A HYBRID NEWTON METHOD FOR SOLVING BOX CONSTRAINED VARIATIONAL INEQUALITY PROBLEMS VIA THE D-GAP FUNCTION

Ji-Ming Peng[1,2], Christian Kanzow[3,4] and Masao Fukushima[5,6]

[1] State Key Laboratory of Scientific and Engineering Computing
Institute of Computational Mathematics and Scientific/Engineering Computing
Academic Sinica
P.O. Box 2719, Beijing 100080, China
e-mail: pjm@lsec.cc.ac.cn

[3] Institute of Applied Mathematics
University of Hamburg
Bundesstrasse 55
D-20146 Hamburg, Germany
e-mail: kanzow@math.uni-hamburg.de

[5] Department of Applied Mathematics and Physics
Graduate School of Engineering
Kyoto University
Kyoto 606-01, Japan
e-mail: fuku@kuamp.kyoto-u.ac.jp

December 30, 1997

Abstract. A box constrained variational inequality problem can be reformulated as an unconstrained minimization problem through the D-gap function. A hybrid Newton-type method is proposed for minimizing the D-gap function. Under suitable conditions, the algorithm is shown to be globally convergent and locally quadratically convergent. Some numerical results are also presented.

Key words: Variational inequality problem, box constraints, D-gap function, Newton's method, unconstrained optimization, global convergence, quadratic convergence.

[2]The research of this author was supported by Project 19601035 of NSFC in China.

[4]Current address (October 1, 1997 — September 30, 1998): Computer Sciences Department, University of Wisconsin — Madison, 1210 West Dayton Street, 53706 Madison, WI; e-mail: kanzow@cs.wisc.edu. The research of this author was supported by the DFG (Deutsche Forschungsgemeinschaft).

[6]The work of this author was supported in part by the Scientific Research Grant-in-Aid from the Ministry of Education, Science and Culture, Japan.

1 Introduction

Let F be a mapping from $\Re^n$ into itself and X be a nonempty closed convex subset of $\Re^n$. The variational inequality problem (VIP) is to find a vector $x^* \in X$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall\, y \in X, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\Re^n$. In this paper, we study the case that X is a box defined by

$$X = \{x \in \Re^n \mid l_i \leq x_i \leq u_i,\ i = 1, \ldots, n\}$$

where $l_i \in \Re \cup \{-\infty\}$ and $u_i \in \Re \cup \{+\infty\}$ with $l_i < u_i$ represent the lower and upper bounds on the variables, respectively. It is not diffult to see [3, Proposition 5.7] that the box structure of the set X enables us to rewrite (1) as

$$F_i(x^*)(y_i - x_i^*) \geq 0,\ i = 1, \ldots, n, \quad \forall\, y \in X. \tag{2}$$

If the constraint set X is the nonnegative orthant $\Re^n_+$, then the VIP reduces to the complementarity problem (CP). This class of special VIPs has numerous important applications in various fields such as mathematical programming, economics, and engineering; see [13, 7] and the references therein.

A useful way to deal with VIP (1) is to reformulate it first as a system of equations or an optimization problem via a merit function, and then solve the resultant system of equations or optimization problem. Recently, much attention has been paid on the reformulation of VIPs and CPs and various merit functions have been proposed and studied [10]. Well known merit functions for VIPs include the *gap function*

$$g(x) = \sup_{y \in X} \langle F(x), x - y \rangle$$

first presented by Auslender [2] and then studied by Marcotte and Dussault [18, 19], the *regularized gap function*

$$f_\alpha(x) = \max_{y \in X} \left\{ \langle F(x), x - y \rangle - \frac{\alpha}{2} \|y - x\|^2 \right\} \tag{3}$$

introduced by Fukushima [9] and Auchmuty [1], and the *D-gap function*

$$g_{\alpha\beta}(x) := f_\alpha(x) - f_\beta(x), \tag{4}$$

proposed by Peng [21] and Yamashita, Taji and Fukushima [27], where α and β are arbitrary positive parameters such that $\alpha < \beta$. Since a box constrained VIP is actually equivalent to a system of KKT mixed complementarity conditions, several merit functions based on the KKT system of VIP have also been proposed and explored extensively, see Qi [23] and the references therein.

