Error Bounds for Inconsistent Linear Inequalities and Programs

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Revised January 18, 1994

Abstract

For any system of linear inequalities, consistent or not, the norm of the violations of the inequalities by a given point, multiplied by a condition constant that is independent of the point, bounds the distance between the point and the nonempty set of points that minimize these violations. Similarly, for a dual pair of possibly infeasible linear programs, the norm of violations of primal-dual feasibility and primal-dual objective equality, when multiplied by a condition constant, bounds the distance between a given point and the nonempty set of minimizers of these violations. These results extend error bounds for consistent linear inequalities and linear programs to inconsistent systems.

Keywords error bounds; linear inequalities; linear programs

Error bounds are playing an increasingly important role in mathematical programming. Beginning with Hoffman's classical error bound for linear inequalities [3], many papers have examined error bounds for linear and convex inequalities, linear and nonlinear programs, as well as linear and nonlinear complementarity problems and affine variational inequalities [28, 29, 17, 26, 24, 23, 20, 14, 18, 8, 10, 11, 22, 8, 21, 16, 9, 19, 5, 2]. These error bounds can be traced to the following classical elementary error bound for an arbitrary point in the n-dimensional real space relative to a nonsingular system of equations $Bx = c$ with solution $\bar{x} = B^{-1}c$:

$$\|x - \bar{x}\| = \|x - B^{-1}c\| = \|B^{-1}(Bx - c)\| \leqq \|B^{-1}\| \cdot \|Bx - c\|$$

Here $\|\cdot\|$ denotes some norm, and the residual $\|Bx-c\|$, when multiplied by the "condition constant" $\|B^{-1}\|$, bounds the distance between x and the solution $\bar{x}$. In a similar, but considerably more general manner, mathematical programming research has concentrated on establishing bounds for computable residuals for various problems. Such residuals evaluated at an arbitrary point, bound a "distance" to the solution set from the given point. Essentially all such research has assumed that the underlying mathematical programming problem, just like $Bx = c$ with nonsingular B, is solvable. In this work we shall not make this rather restrictive assumption, because we want to handle more general situations that correspond to a singular or even rectangular matrix B, for which no exact solution may exist.

In line with this objective, we will begin by showing, for the possibly inconsistent system of linear inequalities

$$Ax \leqq b, \tag{1}$$

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that the residual

$$\|(Ax-b)_+\|, \tag{2}$$

when multiplied by a condition constant $\sigma(A)$, bounds the distance to a closest point in the set of points that minimize some norm of $(Ax-b)_+$. Here A is an $m \times n$ real matrix, b is a vector in the m-dimensional real space R^m, $\|\cdot\|$ denotes any norm on R^m, and $(Ax-b)_+$ denotes the vector $(Ax-b)$ with all its negative components replaced by zeros. When the system (1) is consistent, it is well known [3, 28, 17, 24, 4] that

$$\|x-p(x)\|_\infty \leqq \sigma(A)\|(Ax-b)_+\| \tag{3}$$

Here the projection $p(x)$ is a closest point (using the ∞-norm for simplicity) in the solution set

$$X = \{x | Ax \leqq b\} \tag{4}$$

to the point x and $\sigma(A)$ is the condition constant [24]

$$\sigma(A) := \max \left\{ \|w\|' \;\middle|\; \begin{array}{l} \|A^T w\|_1 = 1,\ w \geqq 0,\ \text{rows of } A \\ \text{corresponding to nonzero elements} \\ \text{of } w \text{ are linearly independent} \end{array} \right\} \tag{5}$$

Here the superscript T denotes the transpose, and $\|\cdot\|'$ is the dual norm to the norm on R^m used in (3), that is $\|u\|' := \max\limits_{\|v\|=1} v^T u$. The norms $\|u\|_p$ and $\|u\|_q$ are dual norms for $1 \leqq p,\ q \leqq \infty,\ \frac{1}{p}+\frac{1}{q}=1$. It is interesting to note that if the constraint $w \geqq 0$ in (5) is omitted, and the norm $\|(Ax-b)_+\|$ in (3) is taken as the ∞-norm, then $\sigma(A) = \|A^{-1}\|_\infty$, for nonsingular A. Error bounds such as (3) have become an important part of the mathematical programming literature. They play an important role as stopping criteria for iterative algorithms as well as in obtaining convergence rates for these algorithms [13, 10, 12, 14, 15, 6, 7, 5]. Our principal concern here is the following question. Do standard residuals, such as (2), signify anything if the solution set is empty? This is not an idle question, since a given problem may not be well posed or its data may have been corrupted by noise so that it does not have a solution. In such cases it would be useful to know whether the residual contains any useful information. Curiously it turns out that it does. In fact we will show that in such cases, the residual will bound the distance to the always-nonempty set of points that minimizes some norm of the violations of (1). In order to handle the case of an empty feasible region X, we define the nonempty set X^1 of minimizers of $\|(Ax-b)_+\|_1$, that is

