THE EXTENDED LINEAR COMPLEMENTARITY PROBLEM*

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Abstract. We consider an extension of the horizontal linear complementarity problem, which we call the extended linear complementarity problem (XLCP). With the aid of a natural bilinear program, we establish various properties of this extended complementarity problem; these include the convexity of the bilinear objective function under a monotonicity assumption, the polyhedrality of the solution set of a monotone XLCP, and an error bound result for a nondegenerate XLCP. We also present a finite, sequential linear programming algorithm for solving the nonmonotone XLCP.

Key words. Complementarity problems, monotonicity, error bound, bilinear program.

AMS subject classifications. 90C30, 90C33

1. Introduction. In the past couple years, the horizontal linear complementarity problem (HLCP) has received an increasing amount of attention among researchers interested in the family of interior-point methods for solving linear programs and complementarity problems. This surge of interest originates from an article by Zhang [17] who used the HLCP as a unifying framework for the convergence analysis of a class of so-called "infeasible-interior-point algorithms". Subsequent work in this area includes [2, 7, 12, 14]. Independently, Sznajder and Gowda [13] have studied some matrix-theoretic properties and their roles in the horizontal and vertical LCPs. Inspired by this flurry of activities and other applications (like the one described in [4, 15]), we became interested in undertaking a further study of the HLCP. In particular, our goal in this paper is twofold: one, to derive some basic results of the HLCP along the line of the classical LCP [3]; and two, to present an alternative solution method for the HLCP (particularly, for the "nonmonotone" problems).

The problem we shall study in this paper is defined as follows. Let M and N be two real matrices of order $m \times n$, and let C be a polyhedral set in R^m. The extended linear complementarity problem, which we shall denote XLCP (M, N, C), is to find a pair of vectors $(x, y) \in R^{2n}_+$ such that

$$Mx - Ny \in C, \quad x \perp y,$$

where the notation $x \perp y$ means that x is orthogonal to y, i.e., $x^T y = 0$. When $m = n$ and C consists of the single vector $p \in R^n$, this problem reduces to the HLCP that has motivated our work. In general, when

$$C = \{Lz + q : z \in R^\ell\}$$

for some matrix $L \in R^{m\times\ell}$ and vector $q \in R^m$, the XLCP (M, N, C) becomes the "general linear complementarity problem" studied by Ye [16]. However, Gowda [6] pointed out that the XLCP can also be obtained from Ye's general linear complementarity problem, and hence the two are equivalent. Ye proposed an interior point approach for this problem, whereas our approach is based on finite bilinear programming.

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The feasible region of XLCP (M, N, C) is denoted $\text{FEA}(M, N, C)$; it is defined to be the set

$$\text{FEA}(M, N, C) \equiv \{(x, y) \in R^{2n}_{+} : Mx - Ny \in C\},$$

which is a polyhedral subset of R^{2n}_{+}. We shall say that XLCP (M, N, C) is *feasible* if $\text{FEA}(M, N, C)$ is nonempty. The set of complementary solutions of the XLCP (M, N, C) is given by

$$\text{SOL}(M, N, C) \equiv \{(x, y) \in \text{FEA}(M, N, C) : x \perp y\}.$$

2. The Equivalent Bilinear Program. Associated with the XLCP (M, N, C) is a natural bilinear program defined on the same feasible region:

$$\begin{array}{ll} \text{minimize} & x^T y \\ \text{subject to} & (x, y) \in \text{FEA}(M, N, C). \end{array}$$

We shall denote this problem by BLP (M, N, C). The BLP (M, N, C) should be contrasted with the "natural" quadratic program that one associates with the standard LCP (q, M) which corresponds to the special case of the XLCP (M, N, C) with $m = n$, $N = I$, and $C = \{-q\}$. The latter quadratic program is [3]

$$\begin{array}{ll} \text{minimize} & x^T(q + Mx) \\ \text{subject to} & x \geq 0, \quad q + Mx \geq 0. \end{array} \tag{1}$$

One important distinction between the BLP (M, N, C) and the quadratic program (1) is that the latter is defined by the variable x only, whereas the former involves the pair (x, y). We shall see shortly that the BLP (M, N, C) plays a similar role in the study of the XLCP (M, N, C) as (1) in the LCP (q, M).