In this paper, we focus our attention to the D-gap function for VIP. It is not difficult to see that $g_{\alpha\beta}(x) \geq 0$ for all $x \in \Re^n$, and $g_{\alpha\beta}(x) = 0$ if and only if x is a solution of the VIP (1). Therefore the VIP can be cast as the following unconstrained optimization problem:

$$\min_{x \in \Re^n}\ g_{\alpha\beta}(x). \tag{5}$$

When the mapping F is differentiable, the D-gap function $g_{\alpha\beta}$ is also differentiable [9, 27]. However, it is not twice differentiable in general. Therefore it is not straightforward to apply conventional second-order methods to problem (5). As a remedy for this inconvenience, Sun, Fukushima and Qi [25] introduced the concept of a computable generalized Hessian of the D-gap function $g_{\alpha\beta}$ and presented a Newton-type method for solving problem (5). Restricting themselves to the box constrained VIP, Kanzow and Fukushima [15] discussed a generalized Hessian of the D-gap function and proposed a Gauss-Newton-type method to minimize it. Further properties of the D-gap function have been investigated in [15, 25, 11].

This work is motivated by the recent paper [22] in which the D-gap function is used to globalize the classical Josephy-Newton method [14] for general VIPs. The main purpose of this work is to further study the algorithm proposed in [22] by restricting ourselves to box constrained VIPs, and test the effectiveness of the algorithm. The paper is organized as follows: In Section 2, we review some basic results that will be used in the paper. In Section 3, we present the algorithm and study its convergence properties. Some numerical results are presented in Section 4. Finally we conclude the paper with some remarks in Section 5.

2 Preliminaries

We first review some concepts related to the VIP and state some properties of the D-gap function defined by (4).

The mapping $F : \Re^n \to \Re^n$ is said to be a *P_0-function* if

$$\max_{\substack{1\le i\le n\\ x_i\neq y_i}} (x_i - y_i)(F_i(x) - F_i(y)) \ge 0 \quad \forall\ x, y \in \Re^n,\ x \neq y,$$

a *P-function* if

$$\max_{1\le i\le n} (x_i - y_i)(F_i(x) - F_i(y)) > 0 \quad \forall\ x, y \in \Re^n,\ x \neq y,$$

and a *uniform P-function (with modulus $\mu > 0$)* if

$$\max_{1\le i\le n} (x_i - y_i)(F_i(x) - F_i(y)) \ge \mu||x - y||^2, \quad \forall\ x, y \in \Re^n.$$

An $n \times n$ matrix M is a *P_0-matrix* if

$$\max_{\substack{1\le i\le n\\ z_i\neq 0}} z_i(Mz)_i \ge 0, \quad \forall\ z \in \Re^n, z \neq 0,$$

and a *P-matrix* if

$$\max_{1\le i\le n} z_i(Mz)_i > 0, \quad \forall\ z \in \Re^n,\ z \neq 0.$$

It is easy to see that, if M is a P-matrix, then there exists a constant $\mu > 0$ such that

$$\max_{1\le i\le n} z_i(Mz)_i \ge \mu\|z\|^2, \quad \forall\ z \in \Re^n. \tag{6}$$

It is known [20, Theorem 5.8] that if F is a differentiable P_0-function, then $\nabla F(x)^T$ is a P_0-matrix for each x. Moreover, if F is a differentiable uniform P-function with modulus $\mu > 0$, then $\nabla F(x)^T$ is a uniform P-matrix with modulus $\mu > 0$ in the sense that

$$\max_{1 \le i \le n} z_i [\nabla F(x)^T z]_i \ge \mu \|z\|^2, \quad \forall\, z \in \Re^n, \quad \forall\, x \in \Re^n. \tag{7}$$

Since we are not aware of any explicit reference for the formula (7), we include a short proof for it.

Lemma 2.1 *Let $F : \Re^n \to \Re^n$ be a differentiable uniform P-function with modulus $\mu > 0$. Then (7) holds.*

Proof Let $x \in \Re^n$ and $z \in \Re^n$ be arbitrary but fixed. Let $\{t_k\} \subseteq \Re$ be a sequence of positive numbers converging to 0. Since F is a uniform P-function and the index set $\{1, \ldots, n\}$ is finite, there exists an index $i_0 \in \{1, \ldots, n\}$ (independent of k) and a subsequence $\{t_k\}_K$ such that $z_{i_0} \neq 0$ and

$$t_k z_{i_0} \left[F_{i_0}(x + t_k z) - F_{i_0}(x)\right] \ge \mu t_k^2 \|z\|^2 \quad \forall k \in K.$$

