Nondegenerate Solutions and Related Concepts in Affine Variational Inequalities

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Dedication. This paper is dedicated to Professor O.L. Mangasarian on the occasion of his 60th birthday. We are happy to submit this paper for publication on Professor Mangasarian's birthday to a journal that he has been associated with for many years, the SIAM Journal on Control and Optimization. He has made many significant contributions to the topics addressed in this paper, namely error bounds, weak sharp minima, minimum principle sufficiency and complementarity problems. We are both indebted to him for his constant encouragement, advice and fruitful collaborations over many years. Without his help and guidance, this paper would not have been possible.

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Abstract.

The notion of a strictly complementary solution for complementarity problems is extended to that of a nondegenerate solution of variational inequalities. Several equivalent formulations of nondegeneracy are given. In the affine case, an existence theorem for a nondegenerate solution is given in terms of several related concepts which are shown to be equivalent in this context. These include a weak sharp minimum, the minimum principle sufficiency, and error bounds. The gap function associated with the variational inequality plays a central role in this existence theorem.

Key Words. Variational inequalities, nondegenerate solutions, weak sharp minima, minimum principle, error bounds.

1 Introduction

Strict complementarity is a familiar notion in the context of optimization problems and/or complementarity theory. A classical result proved in [17, Corollary 2A] shows that a solvable linear complementarity problem defined by a skew-symmetric matrix must possess a strictly complementary solution. In general, the property of strict complementarity of a solution to an optimization or a complementarity problem plays an important role in many aspects of such a problem. Historically, Fiacco and McCormick [14] used this property to develop the first sensitivity theory of nonlinear programs under perturbation. Robinson [40] has introduced a generalized notion of strict complementarity and considered its role in parametric nonlinear programming.

In recent years, the strict complementarity property was given a renewed emphasis in the analysis of many iterative algorithms for solving linear and nonlinear programs and complementarity problems. Dunn [10] and Burke-Moré [6] used a geometric definition of a strictly complementary solution to a nonlinear program and showed how such a solution was essential for the successful identification of active constraints in a broad class of gradient based methods for solving constrained optimization problems. Güler and Ye [19] showed that many interior-point algorithms for linear programs generated a sequence of iterates whose limit points satisfied the strict complementarity condition; they also extended the result to a monotone linear complementarity problem having a strictly complementary solution. Monteiro and Wright [36] demonstrated that the existence of a strictly complementary solution was essential for the fast convergence of these interior-point algorithms for a monotone linear complementarity problem.

The theory of error bounds for inequality systems has in recent years become an active area of research within the field of mathematical programming. In this regard, Hoffman [21] obtained the first error bound for a system of finitely many linear inequalities. The generalizations of Hoffman's result are too numerous to be mentioned here. There are several factors that have motivated this proliferation of activities. In general, an error bound is an inequality that bounds the distance function from a test vector to the solutions of a system of inequalities in terms of a residual function. Part of the importance of an error bound is that it provides the foundation for exact penalization of mathematical programs [24, 30]; this in turn is strongly connected to the theory of optimality conditions for nonlinear programs [4]. Error bounds play an important role in the convergence analysis (particularly in establishing the convergence rates) of many iterative algorithms for solving various mathematical programs. These include the the matrix splitting methods for linear complementarity problems [8, Chapter 5] and affine variational inequalities [25], various descent methods for convex minimization problems [26, 27, 28], and interior-point methods for linear programs and extensions [23, 35, 42]. Error bounds can also be used to design inexact iterative methods [37, 16].

The concept of a weak sharp minimum for a constrained optimization problem was introduced in [11]. The usefulness of this concept in establishing the finite convergence of various iterative algorithms was discussed in several subsequent papers [12, 5, 1]. Among the classes of optimization problems that possess weak sharp minima are linear programs [32] and certain convex quadratic programs and monotone linear complementarity problems [5].

Finally, the minimum principle [29] is a well-known set of conditions that must be satisfied by any local minimum of a nonlinear program with a convex feasible region. One way to state this principle is in terms of the gap function [20] of the nonlinear program; informally, this principle states that a local minimum of a nonlinear program must be a global minimizer of the gap function over the same convex feasible region of the program. In [13], Ferris and Mangasarian studied the "converse" of this principle for the class of convex programs and coined the term "minimum principle sufficiency" when this converse was valid. They also showed (Theorem 6 in the reference) that for a convex quadratic program, the minimum principle sufficiency is equivalent to the existence of weak sharp minima of the program and that of a nondegenerate solution in the primal-dual linear complementarity formulation of the quadratic program. This somewhat unexpected result therefore links up the various concepts that we have discussed so far.

The present research is motivated by the desire to gain a better understanding of the concepts of strict complementarity, error bounds, weak sharp minima, and minimum principle sufficiency for various mathematical programs, and how these concepts are related. The results in [13, 31] suggest that for a monotone linear complementarity problem and its "natural" convex quadratic program [8, Chapter 3], all these concepts are equivalent (to be made precise later). In this paper, we shall extend the equivalences to a monotone affine variational inequality.

By adding appropriate multipliers to the constraints of an affine variational inequality, this problem becomes equivalent to a linear complementarity problem [38]. In view of the results available for the linear complementarity problem [13, 31], this transformation therefore raises the question of whether the intended generalized equivalences for the affine variational inequality are of any significant interest. We shall argue that the results derived herein are potentially useful for two reasons: (i) they do not rely on the multipliers of the constraints, and hence, are independent of the representation of the defining set of the affine variational inequality; and (ii) as it turns out, we shall use a nondifferentiable optimization problem as the bridge to connect the various concepts in question. The latter approach raises the issue of the extent to which these equivalences will remain valid for more general nondifferentiable optimization problems. The full treatment of this last issue is, regrettably, beyond the scope of the present work.

2 Definitions and Review

For a given mapping $F : R^n \to R^n$, the nonlinear complementarity problem, which we shall denote NCP (F), is to find a vector $x \in R^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0.$$

A solution $\hat{x}$ of this problem is said to be *strictly complementary*, or, *nondegenerate*, if $\hat{x} + F(\hat{x}) > 0$. For an optimization problem of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C, \end{array} \tag{1}$$

where $f : R^n \to R$ is continuous and $C \subseteq R^n$ is convex, different forms of nondegeneracy abound in the literature. Dunn [10], Burke and Moré [6] use the relative interior condition:

$$-\nabla f(\hat{x}) \in \operatorname{ri} N_C(\hat{x}), \tag{2}$$

to define an optimal solution $\hat{x}$ of (1) as being nondegenerate. Here $\operatorname{ri} S$ denotes the relative interior of the convex set S and $N_C(x)$ denotes the normal cone to the convex set C at the point $x \in R^n$ which is defined by

$$N_C(x) \equiv \begin{cases} \{y \in R^n \mid y^T(c - x) \leq 0, \text{ for all } c \in C\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

Robinson [40] uses the dual form: $T_C(\hat{x}) \cap \nabla f(\hat{x})^\perp$ is a subspace, where the tangent cone, $T_C(x)$, to C at x is the polar of the normal cone at x; i.e.

$$T_C(x) \equiv \{z \in R^n \mid z^T y \leq 0, \text{ for all } y \in N_C(x)\}.$$

It is easy to show (see [40, Lemma 2.1] for a proof) that the definition (2) is equivalent to the subspace definition. In general, for a convex set $S \subseteq R^n$, the negative of the polar of S is the dual cone of S, which is denoted by S^*.