Since the objective function of BLP (M, N, C) is clearly nonnegative on $\text{FEA}(M, N, C)$, the XLCP (M, N, C) is equivalent to the BLP (M, N, C) in the sense that a pair of vectors (x, y) solves the former problem if and only if (x, y) is a globally optimal solution of the latter problem with a zero objective value. Moreover, by the well-known Frank-Wolfe Theorem of quadratic programming [5], the BLP (M, N, C) always has an optimal solution provided that it is feasible. Of course, it is in general not necessary for an optimal solution of the BLP (M, N, C) to have zero objective value. In what follows, we shall establish several results that pertain to the relationship between the XLCP and the associated BLP.

Proposition 2.1. *Let M and N be $m \times n$ matrices and C a polyhedral set in R^m. The bilinear function $f(x, y) \equiv x^T y$ is convex on the set $FEA(M, N, C)$ if and only if the following implication holds:*

$$[\, (x^i, y^i) \in FEA(M, N, C), i = 1, 2 \,] \Rightarrow (x^1 - x^2)^T (y^1 - y^2) \geq 0. \tag{2}$$

Proof. By an easy calculation, it can be verified that the following identity holds for any two pairs of vectors $(x^i, y^i) \in R^{2n}$ and any scalar τ,

$$\tau(x^1)^T y^1 + (1 - \tau)(x^2)^T y^2 - x(\tau)^T y(\tau) = \tau(1 - \tau)(x^1 - x^2)^T (y^1 - y^2),$$

where

$$\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = \tau \begin{pmatrix} x^1 \\ y^1 \end{pmatrix} + (1 - \tau) \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}.$$

Thus, the claimed equivalence follows easily. □

With the somewhat notorious reputation of the bilinear function, the above proposition is a pleasant surprise in that it exhibits an important instance in which the BLP (M, N, C) is actually a "convex program" (in the sense that it has a convex objective function on the feasible set). Indeed, when one specializes this result to the case of the standard LCP (q, M), one may conclude that if M is a positive semidefinite matrix, then the bilinear form $x^T y$ is a convex function on the set $\{(x, y) \in R^{2n}_+ : Mx - y = q\}$. This fact, though trivial to prove, seems to have been completely overlooked in the LCP literature.

To state the next result which gives a sufficient condition for every Karush-Kuhn-Tucker (KKT) vector of the (general) BLP (M, N, C) to be a solution of the XLCP (M, N, C), we recall that a matrix $L \in R^{m\times m}$ is copositive on a cone $K \subseteq R^m$ if $u^T Lu \geq 0$ for every $u \in K$. Also, we denote the recession cone of the set C by 0^+C; finally, the dual cone of a set $S \subseteq R^m$ is denoted S^*.

PROPOSITION 2.2. *Let M and N be $m \times n$ matrices and C a polyhedral set in R^m. If the matrix $MN^T \in R^{m\times m}$ is copositive on $(0^+C)^*$, then every KKT vector of the BLP (M, N, C), if it exists, solves the XLCP (M, N, C). Thus, if in addition FEA$(M, N, C) \neq \emptyset$, then SOL$(M, N, C) \neq \emptyset$.*

Proof. Without loss of generality, we shall represent the set C in the following form:

$$C = \{u \in R^m : Au \geq b\},$$

for some matrix $A \in R^{\ell\times m}$ and vector $b \in R^\ell$. Then we have

$$(0^+C)^* = \{v \in R^m : v = A^T\lambda \text{ for some } \lambda \in R^\ell_+\},$$

and the BLP (M, N, C) becomes

$$\begin{array}{ll} \text{minimize} & x^T y \\ \text{subject to} & AMx - ANy \geq b \\ & (x, y) \geq 0. \end{array}$$

Now, if (x, y) is a KKT vector of the BLP (M, N, C), then there exist nonnegative vectors $\lambda \in R^\ell$, and $(r, s) \in R^{2n}$ such that

$$y = M^T A^T \lambda + r, \quad x = -N^T A^T \lambda + s$$
$$x^T r = y^T s = \lambda^T (AMx - ANy - b) = 0.$$

Clearly, we have

$$\begin{aligned} x^T y &= x^T y - x^T r - s^T y + s^T r - s^T r \\ &= (x - s)^T (y - r) - s^T r = -\left((\lambda^T A)MN^T(A^T\lambda) + r^T s\right) \leq 0 \end{aligned}$$

where the last inequality follows from the copositivity of MN^T on $(0^+C)^*$. Since $x^T y$ is nonnegative, it follows that $(x, y) \in \mathrm{SOL}(M, N, C)$.