Dividing this expression by t_k^2, taking the limit $k \to \infty$ $(k \in K)$ and using the assumed differentiability of F, we obtain

$$z_{i_0} [\nabla F(x)^T z]_{i_0} \ge \mu \|z\|^2.$$

This implies

$$\max_{1 \le i \le n} z_i [\nabla F(x)^T z]_i \ge \mu \|z\|^2$$

and completes the proof. $\Box$

The following lemma will play an important role in the analyses of this paper. Let X be a box, M be a P-matrix, and $b \in \Re^n$. For any $v \in \Re^n$, let $x(v) \in X$ denote the unique solution of the following affine VIP:

$$\langle b + v + Mx, y - x \rangle \ge 0, \quad \forall\, y \in X, \tag{8}$$

or equivalently

$$(b_i + v_i + [Mx]_i)(y_i - x_i) \ge 0, \; i = 1, \ldots, n, \quad \forall\, y \in X. \tag{9}$$

Lemma 2.2 *Let $x(v)$ be the unique solution of the affine VIP (8), where X is a box and M is a P-matrix. Let $\mu > 0$ be a constant satisfying (6). Then we have*

$$\|x(v) - x(v')\| \le \frac{1}{\mu} \|v - v'\|, \quad \forall\, v, v' \in \Re^n.$$

Proof Let $v, v' \in \Re^n$ be arbitrary. Since $x(v) \in X$, $x(v') \in X$, and X is a box, it follows from (9) that

$$(b_i + v_i + [Mx(v)]_i)(x_i(v') - x_i(v)) \geq 0, \quad i = 1, \ldots, n,$$

and

$$(b_i + v_i' + [Mx(v')]_i)(x_i(v) - x_i(v')) \geq 0, \quad i = 1, \ldots, n.$$

Adding the above two inequalities, we obtain

$$(v_i - v_i')(x_i(v') - x_i(v)) \geq (x_i(v) - x_i(v'))[M(x(v) - x(v'))]_i, \quad i = 1, \ldots, n. \tag{10}$$

Since M is a P-matrix, there exists a constant $\mu > 0$ and some index i_0 such that

$$(x_{i_0}(v) - x_{i_0}(v'))[M(x(v) - x(v'))]_{i_0} \geq \mu \|x(v) - x(v')\|^2.$$

This inequality together with (10) implies that

$$(v_{i_0} - v_{i_0}')(x_{i_0}(v') - x_{i_0}(v)) \geq \mu \|x(v) - x(v')\|^2. \tag{11}$$

Since

$$(v_{i_0} - v_{i_0}')(x_{i_0}(v') - x_{i_0}(v)) \leq \|v - v'\| \cdot \|x(v) - x(v')\|,$$

the inequality (11) yields

$$\|x(v) - x(v')\| \leq \frac{1}{\mu} \|v - v'\|.$$

This completes the proof of the lemma. □

Next we give another result on the continuity of a solution of the box constrained affine VIP with a P-matrix. For any P-matrix M and vectors $b, p \in \Re^n$, let $x(M) \in X$ denote the unique solution of the following affine VIP:

$$\langle b + Mp + Mx, y - x \rangle \geq 0, \quad \forall\, y \in X, \tag{12}$$

or equivalently

$$(b_i + [Mp]_i + [Mx]_i)(y_i - x_i) \geq 0, \; i = 1, \ldots, n, \quad \forall\, y \in X. \tag{13}$$

Lemma 2.3 *Suppose that M and N are P-matrices and X is a box. Let $\mu > 0$ be a constant satisfying (6). Then we have*

$$\|x(M) - x(N)\| \leq \frac{1}{\mu} \|M - N\| \cdot \|x(N) + p\|. \tag{14}$$

Proof The lemma trivially holds if $x(M) = x(N)$. Hence we only need to consider the case where $x(M) \neq x(N)$. Since X is a box, it follows from (13) and the definitions of $x(M)$ and $x(N)$ that

$$(b_i + [Mp]_i + [Mx(M)]_i)(x_i(N) - x_i(M)) \geq 0, \quad i = 1, \ldots, n,$$

and

$$(b_i + [Np]_i + [Nx(N)]_i)(x_i(M) - x_i(N)) \geq 0, \quad i = 1, \ldots, n.$$

Adding the above two inequalities, we get

$$([Mp]_i - [Np]_i + [Mx(M)]_i - [Nx(N)]_i)(x_i(N) - x_i(M)) \geq 0, \quad i = 1, \ldots, n,$$

which implies that

$$[(M - N)(x(N) + p)]_i(x_i(N) - x_i(M)) \geq [M(x(M) - x(N))]_i(x_i(M) - x_i(N)), \quad i = 1, \ldots, n. \tag{15}$$

