm{aff}Z\}^{\perp}}(x - \bar{x}) \neq 0$. In this case, we claim that

$$\left\langle -f(\bar{x}), \pi_{\{\mathrm{aff}Z\}^{\perp}}(x - \bar{x}) \right\rangle = \langle -f(\bar{x}), x - \bar{x} \rangle < 0$$

otherwise from $-f(\bar{x}) \in \mathrm{ri}N(\bar{x} \mid X)$ and $\pi_{\{\mathrm{aff}Z\}^\perp}(x-\bar{x}) \in \{\mathrm{aff}Z\}^\perp$, there exists an $\epsilon > 0$ such that

$$-f(\bar{x}) + \epsilon\pi_{\{\mathrm{aff}Z\}^\perp}(x-\bar{x}) \in N(\bar{x} \mid X)$$

and so

$$\left\langle -f(\bar{x}) + \epsilon\pi_{\{\mathrm{aff}Z\}^\perp}(x-\bar{x}), x-\bar{x} \right\rangle \leq 0$$

which is reduced to

$$\left\langle -f(\bar{x}) + \epsilon\pi_{\{\mathrm{aff}Z\}^\perp}(x-\bar{x}), \pi_{\{\mathrm{aff}Z\}^\perp}(x-\bar{x}) \right\rangle \leq 0$$

by orthogonality between $\mathrm{aff}Z$ and $\{\mathrm{aff}Z\}^\perp$. But we assumed that

$$\left\langle -f(\bar{x}), \pi_{\{\mathrm{aff}Z\}^\perp}(x-\bar{x}) \right\rangle = \langle -f(\bar{x}), x-\bar{x} \rangle = 0$$

so we have

$$0 < \epsilon \left\langle \pi_{\{\mathrm{aff}Z\}^\perp}(x-\bar{x}), \pi_{\{\mathrm{aff}Z\}^\perp}(x-\bar{x}) \right\rangle \leq 0$$

a contradiction.

Hence

$$\langle -f(\bar{x}), x-\bar{x} \rangle < 0 \tag{5.35}$$

for all $x \in X \setminus Z$. This proves (5.33). (5.33) and (5.35) show that Z is the set of maxima for the linear function $\langle -f(\bar{x}), \cdot - \bar{x} \rangle$ over X and is hence an exposed face of X. **Q.E.D.**

A stronger version of (5.33) turns out to be a sufficient condition as demonstrated by the following theorem.

Theorem 5.16 *Let $Z = T^{-1}(0)$. Assume that*

$$Z = \{x \mid x \in X, \langle f(\bar{x}), x-\bar{x} \rangle = 0\} \tag{5.36}$$

for each $\bar{x} \in Z$. Then Z is a sharp zero set of T.

Proof We can actually write

$$Z = \{x \in X | \langle f(z), x - z \rangle \leq 0\}$$

for any $z \in Z$, which implies

$$N(z \mid Z) = \text{cone}\{f(z)\} + N(z \mid X)$$

Let $\mathcal{F}$ be the set of all faces of Z, then

$$\begin{aligned} \bigcup_{z \in Z} N(z \mid Z) &= \bigcup_{F \in \mathcal{F}} N(F \mid Z) \\ &= \bigcup_{F \in \mathcal{F}} \{\text{cone}\{f(z)\} + N(F \mid X)\} \end{aligned}$$

where $z \in \text{ri}F$. We can choose a finite set of $z \in Z$, and an $\alpha > 0$ such that

$$\alpha \mathbf{B} \cap \{\text{cone}\{f(z)\} + N(F \mid X)\} \subset f(z) + N(F \mid X) \qquad \text{for all } F \in \mathcal{F}$$

Then

$$\begin{aligned} \alpha \mathbf{B} \cap \bigcup_{z \in Z} N(z \mid Z) &= \alpha \mathbf{B} \cap \bigcup_{F \in F} N(F \mid Z) \\ &= \alpha \mathbf{B} \cap \bigcup_{F \in \mathcal{F}} \{\text{cone}\{f(z)\} + N(F \mid X)\} \\ &\subset \bigcup_{F \in \mathcal{F}} \{f(z) + N(F \mid X)\} \\ &\subset \bigcup_{z \in Z} \{f(z) + N(z \mid X)\} \end{aligned}$$

i.e. Z satisfies (5.17). But Z is polyhedral, hence Z sharp zero set of T by Theorem 5.11. **Q.E.D.**

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