$$X^1 := \arg\min_x \|(Ax-b)_+\|_1 \tag{6}$$

We also need to define a new condition constant as follows:

$$\tau(A) := \max_{w,\gamma,s} \left\{ \|w\|' \;\middle|\; \begin{array}{l} \|A^T w\|_1 = 1,\ w - e\gamma + s = 0,\ (w,\gamma,s) \geqq 0, \\ \text{rows of } \begin{bmatrix} A & I \\ 0 & -e^T \\ 0 & I \end{bmatrix} \text{corresponding to nonzero} \\ \text{components of } (w,\gamma,s) \text{ are linearly independent} \end{array} \right\} \tag{7}$$

Here I denotes the identity matrix and e a vector of ones, both of appropriate dimension. We immediately note that $\tau(A)$ is a well defined finite real number. In fact the set of feasible (w,γ,s) satisfying the constraints of (7) is compact. Obviously, the set is closed. It is bounded, otherwise there would exist fixed subsets J and K of $\{1,\ldots,m\}$ and a sequence $\{(w^i_J,\, \gamma^i,\, s^i_K)\}$ such that $\{\|w^i_J,\, \gamma^i,\, s^i_K\|\} \to \infty$ and rows of $\begin{bmatrix} A_J & I_J \\ 0 & -e^T \\ 0 & I_K \end{bmatrix}$ are linearly independent, where subscripts J and K denote subsets of rows. Hence a subsequence $\left\{\dfrac{(w^{i_j}_J, \gamma^{i_j}, s^{i_j}_K)}{\|w^{i_j}_J, \gamma^{i_j}, s^{i_j}_K\|}\right\}$ converges to $(\bar{w}_J,\, \bar{\gamma},\, \bar{s}_K) \neq 0$ satisfying $\bar{w}_J^T A_J = 0,\ \bar{w}_J^T I_J - \bar{\gamma} e^T + \bar{s}_K^T I_K = 0$, which contradicts the linear independence of the rows of $\begin{bmatrix} A_J & I_J \\ 0 & -e^T \\ 0 & I_K \end{bmatrix}$.

We are ready now to state and prove our principal result.

Theorem 1 ***(Error bound for possibly inconsistent linear inequalities)*** *For any x in R^n*

$$\|x - p_1(x)\|_\infty \leqq \tau(A)\|(Ax-b)_+\| \tag{8}$$

where $p_1(x)$ is the projection of x (using the ∞-norm) on the error minimizing set X^1, and $\|\cdot\|$ is an arbitrary norm on R^m.

Proof Let $\bar{z} = (A\bar{x} - b)_+$ where $\bar{x}$ is any point in the nonempty set X^1, and let x be an arbitrary fixed point in R^n. Hence $p_1(x)$ is a constituent of the solution $(p_1(x),\, \varepsilon(x),\, z(x))$ of the following solvable linear program:

$$(p_1(x),\, \varepsilon(x),\, z(x)) \in \arg\min_{p,\varepsilon,z} \left\{ \varepsilon \;\middle|\; \begin{array}{c} Ap - b \leqq z,\ z \geqq 0,\ e^T z \leqq e^T \bar{z} \\ -e\varepsilon \leqq p - x \leqq e\varepsilon \end{array} \right\} \tag{9}$$

The dual of this linear program is solved by some $(w(x),\, \gamma(x),\, u(x),\, v(x)\; s(x))$, that is

$$(w(x), \gamma(x), u(x), v(x), s(x)) \in \arg\max_{w,\gamma,u,v,s} \left\{ -b^T w - (e^T\bar{z})\gamma + x^T(u-v) \;\middle|\; \begin{array}{r} A^T w - u + v = 0 \\ w - e\gamma + s = 0 \\ e^T u + e^T v = 1 \\ w, u, v, \gamma, s \geqq 0 \end{array} \right\} \tag{10}$$