The last assertion of the proposition holds because the BLP (M, N, C) must have an optimal solution if it is feasible, and such a minimum solution must also be a complementary solution by what has just been proved. □

In order to combine the above two propositions, we establish a lemma which gives a sufficient condition for the matrix MN^T to be positive semidefinite (hence copositive on any cone).

LEMMA 2.3. *Let C be a polyhedron in R^n and let M and N be square matrices of order n. If $FEA(M, N, C) \neq \emptyset$ and the pair (M, N) satisfies the condition:*

$$\text{(3)} \qquad [\, Mx^i - Ny^i \in C, i = 1, 2\,] \Rightarrow (x^1 - x^2)^T(y^1 - y^2) \geq 0,$$

then MN^T is positive semidefinite.

Proof. We first show that the following implication holds:

$$\text{(4)} \qquad Mx - Ny \in 0^+C \Rightarrow x^T y \geq 0.$$

Indeed, let (x, y) be a pair of vectors satisfying $Mx - Ny \in 0^+C$. For an arbitrary pair of vectors $(\bar{x}, \bar{y}) \in \text{FEA}(M, N, C)$, we have $Mx_\tau - Ny_\tau \in C$ for every scalar $\tau \geq 0$ where $(x_\tau, y_\tau) \equiv (\bar{x}, \bar{y}) + \tau(x, y)$. By the implication (3), it follows easily that $x^T y \geq 0$. Since the origin is always an element in the recession cone, it follows that

$$Mx - Ny = 0 \Rightarrow x^T y \geq 0.$$

Hence, (M, N) is a column monotone pair in the sense defined in [13]. In particular, by Theorem 11 in this reference, it follows that MN^T is positive semidefinite. □

Remark. We need to assume that M and N are square in order to apply the result in [13] to deduce the positive semidefiniteness of MN^T from the column monotonicity of (M, N).

When $m = n$ and C is a singleton, condition (3) is equivalent to the column monotonicity of the pair (M, N) in the following sense. If (M, N) is column monotone, then (3) holds for $C = \{q\}$ for every n-vector q; conversely, by the proof of Lemma 2.3, if (3) holds for $C = \{q\}$ for some n-vector q belonging to the column space of (M, N), then the pair (M, N) must be column monotone. According to [13, Theorem 11], the column monotonicity of (M, N) is in turn equivalent to two conditions: (i) $M + N$ is nonsingular, and (ii) MN^T is positive semidefinite. By this characterization, it is easy to construct pairs of matrices (M, N) for which MN^T is positive semidefinite but (M, N) is not column monotone. If (M, N) is such a pair of matrices, then with an appropriately defined vector q, the BLP $(M, N, \{q\})$ will have the property that every one of its KKT points is a solution of the XLCP $(M, N, \{q\})$ but the BLP itself is not a convex program. A pair of matrices (M, N) with the property that MN^T is positive semidefinite will be called a *monotone product pair*. Unlike a column monotone pair, a monotone product pair (M, N) need not contain any nonsingular column representative matrix (as defined in [13]). Incidentally, Ye [16] showed that if (M, N) is a monotone product pair, then his potential reduction algorithm will compute a solution of the XLCP (M, N, C) in polynomial time. His results also provided a proof that in this case, the feasibility of the XLCP implies its solvability. This conclusion is a special case of Proposition 2.2.

3. Monotone Problems. We say that a pair of $n \times n$ matrices (M, N) is *monotone* with respect to the polyhedral set $C \subseteq R^n$, or in short, (M, N, C) is a *monotone triple*, if the implication (3) holds. (Note that this definition requires that M and N be square.) Summarizing the discussion in the last section, we may state the following result for a XLCP with a monotone triple (M, N, C).

THEOREM 3.1. *Let C be a nonempty polyhedron in R^n and let M and N be square matrices of order n. Suppose that (M, N) is monotone with respect to C and that $FEA(M, N, C) \neq \emptyset$. Then the following statements hold:*

(a) the bilinear function $x^T y$ is convex on $FEA(M, N, C)$;

(b) $SOL(M, N, C) \neq \emptyset$ and $SOL(M, N, C)$ is a polyhedron.