It is not difficult to extend the notion of strict complementarity to the context of a variational inequality (VI) of the form: find $x \in C$ such that

$$F(x)^T(y - x) \geq 0, \quad \text{for all } y \in C, \tag{3}$$

where $C \subseteq R^n$ is a nonempty closed convex set and $F : R^n \to R^n$ is a continuous mapping. We shall denote this problem by VI (F, C); its (possibly empty) solution set is denoted $\mathrm{SOL}(F, C)$. When F is affine and given by $F(x) \equiv q + Mx$ for some vector $q \in R^n$, some matrix $M \in R^{n\times n}$, and all vectors $x \in R^n$, we shall append the word "affine" to describe this VI and denote it by AVI (q, M, C); the notation $\mathrm{SOL}(q, M, C)$ will be used to denote the solution set of this AVI.

Given a vector $\hat{x} \in \mathrm{SOL}(F, C)$, by simply replacing $\nabla f(\hat{x})$ by $F(\hat{x})$ in either (2) or in Robinson's dual definition, we obtain a definition for $\hat{x}$ to be a nondegenerate solution of the VI (F, C). A justification for this definition of nondegeneracy for the VI is the well-known fact that the VI (F, C) is equivalent to the generalized equation

$$0 \in F(x) + N_C(x),$$

or equivalently,

$$-F(x) \in N_C(x),$$

which easily leads to the generalized definition.

When C is a polyhedron, it is possible to give some further characterizations for the nondegeneracy of a solution $\hat{x} \in \mathrm{SOL}(F, C)$. We shall summarize these characterizations in Proposition 1 below. The additional characterizations rely heavily on the face structure of a polyhedral convex set. It is well known that the relative interiors of the faces of a convex set C form a partition of C [41, 18.2]. Throughout this paper, we will use the notation $\mathcal{F}(x)$ to denote the face of C which contains a vector $x \in C$ in its relative interior. The following result was established in [6].

Lemma 1 *The normal cone to a polyhedral convex set C is constant for all $x \in \mathrm{ri}\,\mathcal{F}$, where $\mathcal{F}$ is a face of C; henceforth labeled $\mathcal{N}_{\mathcal{F}}$. Furthermore,*

$$\mathrm{aff}\,\mathcal{F} - x = \mathrm{lin}\,T_C(x) = (\mathrm{aff}\,\mathcal{N}_{\mathcal{F}})^{\perp}.$$

As a consequence of this lemma, it follows that $\mathcal{F} - \mathcal{N}_{\mathcal{F}}$ has full dimension, and hence has a nonempty interior. This observation will be used in the proof of the following proposition. We note that condition (iv) in this proposition has been used by Reinoza [39].

Proposition 1 *Suppose $\hat{x}$ solves VI (F, C) and C is polyhedral. Let $\mathcal{F} = \mathcal{F}(\hat{x})$, so that $-F(\hat{x}) \in \mathcal{N}_{\mathcal{F}}$. The following statements are equivalent:*

(i) $\hat{x} + F(\hat{x}) \in \operatorname{int}(\mathcal{F} - \mathcal{N}_{\mathcal{F}})$,

(ii) $-F(\hat{x}) \in \operatorname{ri} \mathcal{N}_{\mathcal{F}}$,

(iii) $T_C(\hat{x}) \cap F(\hat{x})^{\perp}$ *is a subspace,*

(iv) $\hat{x}$ *is in the relative interior of the face of* C *exposed by* $-F(\hat{x})$.

If in addition, F is monotone, then $\mathrm{SOL}(F, C) \subseteq \mathcal{F}(\hat{x})$.

Proof The equivalence of (ii) and (iii) has been noted before. The equivalence of (ii) and (iv) is by [7, Theorem 2.4]. Since $\hat{x} \in \operatorname{ri} \mathcal{F}$ and $-F(\hat{x}) \in \operatorname{ri} \mathcal{N}_{\mathcal{F}}$, it follows that $\hat{x} + F(\hat{x}) \in \operatorname{ri} \mathcal{F} + \operatorname{ri}(-\mathcal{N}_{\mathcal{F}}) = \operatorname{ri}(\mathcal{F} - \mathcal{N}_{\mathcal{F}})$ which, as we have noted, has a nonempty interior. Thus (ii) implies (i). We now show that (i) implies (ii). First note that $\hat{x} + F(\hat{x}) \in \operatorname{ri} \mathcal{F} + \operatorname{ri}(-\mathcal{N}_{\mathcal{F}})$, so suppose

$$\hat{x} + F(\hat{x}) = y + z$$

with $y \in \operatorname{ri} \mathcal{F}$ and $z \in \operatorname{ri}(-\mathcal{N}_{\mathcal{F}})$. Then $y - \hat{x} \in \operatorname{aff} \mathcal{F} - \hat{x}$, $F(\hat{x}) - z \in \operatorname{aff} \mathcal{N}_{\mathcal{F}}$ and these two subspaces are orthogonal. Hence $y - \hat{x} = 0 = F(\hat{x}) - z$ as required.

For the final statement of the proposition, let $z \in \mathrm{SOL}(F, C)$ be arbitrary. Since F is monotone, it follows that (see e.g. [3])

$$F(c)^T(c - z) \geq 0, \quad \text{for all } c \in C,$$

which implies, since $\hat{x} \in C$, that $F(\hat{x})^T(\hat{x} - z) \geq 0$. However, $\hat{x}$ also solves the VI (F, C), so

$$F(\hat{x})^T(c - \hat{x}) \geq 0, \quad \text{for all } c \in C,$$

implying $F(\hat{x})^T(\hat{x} - z) = 0$. Hence,

$$z \in \{c \in C \mid F(\hat{x})^T(c - \hat{x}) = 0\},$$

which is $\mathcal{F}(\hat{x})$ by [7, Theorem 2.4]. □

In the remainder of this paper, we shall focus on the AVI (q, M, C). As stated before, our goal is to establish the equivalence of the existence of a nondegenerate solution to this problem and a number of related concepts. In what follows, we shall describe each of these concepts more formally.

The notion of a weak sharp minimum was introduced in [11] and extensively analyzed in [5, 13]. The formal definition is as follows.

Definition 1 *Let $f : R^n \to R \cup \{\infty\}$ and $C \subseteq R^n$. A nonempty subset $S \subseteq C$ is a set of* weak sharp minima *for the problem (1) if there is a scalar $\alpha > 0$ such that for all $x \in C$ and all $y \in S$,*

$$f(x) \geq f(y) + \alpha \operatorname{dist}(x \mid S), \tag{4}$$

where

$$\operatorname{dist}(x \mid S) \equiv \inf\{\|z - x\| : z \in S\}$$

is the distance from the point x to S measured by any norm.

Note that a set of weak sharp minima for (1), if it exists, must be equal to the set of global minimizers of f over C. In general, for the problem (1), it would be useful to know when a set of weak sharp minima exists. As mentioned in the introduction, an affirmative answer to this question is known for a linear program and certain convex quadratic programs.

Observe that if the problem (1) has a weak sharp minimum, then the inequality (4), which is equivalent to

$$\operatorname{dist}(x \mid S) \leq \alpha^{-1}(f(x) - f_{\min}), \quad \text{for all } x \in C, \tag{5}$$

where $f_{\min}$ is the minimum value of f on C, can be interpreted as providing an *error bound* for an arbitrary feasible point x to the set of minimizers of (1), with the residual given by the deviation of the objective value $f(x)$ from its minimum value. Consequently, a necessary and sufficient condition for the existence of a weak sharp minimum for the problem (1) is the existence of an error bound of the type (5) where S is the set of minimizers of (1).