Assuming that $\varepsilon(x) > 0$, else $x - p_1(x) = 0$ and (8) holds trivially, it follows from the complementarity condition

$$u(x)(p_1(x) + e\varepsilon(x) - x) + v(x)(-p_1(x) + e\varepsilon(x) + x) = 0$$

that $u(x)v(x) = 0$. Hence from the constraint conditions

$$A^T w(x) - u(x) + v(x) = 0, \quad e^T u(x) + e^T v(x) = 1$$

we have that $\|A^T w(x)\|_1 = 1$. By the basic feasible solution theorem [25], there exists a solution $(w(x),\, \gamma(x),\, u(x),\, v(x),\, s(x))$ such that the columns of the matrix

$$\begin{bmatrix} A^T & 0 & -I & I & 0 \\ I & -e & 0 & 0 & I \\ 0 & 0 & e^T & e^T & 0 \end{bmatrix}$$

corresponding to nonzero components of $(w(x), \gamma(x), u(x), v(x), s(x))$ are linearly independent. Hence so are the columns of the matrix

$$\left[\begin{array}{ccc} A^T & 0 & 0 \\ I & -e & I \end{array}\right]$$

corresponding to nonzero components of $(w(x), \gamma(x), s(x))$. Consequently it follows from (7) that

$$\|w(x)\|' \leqq \tau(A)$$

We then have

$$\begin{array}{rcll} \varepsilon(x) = \|x - p_1(x)\|_\infty & = & -b^T w(x) - e^T \bar{z}\gamma(x) + x^T(u(x) - v(x)) & \\ & = & w(x)^T(Ax - b) - e^T \bar{z}\gamma(x) & \\ & \leqq & w(x)^T(Ax - b)_+ & \text{(Since } w(x) \geqq 0,\ e^T\bar{z} \geqq 0,\ \gamma(x) \geqq 0) \\ & \leqq & \|w(x)\|'\, \|(Ax - b)_+\| & \text{(By generalized Cauchy-Schwarz inequality)} \\ & \leqq & \tau(A)\, \|(Ax - b)_+\| & \end{array}$$

$\square$

We turn our attention now to the pair of dual linear programs

$$\begin{array}{ll} \max\limits_x c^T x & s.t.\ Ax \leqq b,\ x \geqq 0 \\ \min\limits_u b^T u & s.t.\ A^T u \geqq c,\ u \geqq 0 \end{array} \tag{11}$$

neither of which may be feasible. This pair is equivalent to the skew-symmetric linear complementarity problem (LCP)

$$Mz + q \geqq 0,\ z \geqq 0,\ z(Mz + q) = qz \leqq 0 \tag{12}$$

where

$$M = \begin{pmatrix} 0 & A^T \\ -A & 0 \end{pmatrix},\quad q = \begin{pmatrix} -c \\ b \end{pmatrix},\quad z = \begin{pmatrix} x \\ u \end{pmatrix} \tag{13}$$

$$Z^1 := \arg\min_z \left\{ \left\| \begin{pmatrix} -Mz - q \\ -z \\ qz \end{pmatrix}_+ \right\|_1 \right\} \tag{14}$$

By applying Theorem 1 to the LCP (12) representing the dual linear programs (11) we obtain the following error bound result.

Theorem 2 ***(Error bound for possibly infeasible linear programs)*** *For any $(x, u) \in R^{n+m}$*

$$\|(x, u) - p_1(x, u)\|_\infty \leqq \tau \begin{pmatrix} -M \\ -I \\ q \end{pmatrix} \|(Ax - b,\ -x,\ -A^T u + c,\ -u,\ -cx + bu)_+\| \tag{15}$$

Here $p_1(x, u)$ is the projection of $z = (x, u)$ (using the ∞-norm) on the error minimizing set Z^1 defined by (14), $\|\cdot\|$ is an arbitrary norm on $R^{2(m+n)+1}$ and τ is defined by (7).

We conclude by noting that the idea of an error bound for inconsistent linear inequalities derived here can be extended to unsolvable linear complementarity problems in a manner similar to that of [23, 22] for solvable LCPs. It may also be possible to establish convergence rates for iterative methods for approximately solving unsolvable LCPs similar to the results of [27, 10] for solvable LCPs. It is also worth noting that the error bound inequality (8) can be sharpened to the following, by using the 1-norm in (8), by not dropping the term $-e^T\bar{z}\gamma(x)$ from the string of inequalities at the end of the proof of Theorem 1, and by noting that $\gamma(x) \geqq \|w(x)\|_\infty$:

$$\|x - p_1(x)\|_\infty \leqq \tau(A)(\|(Ax-b)_+\|_1 - \|(A\bar{x}-b)_+\|_1), \quad \bar{x} \in X^1 \tag{16}$$

This inequality can then be interpreted as the residual $\|(Ax-b)_+\|_1$ having a weak sharp minimum in the sense of [1]. However this inequality is not useful as an error bound without dropping the unknown term $\|(A\bar{x}-b)_+\|_1$. Thus although our results could have been derived as a consequence of weak sharp minimum theory, our approach here gives an explicit expression (7) for the condition constant $\tau(A)$, which in general is not given by weak sharp minimum theory.

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