Proof. Only the polyhedrality of $\text{SOL}(M, N, C)$ requires a proof. We observe that $\text{SOL}(M, N, C)$ is a convex set by (a). Since the BLP (M, N, C) is a quadratic program and the set of optimal solutions of any quadratic program is equal to the union of a finite number of convex polyhedra [10], the convexity of SOL(M, N, C) must imply its polyhedrality. $\square$

Under the assumptions of Theorem 3.1, it is possible to give an explicit (polyhedral) representation for SOL(M, N, C). Instead of presenting such an expression in its fullest generality, we shall devote the remainder of this section to discuss the HLCP which has $C = \{-q\}$. For this case, we first introduce a special set associated with a column monotone pair. (A remark: Although the next three results can be proved by invoking the close connection between a column monotone pair and a positive semidefinite matrix, our derivation is more direct and reveals some interesting features of the HLCP.)

PROPOSITION 3.2. *Let (M, N) be a column monotone pair of $n \times n$ matrices. Let*

$$\mathcal{K}(M, N) \equiv \{(u, v) \in R^{2n} : Mu - Nv = 0, u \perp v\}.$$

Then $(u, v) \in \mathcal{K}(M, N)$ if and only if there exists a vector λ in the null space of $MN^T + NM^T$ such that

$$u = -N^T\lambda, \quad \text{and} \quad v = M^T\lambda. \tag{5}$$

Thus, $\mathcal{K}(M, N)$ is a linear subspace of R^{2n}.

Proof. The column monotonicity of (M, N) implies that $(\bar{u}, \bar{v}) \in \mathcal{K}(M, N)$ if and only if $(\bar{u}, \bar{v})$ is an optimal solution of the (equality constrained) quadratic program:

$$\begin{aligned} &\text{minimize} \quad u^Tv \\ &\text{subject to} \quad Mu - Nv = 0, \end{aligned}$$

and $\bar{u}^T\bar{v} = 0$. Thus, if $(\bar{u}, \bar{v}) \in \mathcal{K}(M, N)$, then there must exist a vector λ such that $\bar{u} = -N^T\lambda$ and $\bar{v} = M^T\lambda$. Moreover, we must have

$$\lambda^T MN^T\lambda = -\bar{u}^T\bar{v} = 0.$$

Since MN^T is positive semidefinite, it follows that $(MN^T + NM^T)\lambda = 0$. The converse is easily proved. From this characterization of the set $\mathcal{K}(M, N)$, it follows trivially that this set must be a linear subspace. $\square$

In the next result, we shall give two representations of the solution set of the "monotone" HLCP:

$$\begin{aligned} &Mx - Ny + q = 0 \\ &(x, y) \geq 0, \quad x \perp y, \end{aligned} \tag{6}$$

where (M, N) is a column monotone pair. One representation is valid in general, and the other is valid in the case when the problem is *nondegenerate*, i.e. when it has a solution $(\bar{x}, \bar{y})$ satisfying $\bar{x} + \bar{y} > 0$. Throughout the remainder of the paper, we shall write (M, N, q) for $(M, N, \{-q\})$.

PROPOSITION 3.3. *Let (M, N) be a column monotone pair of $n \times n$ matrices and let $(x^0, y^0) \in SOL(M, N, q)$ be arbitrary. Then*

$$\begin{aligned} SOL(M, N, q) = \{ \; &(x, y) \in FEA(M, N, q) : \\ &x^Ty^0 + y^Tx^0 = 0, (x, y) \in (x^0, y^0) + \mathcal{K}(M, N)\}. \end{aligned} \tag{7}$$

If the HLCP (M, N, q) is nondegenerate, then

$$SOL(M, N, q) = \{(x, y) \in FEA(M, N, q) : x^Ty^0 + y^Tx^0 = 0\}. \tag{8}$$

Proof. Since $(x^0)^T y^0 = 0$, we may write

$$x^T y = x^T y^0 + y^T x^0 + (x - x^0)^T (y - y^0).$$

By the column monotonicity of (M, N), it follows that $(x, y) \in \mathrm{SOL}(M, N, q)$ if and only if $(x, y) \in \mathrm{FEA}(M, N, q)$, $x^T y^0 + y^T x^0 = 0$, and $(x - x^0)^T (y - y^0) = 0$, or by Proposition 3.2, if and only if (x, y) belongs to the right-hand set in (7).