The notion of minimum principle sufficiency was introduced in [13]. The minimum principle is a well-known necessary optimality condition for a program of the form (1), where C is convex; this principle states that, for a continuously differentiable function f, if $\bar{x}$ solves (1) then $\bar{x} \in \mathrm{SOL}(\nabla f, C)$. Roughly speaking, minimum principle sufficiency is the converse assumption; nevertheless, in order to make this precise, it will be necessary for us to introduce the gap function associated with the VI (F, C). Specifically, the gap function for the latter problem is the extended-valued function $g : R^n \to R \cup \{\infty\}$ given by

$$g(x) \equiv x^T F(x) - \omega(x), \quad \text{for all } x \in R^n, \tag{6}$$

where

$$\omega(x) \equiv \inf\{z^T F(x) : z \in C\}. \tag{7}$$

The function ω was introduced in [18] where it was used for stability analysis of the AVI. Let

$$\Omega(x) \equiv \text{ argmin}\{z^T F(x) : z \in C\};$$

it is understood that if the minimum value in $\omega(x)$ is not attained, then $\Omega(x)$ is defined to be the empty set. We note that if C is polyhedral, then $\omega(x)$ is the optimum objective value of a linear program.

The following proposition summarizes some important properties of the two functions g and ω. No proof is needed for these properties.

Proposition 2 *Let $F : R^n \to R^n$ be a mapping and C be a closed convex subset of R^n. The following statements are valid.*

(i) The function $\omega : R^n \to R \cup \{-\infty\}$ is concave and extended-valued; if F is a monotone affine function, then g is convex.

(ii) The function g is nonnegative on C.

(iii) A vector $x \in \mathrm{SOL}(F, C)$ if and only if $x \in \Omega(x)$, or equivalently, $x \in C$ and $g(x) = 0$.

(iv) If C is polyhedral, then

$$\begin{aligned} \operatorname{dom}\omega &\equiv \{x \in R^n \mid \omega(x) > -\infty\} \\ &= \{x \in R^n \mid F(x) \in (\operatorname{rec} C)^*\}, \end{aligned}$$

where $(\operatorname{rec} C)^*$ is the dual of the recession cone of C.

(v) If C is polyhedral and F is affine, then ω is piecewise linear and g is piecewise quadratic.

Returning to the problem (1) and letting $F \equiv \nabla f$, we see that the minimum principle for this problem can be stated simply as: if x is a local minimizer of (1), then $x \in \Omega(x)$. Obviously, if f is a convex function, then every vector $x \in C$ with the property that $x \in \Omega(x)$ must be a global minimizer of (1). For a convex function f, the minimum principle sufficiency stipulates that for all optimal solutions x of (1), or equivalently, for all x such that $x \in \Omega(x)$, if $x' \in \Omega(x)$, then x' is a also global minimizer of (1). In what follows, we shall give several equivalent formulations for this sufficiency property, one of which will be the basis for generalization to a nondifferentiable function f.

Proposition 3 *Let $f : R^n \to R$ be a continuously differentiable convex function and $C \subseteq R^n$ be a closed convex set. Assume that $\mathcal{S} \equiv \operatorname{argmin}\{f(x) : x \in C\} \neq \emptyset$. The following statements are equivalent.*

(a) The minimum principle sufficiency holds for the minimization problem (1).

(b) For all $x \in \mathcal{S}$, $\mathcal{S} = \Omega(x)$ where $\Omega(x) \equiv \operatorname{argmin}\{z^T \nabla f(x) : z \in C\}$.

(c) For all $x \in \mathcal{S}$,

$$[z \in C,\ \nabla f(x)^T (z - x) = 0] \Rightarrow z \in \mathcal{S}.$$

If in addition, C is polyhedral and $\mathcal{S} \neq \emptyset$, then any one of the above statements is further equivalent to:

(d) $\mathcal{S}$ is a set of weak sharp minima for (1).

Proof Since $\mathcal{S} \subseteq \Omega(x)$ for all optimal solutions x of (1), the equivalence of (a) and (b) is obvious. That (c) is also equivalent to (a) or (b) is equally obvious because x solves (1) if and only if $x \in \Omega(x)$. Finally, the equivalence of (d) and the above statements was proved in [5, Theorem 4.2]. $\square$

Remark. Theorem 4.2 in [5] shows that in the above proposition, (d) always implies (a) for an arbitrary closed convex set C; nevertheless, Example 4.3 shows that the polyhedrality of C is needed for the reverse implication.

3 Miscellaneous Preliminary Results

We have now defined all the concepts we shall deal with in this paper. Our ultimate goal is to link them together for the monotone AVI (q, M, C) where M is assumed to be positive semidefinite and C is polyhedral. The linkage is via the gap function g for this AVI. Motivation for using this function

g stems partly from statement (iii) in Proposition 2 which suggests that g is a likely candidate for a residual function for the AVI. This choice is also supported by some error bound results in [18] which are derived with the aid of some additional properties of the monotone AVI. In what follows, we shall summarize the relevant results for later use. Throughout the rest of this paper, we shall fix the vector $q \in R^n$, the matrix $M \in R^{n \times n}$, and the set $C \subseteq R^n$. We shall assume that M is positive semidefinite and C is a polyhedral. We shall further assume that $\mathrm{SOL}(q, M, C) \neq \emptyset$.

There are two important constants associated with the solution set of the monotone AVI (q, M, C). Indeed, by results in [18], there exist a vector $d \in R^n$ and a scalar $\sigma \in R_+$, both dependent on the data (q, M, C), such that

$$d = (M + M^T)x, \quad \sigma = x^T M x, \tag{8}$$

for all $x \in \mathrm{SOL}(q, M, C)$. Furthermore, $\mathrm{SOL}(q, M, C)$ can be characterized using these constants as

$$\mathrm{SOL}(q, M, C) = \{x \in C \mid \omega(x) - (q^T x + \sigma) \geq 0, (M + M^T)x = d\}.$$

Since, for every $x \in \mathrm{SOL}(q, M, C), \omega(x) \leq (q + Mx)^T x = (q + d - M^T x)^T x = (q + d)^T x - \sigma$, simple algebra gives the alternative characterization:

$$\begin{aligned} \mathrm{SOL}(q, M, C) &= \{x \in C \mid \omega(x) - (q + d)^T x + \sigma \geq 0, (M + M^T)x = d\} \\ &= \{x \in C \mid \omega(x) - (q + d)^T x + \sigma = 0, (M + M^T)x = d\}. \end{aligned}$$

For a given polyhedral cone $K \subseteq R^n$, the AVI (q, M, K) is equivalent to a generalized linear complementarity problem which is to find a vector $y \in R^n$ such that

$$y \in K, \quad q + My \in K^*, \quad \text{and} \quad y^T(q + My) = 0,$$

where

$$K^* \equiv \{y \in R^n \mid y^T x \geq 0, \forall x \in K\}$$

is the dual cone of K. In this case, we shall use the prefix "GLCP" instead of "AVI" to describe the problem. The feasible region of GLCP (q, M, K) is given by

$$\mathrm{FEA}(q, M, K) \equiv \{y \in K \mid q + My \in K^*\}. \tag{9}$$

Since $\mathcal{F} - \mathcal{N}_{\mathcal{F}} \subseteq K + K^*$ and both have full dimension, it follows from Proposition 1 that if $\hat{y}$ is a strictly complementary solution of the GLCP (q, M, K), then

$$\hat{y} + q + M\hat{y} \in \mathrm{int}\,(K + K^*).$$

It is known [38] that the AVI (q, M, C) is equivalent to a mixed linear complementarity problem in higher dimensions. In what follows, we shall establish a connection between the nondegenerate solutions of these two problems. For this purpose, we shall represent C as

$$C = \{x \in R^n \mid Ax \geq b\} \tag{10}$$

for some matrix $A \in R^{m \times n}$ and vector $b \in R^m$. Then a vector $x \in C$ is a solution of AVI (q, M, C) if and only if there exists a vector $\lambda \in R^m$ such that the following conditions hold:

$$\begin{aligned} &0 = q + Mx - A^T \lambda \\ &w = Ax - b \\ &w \geq 0, \quad \lambda \geq 0, \quad w^T \lambda = 0. \end{aligned}$$

These conditions define the GLCP (p, N, K) where the variable y and the data (p, N, K) are given by

$$y \equiv \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad p \equiv \begin{pmatrix} q \\ -b \end{pmatrix}, \quad N \equiv \begin{bmatrix} M & -A^T \\ A & 0 \end{bmatrix}, \tag{11}$$

and $K \equiv R^n \times R^m_+$. Specializing Proposition 1 to the latter GLCP, we can show that a solution $(\hat{x}, \hat{\lambda})$ of GLCP (p, N, K) is nondegenerate if and only if $\hat{w} + \hat{\lambda} > 0$, where $\hat{w} \equiv A\hat{x} - b$. Based on this observation, the following result is easy to prove.

Proposition 4 *Let C be given by (10). A solution $\hat{x}$ of the AVI (q, M, C) is nondegenerate if and only if for some $\hat{\lambda}$, $(\hat{x}, \hat{\lambda})$ is a nondegenerate solution of the GLCP (p, N, K).*

Proof Let

$$\mathcal{I} \equiv \{i \mid (A\hat{x} = b)_i\}$$

be the index set of active constraints at $\hat{x}$. By the definition of $\mathcal{F} = \mathcal{F}(\hat{x})$, we have

$$\mathcal{F} = \{x \in C \mid (Ax = b)_i, \text{ for all } i \in \mathcal{I}\},$$

and $\hat{x} \in \operatorname{ri} \mathcal{F}$. Hence,

$$\mathcal{N}_{\mathcal{F}} = \{A^T \lambda \mid \lambda \in R^m_-, \lambda_i = 0, \text{ for all } i \notin \mathcal{I}\}.$$

From the theory of convex polyhedra, particularly [41, 6.6], we have

$$\begin{aligned}
\operatorname{ri} \mathcal{F} &= \{ x \in \mathcal{F} \mid (Ax > b)_i, \text{ for all } i \notin \mathcal{I} \} \\
\operatorname{ri} \mathcal{N}_{\mathcal{F}} &= \{ A^T \lambda \mid \lambda_i < 0, \text{ for all } i \in \mathcal{I};\ \lambda_i = 0, \text{ for all } i \notin \mathcal{I} \}.
\end{aligned}$$

Hence, according to Proposition 1, $\hat{x}$ is nondegenerate if and only if $\hat{x} \in \operatorname{ri} \mathcal{F}$ and $-(q + M\hat{x}) \in \operatorname{ri} \mathcal{N}_{\mathcal{F}}$. From this, the existence of the desired $\hat{\lambda}$ is obvious. □

The GLCP (p, N, K) defined above is related to the linear program defining the function $\omega(x)$ which is given by:

$$\omega(x) \equiv \min\{ z^T(q + Mx) : z \in C \};$$

see (7). The dual of this linear program, denoted $\Delta(x)$, is

$$\begin{array}{ll}
\text{maximize} & b^T \lambda \\
\text{subject to} & q + Mx - A^T \lambda = 0, \quad \lambda \geq 0.
\end{array}$$

We shall let $\Lambda(x)$ denote the (possibly empty) optimal solution set of $\Delta(x)$. The following result summarizes an important relation between the dual program $\Delta(x)$ and the GLCP (p, N, K) as well as two properties of $\Delta(x)$ as a parametric linear program with a changing right-hand side in the constraints.

Proposition 5 *The following three statements hold:*

(a) if $\hat{x} \in \mathrm{SOL}(q, M, C)$, then a pair $(\hat{x}, \hat{\lambda})$ solves the GLCP (p, N, K) if and only if $\hat{\lambda} \in \Lambda(\hat{x})$;

(b) there exists a constant $\alpha > 0$ such that for all $x \in R^n$ with $\Lambda(x) \neq \emptyset$ and all λ feasible to $\Delta(x)$,

$$-b^T \lambda + \omega(x) \geq \alpha \operatorname{dist}(\lambda \mid \Lambda(x)); \tag{12}$$

(c) there exists a constant $\beta > 0$ such that for all x and x' in R^n with $\Lambda(x) \neq \emptyset$ and $\Lambda(x') \neq \emptyset$,

$$\Lambda(x) \subseteq \Lambda(x') + \beta \|x - x'\| \mathcal{B}(0, 1),$$

where $\mathcal{B}(0, 1)$ is the unit Euclidean ball in R^m.

Proof Statement (a) is obvious. For statement (b), observe that if $\Lambda(x) \neq \emptyset$ for some x, then $\omega(x)$ is finite and equal to the optimal objective value of $\Delta(x)$. By [32, Lemma A.1], every solvable linear program has a nonempty set of weak sharp minima. A careful look at the proof of this result reveals that the constant associated with such a set of weak sharp minima is independent of the right-hand side in the constraints of the program. Thus (b) follows. Statement (c) follows from the Lipschitzian property of the solutions to a parametric right-hand sided linear program as proved in [33, Theorem 2.4]. $\square$

We shall associate the following optimization problem with the AVI (q, M, C):

$$\begin{array}{ll} \text{minimize} & g(x) \\ \text{subject to} & x \in C \end{array} \tag{13}$$

where g is the gap function defined in (6) with $F(x) \equiv q + Mx$. By Proposition 2, the function g is convex, piecewise quadratic, and possibly extended-valued; it is in general not Fréchet differentiable. We should mention that recently, there have been several differentiable optimization problems introduced for the study of a monotone VI [2, 15, 34]; since the objective functions of the latter optimization problems are not known to be convex even for a monotone AVI, it is therefore not clear whether our results can be extended to these other (possibly nonconvex) optimization formulations of the AVI.

Since C is polyhedral, it can be represented as

$$C = \operatorname{conv} G + \operatorname{rec} C \tag{14}$$

for some finite point set $G \subseteq R^n$ where $\operatorname{conv} G$ denotes the convex hull of G and $\operatorname{rec} C$ denotes the recession cone of C. When C is a cone, we have $G = \{0\}$ and $C = \operatorname{rec} C$. Clearly, the problem (13) can be equivalently stated as

$$\begin{array}{ll} \text{minimize} & x^T(q + Mx) - \tilde{\omega}(x) \\ \text{subject to} & x \in C, \quad q + Mx \in (\operatorname{rec} C)^*, \end{array} \tag{15}$$

where

$$\tilde{\omega}(x) \equiv \min\{z^T(q + Mx) : z \in G\}. \tag{16}$$

When C is a cone, the latter formulation reduces to

$$\begin{array}{ll} \text{minimize} & x^T(q + Mx) \\ \text{subject to} & x \in \text{FEA}(q, M, C), \end{array}$$

since $C = \operatorname{rec} C$ and $\tilde{\omega}(x)$ is identically equal to zero in this case; see (9) for the definition of $\mathrm{FEA}(q, M, C)$.