Suppose that the HLCP (M, N, q) is nondegenerate. It suffices to verify that the right-hand set in (8) is contained in $\mathrm{SOL}(M, N, q)$. Take any vector (x, y) belonging to this right-hand set. Let $(\bar{x}, \bar{y})$ be a nondegenerate solution of the HLCP (M, N, q); then $(x^0, y^0) \in (\bar{x}, \bar{y}) + \mathcal{K}(M, N)$. Since $(x, y) \in \mathrm{FEA}(M, N, q)$, we can verify, by the characterization of the set $\mathcal{K}(M, N)$ in Proposition 3.2, that

$$x^T y^0 + y^T x^0 = x^T \bar{y} + y^T \bar{x}. \tag{9}$$

Indeed, for some vector λ, we have

$$x^0 = \bar{x} - N^T \lambda$$
$$y^0 = \bar{y} + M^T \lambda.$$

Multiplying the first equation by $(y - y^0)^T$ and the second equation by $(x - x^0)^T$, adding the resulting equations, and using the fact that $(x^0)^T y^0 = \bar{x}^T y^0 + \bar{y}^T x^0 = 0$ and $M(x - x^0) - N(y - y^0) = 0$, we immediately deduce the desired equation (9). Consequently, we have $x^T \bar{y} + y^T \bar{x} = 0$ which easily implies $x^T y = 0$ by the nondegeneracy of the solution $(\bar{x}, \bar{y})$. □

The polyhedral representations (7) and (8) allow us to obtain some error bounds for the monotone HLCP. (The polyhedrality of (7) follows from Proposition 3.2.) Although some such bounds have been obtained in [9] for the general HLCP, they are valid only for test vectors that lie in a compact set. In what follows, we shall use (8) to obtain a sharpened error bound for the nondegenerate, monotone, HLCP.

COROLLARY 3.4. *Let (M, N) be a column monotone pair of $n \times n$ matrices. If the HLCP (M, N, q) has a nondegenerate solution, then there exists a constant $\sigma > 0$, dependent on (M, N, q), such that for all $(x, y) \in FEA(M, N, q)$,*

$$dist((x, y), SOL(M, N, q)) \leq \sigma x^T y,$$

where "dist" denotes the distance (measured by any norm) from a vector to a set.

Proof. It suffices to apply the well-known error bound for polyhedra [8, 11] to the representation (8) and to note that for any solution $(x^0, y^0) \in \mathrm{SOL}(M, N, q)$ and feasible vector $(x, y) \in \mathrm{FEA}(M, N, q)$, we have

$$x^T y^0 + y^T x^0 = x^T y - (x - x^0)^T (y - y^0) \leq x^T y.$$

This establishes the corollary. □

4. A Finite SLP Algorithm. We now return to the general XLCP (M, N, C). The bilinear programming formulation of this problem allows us to compute a solution by solving a finite sequence of linear programs (SLP) when the triple (M, N, C) satisfies the assumptions of Proposition 2.2. Since these assumptions are considerably more general than the column monotonicity property (for one thing, M and N need not be square matrices), the SLP procedure is applicable to a broader class of XLCPs than the (square) monotone class.

The algorithm described below was formulated in [1] and its finite termination was established for bilinear programs, not necessarily convex. We shall rephrase the algorithm for the BLP (M, N, C) and use the convergence results from the reference to establish its finite termination. In essence, this algorithm is a modified Frank-Wolfe algorithm for solving the BLP (M, N, C) as a quadratic program, whose convergence was originally proved for convex functions [5].

An SLP Algorithm. Start with any feasible $(x^0, y^0) \in \text{FEA}(M, N, C)$. In general, determine (x^{i+1}, y^{i+1}) from (x^i, y^i) as follows:

- Let (u^i, v^i) be a vertex optimal solution of the linear program:

$$\begin{array}{ll} \text{minimize} & x^T y^i + y^T x^i \\ \text{subject to} & (x, y) \in \text{FEA}(M, N, C). \end{array}$$