Unlike the function $\omega(x), \tilde{\omega}(x)$ is finite valued for all $x \in R^n$ and it is dependent on the point set G (in particular, on the representation of C). Nevertheless, $\omega(x) = \tilde{\omega}(x)$ for all $x \in \operatorname{dom} \omega$; recall that by Proposition 2, $\operatorname{dom} \omega$ consists of all vectors x satisfying $q + Mx \in (\operatorname{rec} C)^*$. The function $\tilde{\omega}(x)$ will play an important part in the proofs (but not the statements) of the results involving the AVI (q, M, C). We shall let $\mathrm{FEA}(q, M, C)$ denote the feasible region of the problem (15). This coincides with the previous definition (9) when C is a cone. Trivially, we have $\mathrm{SOL}(q, M, C) \subseteq \mathrm{FEA}(q, M, C)$. Moreover, the problem (13) is equivalent to

$$\begin{array}{ll} \text{minimize} & g(x) \\ \text{subject to} & x \in \mathrm{FEA}(q, M, C). \end{array} \tag{17}$$

Although the function g is not Fréchet differentiable, it is directionally differentiable at every vector in $\mathrm{FEA}(q, M, C)$ along all feasible directions. This fact is made precise in the following result.

Proposition 6 *Let $q \in R^n$ and $M \in R^{n\times n}$ be arbitrary; let $C \subseteq R^n$ be a polyhedral set. For any vectors $\bar{x}$ and x in $\mathrm{FEA}(q, M, C)$, the directional derivative*

$$\omega'(\bar{x}; x - \bar{x}) \equiv \lim_{\tau \downarrow 0} \frac{\omega(\bar{x} + \tau(x - \bar{x})) - \omega(\bar{x})}{\tau}$$

exists, is finite, and is equal to

$$\min\{u^T M(x - \bar{x}) : u \in \Omega(\bar{x})\},$$

where $\Omega(\bar{x}) \equiv \operatorname{argmin}\{z^T(q + M\bar{x}) : z \in C\}$; hence, $g'(\bar{x}; x - \bar{x})$ exists and is equal to

$$(x - \bar{x})^T(q + (M + M^T)\bar{x}) - \omega'(\bar{x}; x - \bar{x}).$$

Proof Since $q + M\bar{x} \in (\operatorname{rec} C)^*$, $\Omega(\bar{x}) \neq \emptyset$. It suffices to verify that

$$\omega'(\bar{x}; x - \bar{x}) = \min\{u^T M(x - \bar{x}) : u \in \Omega(\bar{x})\},$$

and that this derivative is finite. Since both $\bar{x}$ and x are in $\mathrm{FEA}(q, M, C)$, it follows that

$$\omega(\bar{x} + \tau(x - \bar{x})) = \tilde{\omega}(\bar{x} + \tau(x - \bar{x}))$$

for all $\tau \in [0,1]$. Hence, we have

$$\omega'(\bar{x}; x-\bar{x}) = \tilde{\omega}'(\bar{x}; x-\bar{x}) = \min\{u^T M(x-\bar{x}) : u \in \tilde{\Omega}(\bar{x})\},$$

where

$$\tilde{\Omega}(\bar{x}) \equiv \operatorname{argmin}\{z^T(q+M\bar{x}) : z \in G\}$$

is a nonempty, finite subset of $\Omega(\bar{x})$. Since $\tilde{\omega}'(\bar{x}; x-\bar{x})$ is finite, thus so is $\omega'(\bar{x}; x-\bar{x})$. Moreover, we have

$$\omega'(\bar{x}; x-\bar{x}) \geq \min\{u^T M(x-\bar{x}) : u \in \Omega(\bar{x})\}. \tag{18}$$

Since

$$\begin{aligned} \omega(\bar{x}+\tau(x-\bar{x})) &= \min\{z^T(q+M\bar{x}) + \tau z^T M(x-\bar{x}) : z \in C\} \\ &\leq \min\{z^T(q+M\bar{x}) + \tau z^T M(x-\bar{x}) : z \in \Omega(\bar{x})\} \\ &= \omega(\bar{x}) + \tau \min\{u^T M(x-\bar{x}) : u \in \Omega(\bar{x})\}, \end{aligned}$$

it follows that the reverse inequality in (18) also holds. Consequently, equality holds in (18). $\square$

Note that if $\bar{x} \in \mathrm{SOL}(q,M,C)$ and M is positive semidefinite, then Proposition 6 yields

$$g'(\bar{x}; x-\bar{x}) = (x-\bar{x})^T(q+d) - \omega'(\bar{x}; x-\bar{x}) \tag{19}$$

for all $x \in \mathrm{FEA}(q,M,C)$, where $d = (M+M^T)\bar{x}$ is one of the two constants associated with the solutions of the AVI (q,M,C). With the above proposition, we can now discuss the extension of the minimum principle sufficiency to the nondifferentiable gap minimization problem (13), or equivalently, to (17). Some related work on error bounds for convex, piecewise quadratic minimization problems, of which (13) is a special case, can be found in [22]. The following result establishes two properties of solutions to the AVI (q,M,C).

Proposition 7 *Let $q \in R^n$ be arbitrary, $M \in R^{n\times n}$ be positive semidefinite, and $C \subseteq R^n$ be a polyhedral set. If x and $\bar{x}$ are any two vectors in* $\mathrm{SOL}(q,M,C)$, *then $g'(\bar{x}; x-\bar{x}) = 0$ and*

$$\omega(x) = \omega(\bar{x}) + \omega'(\bar{x}; x-\bar{x}). \tag{20}$$

Proof Since $\mathrm{SOL}(q,M,C)$ is convex, $\bar{x}+\tau(x-\bar{x}) \in \mathrm{SOL}(q,M,C)$ for all $\tau \in [0,1]$. Hence for all such τ,

$$g(\bar{x}+\tau(x-\bar{x})) = 0,$$

which easily implies $g'(\bar{x}; x-\bar{x}) = 0$.

Since x and $\bar{x}$ belong to $\mathrm{SOL}(q,M,C)$, we have

$$\begin{aligned}
&\omega(x) = x^T(q+Mx) \\
&= x^T(q+M\bar{x}) + x^T M(x-\bar{x}) \\
&= \bar{x}^T(q+M\bar{x}) + (x-\bar{x})^T(q+M\bar{x}) + \bar{x}^T M(x-\bar{x}) + (x-\bar{x})^T M(x-\bar{x}) \\
&\geq \omega(\bar{x}) + \min\{u^T M(x-\bar{x}) : u \in \Omega(\bar{x})\} \\
&= \omega(\bar{x}) + \omega'(\bar{x}; x-\bar{x}) \geq \omega(x),
\end{aligned}$$

where the last inequality follows from the concavity of ω. $\square$

Alternatively stated, the above proposition says that for a monotone AVI (q,M,C) and any $\bar{x} \in \mathrm{SOL}(q,M,C)$, we have

$$\begin{aligned}
&\mathrm{SOL}(q,M,C) \subseteq \\
&\{x \in \mathrm{FEA}(q,M,C) \mid g'(\bar{x}; x-\bar{x}) = 0, \omega(x) = \omega(\bar{x}) + \omega'(\bar{x}; x-\bar{x})\}.
\end{aligned} \tag{21}$$

We say that the *restricted minimum principle sufficiency* (RMPS) holds for the problem (17) if for any $\bar{x} \in \mathrm{SOL}(q,M,C)$, equality holds in (21); or equivalently, the implication holds:

$$\left.\begin{aligned} x \in \mathrm{FEA}(q,M,C), g'(\bar{x}; x-\bar{x}) = 0 \\ \omega(x) = \omega(\bar{x}) + \omega'(\bar{x}; x-\bar{x}) \end{aligned}\right\} \Rightarrow x \in \mathrm{SOL}(q,M,C). \tag{22}$$

The word "restricted" that describes this property reflects the additional restriction—equation (20)—that the vector x has to satisfy in order for it to be a solution of AVI (q,M,C). If ω is a smooth (linear) function on $\mathrm{FEA}(q,M,C)$ (instead of a piecewise linear function), the latter restriction is redundant. In particular, this is the case when C is a cone.

The following two results give some necessary and sufficient conditions for the two conditions, $g'(\bar{x}; x-\bar{x}) = 0$ and (20), to hold separately. Although these results are not needed in the proof of the main theorem in the next section, they give some insights into the RMPS property of the AVI.

Proposition 8 *Let $q \in R^n$ and $M \in R^{n\times n}$ be arbitrary; let $C \subseteq R^n$ be a polyhedral set. Let $\bar{x} \in \mathrm{SOL}(q, M, C)$ and $x \in \mathrm{FEA}(q, M, C)$ be given. Then $g'(\bar{x}; x - \bar{x}) = 0$ if and only if $x \in \Omega(\bar{x})$ and*

$$(u - \bar{x})^T (q + Mx) \geq 0, \quad \textit{for all } u \in \Omega(\bar{x}). \tag{23}$$

Proof Indeed, by Proposition 6, we have $g'(\bar{x}; x - \bar{x}) = 0$ if and only if

$$(x - \bar{x})^T \left(q + (M + M^T)\bar{x} \right) = \min\{u^T M(x - \bar{x}) : u \in \Omega(\bar{x})\},$$

or equivalently,

$$(x - \bar{x})^T (q + M\bar{x}) = \min\{(u - \bar{x})^T M(x - \bar{x}) : u \in \Omega(\bar{x})\}.$$

Since $\bar{x} \in \mathrm{SOL}(q, M, C)$ and $x \in C$, the left-hand side is nonnegative whereas the right-hand side is nonpositive because $\bar{x} \in \Omega(\bar{x})$. Consequently, $g'(\bar{x}; x - \bar{x}) = 0$ if and only if

$$0 = (x - \bar{x})^T (q + M\bar{x}) = \min\{(u - \bar{x})^T M(x - \bar{x}) : u \in \Omega(\bar{x})\}.$$

The first equality is equivalent to $x \in \Omega(\bar{x})$. Moreover, for all $u \in \Omega(\bar{x})$, we have $(u - \bar{x})^T (q + M\bar{x}) = 0$; hence,

$$(u - \bar{x})^T M(x - \bar{x}) = (u - \bar{x})^T (q + Mx).$$

Consequently,

$$\min\{(u - \bar{x})^T M(x - \bar{x}) : u \in \Omega(\bar{x})\} = 0$$

if and only if (23) holds. $\square$

Proposition 9 *Let $q \in R^n$ and $M \in R^{n\times n}$ be arbitrary; let $C \subseteq R^n$ be a polyhedral set. Let $\bar{x}$ and x be any two vectors in $\mathrm{FEA}(q, M, C)$. Then the following are equivalent:*

(i) *(20) holds,*

(ii) $\Omega(x) \cap \Omega(\bar{x}) \neq \emptyset$,

(iii) *for all $\lambda \in (0, 1)$*

$$\omega(\lambda x + (1 - \lambda)\bar{x}) = \lambda\omega(x) + (1 - \lambda)\omega(\bar{x}).$$

Proof Suppose (20) holds. Then for any $u \in \Omega(\bar{x})$ such that $u^T M(x - \bar{x}) = \omega'(\bar{x}; x - \bar{x})$, we have

$$\begin{aligned} \omega(x) &\leq u^T(q + Mx) \\ &= u^T(q + M\bar{x}) + u^T M(x - \bar{x}) \\ &= \omega(\bar{x}) + \omega'(\bar{x}; x - \bar{x}). \end{aligned}$$

Hence, $\omega(x) = u^T(q + Mx)$ which implies $u \in \Omega(x) \cap \Omega(\bar{x})$. Conversely, if $u \in \Omega(x) \cap \Omega(\bar{x})$, then

$$\begin{aligned} \omega(x) &= u^T(q + Mx) \\ &= u^T(q + M\bar{x}) + u^T M(x - \bar{x}) \\ &\geq \omega(\bar{x}) + \omega'(\bar{x}; x - \bar{x}). \end{aligned}$$

By the concavity of ω, we have

$$\omega(x) \leq \omega(\bar{x}) + \omega'(\bar{x}; x - \bar{x}).$$

Thus (i) is equivalent to (ii). The equivalence of (ii) and (iii) follows from the fact that $\Omega(x)$ is the subdifferential of the support function of C at $-(Mx + q)$ and [9, Lemma 5.3]. □

4 The Main Result

We are now ready to state the main result of this paper. This result gives various necessary and sufficient conditions for the existence of a nondegenerate solution for a monotone AVI.

Theorem 1 *Let $q \in R^n$ be arbitrary, $M \in R^{n \times n}$ be positive semidefinite, and $C \subseteq R^n$ be a polyhedral set. Suppose* $\mathrm{SOL}(q, M, C) \neq \emptyset$. *Let $d \in R^n$ and $\sigma \in R_+$ be the two constants associated with the AVI (q, M, C). The following statements are equivalent:*

(a) The AVI (q, M, C) has a nondegenerate solution.

(b) The set $\mathrm{SOL}(q, M, C)$ *is a set of weak sharp minima for the problem (13).*

(c) There exists a constant $\gamma > 0$ such that for all $x \in C$,

$$\mathrm{dist}\,(x \mid \mathrm{SOL}(q, M, C)) \leq \gamma g(x). \tag{24}$$

(d) The representation holds:

$$\mathrm{SOL}(q, M, C) = \{x \in C \mid \omega(x) - (q+d)^T x + \sigma \geq 0\}. \qquad (25)$$

(e) The restricted minimum principle sufficiency holds for the problem (17); i.e. the implication (22) holds.

As it turns out, the proof of this theorem, except for the equivalence of (b) and (c), is rather complicated. We shall divide the entire proof into several parts. Throughout the proof we will assume, if necessary, that C is written in the form (10) or (14). Notice that since the function ω is in general not differentiable, the equivalence of (b) and (e) does not follow from Proposition 3.

The easiest part is the equivalence of (b) and (c); this follows from the remark made after Definition 1 and the observation that $g_{\min} = 0$. Note that effectively, the inequality (24) concerns only those vectors $x \in \mathrm{FEA}(q, M, C)$; indeed, since $g(x) = \infty$ for all $x \in C \setminus \mathrm{FEA}(q, M, C)$, (24) trivially holds for the latter vectors x.

The following lemma establishes (a) $\Rightarrow$ (d).