- Stop if $(u^i)^T y^i + (v^i)^T x^i = 2(x^i)^T y^i$.
- Otherwise, let

$$\begin{pmatrix} x^{i+1} \\ y^{i+1} \end{pmatrix} = (1 - \tau_i) \begin{pmatrix} x^i \\ y^i \end{pmatrix} + \tau_i \begin{pmatrix} u^i \\ v^i \end{pmatrix},$$

where

$$\tau_i \in \text{argmin}_{\tau \in [0,1]} (x^i + \tau(u^i - x^i))^T (y^i + \tau(v^i - y^i)).$$

Theorem 4.1. *Let M and N be $m \times n$ matrices and C a polyhedral set in R^m. Suppose $FEA(M, N, C) \neq \emptyset$. If the BLP (M, N, C) has the property that every one of its KKT points solves the XLCP (M, N, C), then in a finite number of iterations, the above algorithm will produce a vertex $(u^i, v^i) \in FEA(M, N, C)$ satisfying $(u^i)^T v^i = 0$.*

Proof. Note that the sequence $\{(x^i, y^i)\}$ generated by the SLP algorithm is bounded because it lies in the convex hull of the vertices of $\text{FEA}(M, N, C)$ and (x^0, y^0). Hence $\{(x^i, y^i)\}$ has at least one accumulation point $(\bar{x}, \bar{y})$ which must satisfy the minimum principle necessary optimality condition [1, Theorem A.1], and hence the KKT conditions for the BLP (M, N, C). By assumption, it follows that $(\bar{x}, \bar{y})$ solves the XLCP (M, N, C). Consequently, $\bar{x}^T \bar{y} = 0$. By [1, Theorem A.2], a vertex (u^i, v^i) generated by the SLP algorithm solves the BLP (M, N, C) with zero minimum. Hence this vertex also solves the XLCP (M, N, C). $\square$

4.1. Sufficient pairs of matrices. Specializing Theorem 4.1 to the HLCP (M, N, q), we obtain the following corollary.

Corollary 4.2. *Let (M, N) be a monotone product pair of $n \times n$ matrices. If the HLCP (M, N, q) is feasible, then in a finite number of iterations, the SLP algorithm will produce a vertex solution of this HLCP.*

Inspired by the class of (row/column) sufficient matrices [3, Section 3.5], we can broaden the class of matrix pairs (M, N) for which the above corollary is valid. Specifically, we say that a pair of $n \times n$ matrices (M, N) is *row sufficient* if the following implication holds: with $A \equiv (M, N) \in R^{n \times 2n}$,

$$\left[\begin{pmatrix} u \\ v \end{pmatrix} \in \text{range } A^T, \ u \circ v \leq 0 \right] \Rightarrow u \circ v = 0,$$

where "range" denotes the column space of a matrix and $\circ$ denotes the Hadamard product of two vectors; i.e., $x \circ y$ is the vector whose components are the products of the corresponding components of x and y. Similarly, (M, N) is said to be *column sufficient* if the following implication holds: with $\tilde{A} \equiv (M, -N) \in R^{n\times 2n}$,

$$\left[\begin{pmatrix} u \\ v \end{pmatrix} \in \text{null } \tilde{A}, \ u \circ v \leq 0 \right] \Rightarrow u \circ v = 0,$$

where "null" denotes the null space of a matrix. Finally, the pair (M, N) is said to be *sufficient* if it is both row and column sufficient.

While a monotone product pair must be row sufficient but not necessarily column sufficient, a column monotone pair must be (both row and column) sufficient. The role played by the (row/column) sufficient pairs in the HLCP is similar to that by the (row/column) sufficient matrices in the standard LCP. For the sake of completeness, we state the following characterization result for the HLCP.

THEOREM 4.3. *Let (M, N) be a pair of $n \times n$ matrices.*

(a) The pair (M, N) is row sufficient if and only if for every vector $q \in R^n$ for which the HLCP (M, N, q) is feasible, every KKT vector of the BLP (M, N, q) solves the HLCP (M, N, q).

(b) The pair (M, N) is column sufficient if and only if for every vector $q \in R^n$, the solution set of the HLCP (M, N, q), if nonempty, is convex.