Lemma 2 *Under the assumptions of Theorem 1, statement (a) implies statement (d).*

Proof Let S denote the right-hand set in (25). It suffices to verify $S \subseteq \mathrm{SOL}(q, M, C)$, as the reverse inclusion is always valid. Let $x \in S$ and let $\hat{x}$ be a nondegenerate solution of AVI (q, M, C). Since $\omega(x)$ is finite, its dual program $\Delta(x)$ has an optimal solution λ that satisfies

$$b^T \lambda = \omega(x).$$

Since $\hat{x} \in \mathrm{SOL}(q, M, C)$ is nondegenerate, by Propositions 4 and 5, there exists a $\hat{\lambda} \in \Lambda(\hat{x})$ satisfying

$$\hat{\lambda}^T (A\hat{x} - b) = 0, \quad \text{and} \quad \hat{\lambda} + A\hat{x} - b > 0.$$

We have,

$$\begin{aligned} \omega(x) &\geq (q+d)^T x - \sigma \\ &= (q + (M + M^T)\hat{x})^T x - \hat{x}^T M \hat{x} \\ &= (q + M\hat{x})^T x + \hat{x}^T M (x - \hat{x}) \\ &= \hat{\lambda}^T A x + (\lambda - \hat{\lambda})^T A \hat{x} \\ &= \hat{\lambda}^T (Ax - b) + \lambda^T (A\hat{x} - b) + \lambda^T b, \end{aligned}$$

which yields,

$$0 \geq \hat{\lambda}^T(Ax - b) + \lambda^T(A\hat{x} - b) \geq 0.$$

Since $\hat{\lambda} + A\hat{x} - b > 0$ and λ and $Ax - b$ are both nonnegative, it follows easily that $\lambda^T(Ax - b) = 0$. Thus $x \in \mathrm{SOL}(q, M, C)$ as desired. □

Next, we prove (d) ⇒ (c). The proof of this implication uses the following consequence of the famous Hoffman error bound for systems of linear inequalities [21]. Let P be a polyhedral set in R^n, and let E and f be, respectively, a matrix and vector of compatible dimensions. If the polyhedron

$$S \equiv \{x \in P \mid : Ex \geq f\}$$

is nonempty, then there exists a constant $c > 0$ such that

$$\mathrm{dist}\,(x \mid S) \leq c\|(Ex - f)_-\|_\infty, \quad \text{for all } x \in P,$$

where the subscript "$_-$" denotes the nonpositive part of a vector.

Lemma 3 *Under the assumptions of Theorem 1, statement (d) implies statement (c).*

Proof Invoking the function $\tilde{\omega}(x)$ defined in (16), we can express (25) equivalently as

$$\begin{aligned}&\mathrm{SOL}(q, M, C) = \\ &\{x \in \mathrm{FEA}(q, M, C) \mid z^T(q + Mx) - (q + d)^T x + \sigma \geq 0, \forall z \in G\}.\end{aligned}$$

By the aforementioned consequence of Hoffman's result, we deduce the existence of a constant $\gamma > 0$ such that for all $x \in \mathrm{FEA}(q, M, C)$,

$$\mathrm{dist}\,(x \mid \mathrm{SOL}(q, M, C)) \leq \gamma \max_{z \in G} \left(z^T(q + Mx) - (q + d)^T x + \sigma \right)_- .$$

To complete the proof, it remains to verify that for all $x \in \mathrm{FEA}(q, M, C)$ and all $z \in G$,

$$\left(z^T(q + Mx) - (q + d)^T x + \sigma \right)_- \leq x^T(q + Mx) - \omega(x).$$

Since $x^T(q + Mx) \geq \omega(x)$ for all $x \in C$, it suffices to show that for all $x \in \mathrm{FEA}(q, M, C)$ and $z \in G$,

$$(q + d)^T x - \sigma - z^T(q + Mx) \leq x^T(q + Mx) - \omega(x);$$

in turn, since $z^T(q + Mx) \geq \omega(x)$, it suffices to verify

$$(q + d)^T x - \sigma \leq x^T(q + Mx).$$

For some $\bar{x} \in \mathrm{SOL}(q, M, C)$, the left-hand side of the above inequality is equal to,

$$\begin{aligned}&(q + (M + M^T)\bar{x})^T x - \bar{x}^T M \bar{x} = \\ &(q + Mx)^T x - (\bar{x} - x)^T M(\bar{x} - x) \leq (q + Mx)^T x,\end{aligned}$$

where the last inequality follows by the positive semidefiniteness of M. $\square$

We next show that (d) and (e) are equivalent. The proof of this equivalence is based on the following lemma which shows that the two sets on the right-hand sides of (21) and (25) are equal.

Lemma 4 *Under the assumptions of Theorem 1,*

$$\begin{aligned}&\{x \in \mathrm{FEA}(q, M, C) \mid \omega(x) - (q + d)^T x + \sigma \geq 0\} = \\ &\{x \in \mathrm{FEA}(q, M, C) \mid g'(\bar{x}; x - \bar{x}) = 0, \omega(x) = \omega(\bar{x}) + \omega'(\bar{x}; x - \bar{x})\}\end{aligned} \tag{26}$$

for any $\bar{x} \in \mathrm{SOL}(q, M, C)$; hence statements (d) and (e) are equivalent.

Proof Let x be any vector belonging to the right-hand set in (26). Combining (19) and (20), we deduce,

$$\omega(x) = \omega(\bar{x}) + (x - \bar{x})^T(q + d).$$

Thus

$$\omega(x) - (q + d)^T x + \sigma = \omega(\bar{x}) - (q + d)^T \bar{x} + \sigma = 0,$$

where the last equality holds because $\bar{x} \in \mathrm{SOL}(q, M, C)$. This establishes one inclusion in (26). To show the reverse inclusion, let x belong to the left-hand set in (26). By the concavity of ω, we have

$$\begin{aligned} 0 &\leq \omega(x) - (q + d)^T x + \sigma \\ &\leq \omega(\bar{x}) - (q + d)^T \bar{x} + \sigma - (q + d)^T(x - \bar{x}) + \omega'(\bar{x}; x - \bar{x}) \\ &= -g'(\bar{x}; x - \bar{x}) \leq 0. \end{aligned}$$

Thus equality holds throughout and (26) follows. The equivalence of statements (d) and (e) is now obvious. $\square$

Finally, we show that (c) $\Rightarrow$ (a). Before presenting the details of the proof, we explain the key steps involved. First, we recall the GLCP (p, N, K) that is equivalent to the AVI (q, M, C); see (11) for the definition of this GLCP. Consider the convex quadratic program in the variable (x, λ):

$$\begin{array}{ll} \text{minimize} & x^T(q + Mx) - b^T\lambda \\ \text{subject to} & 0 = q + Mx - A^T\lambda \\ & Ax - b \geq 0, \quad \lambda \geq 0; \end{array} \tag{27}$$

this is the "natural" quadratic program associated with the GLCP (q, N, K). We will show that condition (c) in Theorem 1 implies that this program has a nonempty set of weak sharp minima; the proof of this implication will use Proposition 5. Thus by Proposition 3, the minimum principle sufficiency holds for (27). Next by using a similar proof technique as in [13, Theorem 13], we will establish that the GLCP (p, N, K) has a nondegenerate solution. Proposition 4 will then imply that the AVI (q, M, C) has a nondegenerate solution.

In what follows, let $y \equiv (x, \lambda)$; also let $f(y)$ denote the objective function of (27). Note that $f(y) = y^T(p + Ny)$ and the matrix N is positive semidefinite; moreover, the feasible region of (27) is precisely $\mathrm{FEA}(p, N, K)$.

Lemma 5 *Under the assumptions of Theorem 1, statement (c) implies that*

$$\mathrm{SOL}(p, N, K) = \{y \in \mathrm{FEA}(p, N, K) \mid \nabla f(\bar{y})^T(y - \bar{y}) \leq 0\}, \tag{28}$$

for any $\bar{y} \in \mathrm{SOL}(p, N, K)$.

Proof Since $\mathrm{SOL}(q, M, C) \neq \emptyset$, it follows that $\mathrm{SOL}(p, N, K) \neq \emptyset$; moreover, the optimal solution set of (27) is equal to $\mathrm{SOL}(p, N, K)$. The claimed equation (28) is a consequence of the minimum principle sufficiency holding for (27); see Proposition 3. Thus by the analysis made above, it suffices to show that condition (c) in Theorem 1 implies that there exists a constant $\gamma' > 0$ such that

$$x^T(q + Mx) - b^T\lambda \geq \gamma' \operatorname{dist}(y \mid \mathrm{SOL}(p, N, K)), \tag{29}$$

for all $y \equiv (x, \lambda) \in \mathrm{FEA}(p, N, K)$. Let y be any such vector. Then $x \in \mathrm{FEA}(q, M, C)$ and λ is feasible to $\Delta(x)$. Thus $\Lambda(x) \neq \emptyset$ and the inequality (12) is valid for this pair (x, λ). We have

$$\begin{aligned} x^T(q + Mx) - b^T\lambda &= g(x) + \omega(x) - b^T\lambda \\ &\geq \gamma^{-1} \operatorname{dist}(x \mid \mathrm{SOL}(q, M, C)) + \alpha \operatorname{dist}(\lambda \mid \Lambda(x)), \end{aligned}$$

where the last inequality follows from (12) and (24). Now choose $(x', \lambda') \in \mathrm{SOL}(q, M, C) \times \Lambda(x)$ such that

$$\|x - x'\| = \operatorname{dist}(x \mid \mathrm{SOL}(q, M, C)) \quad \text{and} \quad \|\lambda - \lambda'\| = \operatorname{dist}(\lambda \mid \Lambda(x)).$$

Since $x' \in \mathrm{SOL}(q, M, C)$, it follows that $\omega(x')$ is finite and thus $\Lambda(x') \neq \emptyset$. By part (c) of Proposition 5, there exists $\tilde{\lambda} \in \Lambda(x')$ satisfying

$$\|\lambda' - \tilde{\lambda}\| \leq \beta \|x - x'\|.$$

By part (a) of the same proposition, the pair $(x', \tilde{\lambda}) \in \mathrm{SOL}(p, N, K)$. Consequently, we have

$$\begin{aligned}
&\operatorname{dist}(y \mid \mathrm{SOL}(q, N, K)) \\
&\leq \|x - x'\| + \|\lambda - \tilde{\lambda}\| \\
&\leq \operatorname{dist}(x \mid \mathrm{SOL}(q, M, C)) + \operatorname{dist}(\lambda \mid \Lambda(x)) + \|\lambda' - \tilde{\lambda}\| \\
&\leq (1 + \beta) \operatorname{dist}(x \mid \mathrm{SOL}(q, M, C)) + \operatorname{dist}(\lambda \mid \Lambda(x)).
\end{aligned}$$

Thus by letting

$$\gamma' \equiv \min\left(\frac{1}{\gamma(1+\beta)}, \alpha\right),$$

it is easy to see that (29) must hold. □

Lemma 6 *Under the assumptions of Theorem 1, statement (c) implies statement (a).*

Proof It suffices to show that the GLCP (p, N, K) has a nondegenerate solution. By the expression of $\mathrm{SOL}(p, N, K)$ given in Lemma 5 and by expanding $\nabla f(\bar{y})^T (y - \bar{y})$, such a solution exists if and only if the following linear program in the variables $(x, \lambda, \varepsilon)$:

$$\begin{array}{ll}
\text{minimize} & -\varepsilon \\
\text{subject to} & 0 = q + Mx - A^T \lambda \\
& Ax - b \geq 0, \quad \lambda \geq 0 \\
& \left(q + (M + M^T)\bar{x}\right)^T (x - \bar{x}) - b^T(\lambda - \bar{\lambda}) \leq 0 \\
& \lambda + Ax - b \geq \varepsilon e,
\end{array}$$

has a feasible solution with a negative objective value, where e is the vector of all ones and $(\bar{x}, \bar{\lambda})$ is an arbitrary solution of the GLCP (p, N, K). Assume that the GLCP (p, N, K) does not have a nondegenerate solution. Since the above linear program is feasible, with $(x, \lambda, \varepsilon) \equiv (\bar{x}, \bar{\lambda}, 0)$ as a feasible solution, the assumption implies that the program has an optimal solution with zero objective value. By letting (u, v, ζ, w) be an optimal dual solution, we have

$$\begin{aligned}
&M^T u + A^T(v + w) - \zeta \left(q + (M + M^T)\bar{x}\right) = 0 \\
&-Au + b\zeta + w \leq 0 \\
&e^T w = 1 \\
&v, \zeta, w \geq 0 \\
&-q^T u + b^T(v + w) + \zeta \left(b^T\bar{\lambda} - \bar{x}^T(q + (M + M^T)\bar{x})\right) = 0.
\end{aligned}$$

Premultiplying the first equation by u^T, the second constraint by $(v + w)^T$, and the last equation by ζ, adding the resulting constraints, using the fact that $b^T\bar{\lambda} - \bar{x}^T(q + M\bar{x}) = 0$, and simplifying, we deduce

$$(u - \zeta\bar{x})^T M(u - \zeta\bar{x}) + (v + w)^T w \leq 0.$$

Since M is positive semidefinite, and both w and v are nonnegative, the last inequality implies that $w = 0$, which contradicts the equation $e^T w = 1$. $\square$

Combining the above lemmas, we have the following proof of Theorem 1.

Proof of Main Theorem From Lemmas 2, 3, 4, 6, as well as the previously mentioned equivalence of (b) and (c), we see that the following implications are valid:

$$\begin{array}{ccccc}
(\mathrm{a}) & \Rightarrow & (d) & \Leftrightarrow & (e) \\
 & & \Downarrow & & \\
 & & (c) & \Rightarrow & (a) \\
 & & \Updownarrow & & \\
 & & (b) & &
\end{array}$$

Consequently, all five statements (a)—(e) are equivalent. $\square$

In summary, Theorem 1 has shown that for a monotone AVI, the following five properties are equivalent: (a) existence of a nondegenerate solution,

(b) existence of a nonempty set of weak sharp minima for the gap minimization problem, (c) validity of an error bound in terms of the gap function alone, (d) a simplified representation of the solution set, and (e) validity of the restricted minimum principle sufficiency for the gap minimization problem.

We conclude this paper by giving an application of Theorem 1 that generalizes the classical result of Goldman and Tucker [17] mentioned in the opening of this paper.

Corollary 1 *Let $q \in R^n$ be arbitrary, $M \in R^{n \times n}$ be positive semidefinite, and $C \subseteq R^n$ be a polyhedral set. Suppose* $\mathrm{SOL}(q, M, C) \neq \emptyset$ *and* $\mathrm{FEA}(q, M, C)$ *is contained in the null space of $M + M^T$. Then the AVI (q, M, C) has a nondegenerate solution.*

Proof Since $\mathrm{FEA}(q, M, C)$ is contained in the null space of $M + M^T$, it follows that $x^T M x = 0$ for all $x \in \mathrm{FEA}(q, M, C)$. Thus the two constants, d and σ, of the AVI (q, M, C) are both equal to zero. Moreover, it is easy to verify that the right-hand set in (25) reduces to

$$\{x \in C \mid u^T(q + Mx) - x^T(q + Mx) \geq 0, \text{ for all } u \in C\},$$

which is exactly $\mathrm{SOL}(q, M, C)$. Thus, property (d) of Theorem 1 holds, and the corollary is established. $\square$

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