Proof. Assume that (M, N) is a row sufficient pair. Suppose that (x, y) is a KKT vector of the BLP (M, N, q). Then there exist vectors $\lambda \in R^n$, and $(r, s) \in R^{2n}_+$ such that

$$y = M^T\lambda + r, \quad x = -N^T\lambda + s$$
$$x^T r = y^T s = 0.$$

By a similar derivation as in the proof of Proposition 2.2, we can show that

$$x \circ y = -((M^T\lambda) \circ (N^T\lambda) + r \circ s).$$

Thus, $(M^T\lambda) \circ (N^T\lambda) \leq 0$. The row sufficiency of (M, N) therefore implies that $(M^T\lambda) \circ (N^T\lambda) = 0$ which in turn yields $x \circ y = 0$.

To prove the converse in (a), suppose that the pair (M, N) is not row sufficient. Then, for some vector $\lambda \in R^n$, we have $(M^T\lambda) \circ (N^T\lambda) \leq 0$ and $(M^T\lambda)_i(N^T\lambda)_i < 0$ for at least one component i. Without loss of generality, we may assume that $(M^T\lambda)_i > 0$ and $(N^T\lambda)_i < 0$. Let

$$y \equiv (M^T\lambda)^+, \quad r \equiv (M^T\lambda)^-$$
$$s \equiv (N^T\lambda)^+, \quad x \equiv (N^T\lambda)^-$$

where v^+ and v^- denote, respectively, the nonnegative and nonpositive part of a vector v. Also let $q = Ny - Mx$. It is then easy to verify that (x, y) is a KKT vector of the BLP (M, N, q) with (r, s) as the corresponding multipliers; nevertheless, x is not complementary to y. Thus (a) holds.

To prove (b), suppose the pair (M, N) is column sufficient. Let (x^i, y^i) for $i = 1, 2$ be two solutions of the HLCP (M, N, q). It is then easy to verify for all components $k = 1, \ldots, n$, we have

$$(x^1 - x^2)_k(y^1 - y^2)_k = -(x^1_k y^2_k + x^2_k y^1_k) \leq 0.$$

Since we also have $M(x^1 - x^2) - N(y^1 - y^2) = 0$, it follows that $x^1_k y^2_k = x^2_k y^1_k = 0$ for all k. In turn, this easily implies that

$$(\tau x^1 + (1-\tau)x^2)^T(\tau y^1 + (1-\tau)y^2) = 0$$

for all $\tau \in [0,1]$. Thus, the convexity of SOL(M,N,q) follows. Conversely, suppose that (M,N) is not column monotone. Then there exists a vector $(x,y) \in R^{2n}$ satisfying $Mx - Ny = 0, x \circ y \leq 0$, and $x_i y_i < 0$ for at least one index i. Let

$$-q \equiv Mx^+ - Ny^+ = Mx^- - Ny^-.$$

It is then easy to verify that (x^+, y^+) and (x^-, y^-) are solutions of the HLCP (M,N,q) but that these solutions are not "cross complementary"; i.e., either $(x^+)^T y^- > 0$ or $(x^-)^T y^+ > 0$. The latter cross complementarity property is easily seen to be both necessary and sufficient for the solution set of any HLCP to be convex. $\square$

It follows immediately from Corollary 4.2 and Theorem 4.3 that if (M,N) is a row sufficient pair, then the SLP algorithm will compute a solution to the HLCP (M,N,q) for every q for which FEA$(M,N,q) \neq \emptyset$.

In [13], a pair of square matrices (M,N) was defined to be *row monotone* if (M^T, N^T) is column monotone. We have previously mentioned that a column monotone pair must be (column and row) sufficient. Nevertheless, a row monotone pair need not be either column or row sufficient. Indeed, borrowing from [13, Example 10], let us consider the pair

$$M = \begin{bmatrix} 3/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

which is obtained by transposing respectively the matrices C and D in the cited example. The pair (M,N) is row monotone because as shown in the reference (C,D) is column monotone. But the pair (M,N) is neither column nor row sufficient. Column sufficiency is violated with

$$u = (2,0), \quad v = (-1,5);$$

whereas row sufficiency is violated with

$$u = (-1/2, -1/2), \quad v = (1,0).$$

The reason for this dichotomy is that the definition of row monotonicity in [13] was relevant for the *vertical* LCP and was not shown to have any relation to the HLCP. On the other hand, the column and row sufficiency defined herein have direct implications for the HLCP. Thus, it is not surprising that these (column/row) sufficiency and monotonicity concepts for matrix pairs would be quite different. The reader is referred to [6] for more discussion on these concepts.

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