Since M is a P-matrix, it follows from (6) and $x(M) - x(N) \neq 0$ that there exists an index i_0 such that

$$[M(x(M) - x(N))]_{i_0}(x_{i_0}(M) - x_{i_0}(N)) \geq \mu\|x(M) - x(N)\|^2. \tag{16}$$

Then (15) and (16) imply

$$\begin{aligned} \mu\|x(M) - x(N)\|^2 &\leq [(M - N)(x(N) + p)]_{i_0}(x_{i_0}(N) - x_{i_0}(M)) \\ &\leq \|M - N\| \cdot \|x(N) + p\| \cdot \|x(N) - x(M)\|. \end{aligned}$$

This completes the proof. $\square$

Now let x be a given point in $\Re^n$ and consider the linearized variational inequality problem of finding a point $z \in X$ such that

$$\langle F(x) + \nabla F(x)^T(z - x), y - z\rangle \geq 0, \quad \forall\, y \in X, \tag{17}$$

or equivalently

$$(F_i(x) + [\nabla F(x)^T(z - x)]_i)(y_i - z_i) \geq 0, \ i = 1, \ldots, n, \quad \forall\, y \in X. \tag{18}$$

If $\nabla F(x)$ is a P-matrix, then problem (17) has a unique solution, which we denote $z(x)$. The following lemma is a refinement of Proposition 2.2 in [26].

Lemma 2.4 *Suppose that F is a continuously differentiable uniform P-function and X is a box. Then the solution $z(x)$ of the affine VIP (17) is continuous as a function of x. Moreover, x is a solution of the VIP (1) if and only if $x = z(x)$.*

Proof First note that, since F is a uniform P-function, (7) is satisfied with some constant $\mu > 0$ independent of x. For two arbitrary points $x, x' \in \Re^n$, let $z(x)$ and $z(x')$ be the unique solutions of the linearized VIPs (17) at x and x', respectively. Also let $\bar{z}$ denote the unique solution of the affine VIP

$$\langle F(x) + \nabla F(x')^T(z - x), y - z\rangle \geq 0, \quad \forall\, y \in X.$$

It then follows from Lemma 2.2 with $v := F(x) - \nabla F(x')^T x$, $v' := F(x') - \nabla F(x')^T x'$, $b := 0$ and $M := \nabla F(x')^T$ that

$$\|\bar{z} - z(x')\| \leq \frac{1}{\mu}\|F(x) - F(x') + \nabla F(x')^T(x' - x)\|.$$

On the other hand, by Lemma 2.3 with $b := F(x), p := -x, N := \nabla F(x)^T$ and $M := \nabla F(x')^T$, we have

$$\|\bar{z} - z(x)\| \leq \frac{1}{\mu}\|\nabla F(x')^T - \nabla F(x)^T\| \cdot \|z(x) - x\|.$$

It then follows that

$$\begin{aligned} & \|z(x) - z(x')\| \\ \leq\ & \|\bar{z} - z(x')\| + \|\bar{z} - z(x)\| \\ \leq\ & \frac{1}{\mu}\left(\|F(x) - F(x') + \nabla F(x')^T(x' - x)\| + \|\nabla F(x')^T - \nabla F(x)^T\| \cdot \|z(x) - x\|\right). \end{aligned}$$

Consequently, for any fixed $x \in \Re^n$, we obtain

$$\lim_{x' \to x} \|z(x) - z(x')\| = 0.$$

This proves the first half of the lemma.

To prove the second half, suppose first that $z(x) = x$. Then it follows immediately from (17) that x solves (1). Conversely suppose that x is a solution of (1). Since X is a box and $z(x) \in X$, (2) yields

$$F_i(x)(z_i(x) - x_i) \geq 0, \quad i = 1, \ldots, n. \tag{19}$$

Similarly, from $x \in X$ and (18), we have

$$F_i(x)(x_i - z_i(x)) + [\nabla F(x)^T(z(x) - x)]_i(x_i - z_i(x)) \geq 0, \quad i = 1, \ldots, n. \tag{20}$$

The inequalities (19) and (20) give

$$(z_i(x) - x_i)[\nabla F(x)^T(z(x) - x)]_i \leq 0, \quad i = 1, \ldots, n. \tag{21}$$

Since $\nabla F(x)^T$ is a P-matrix, it follows from (7) that there exists an index i_0 such that

$$(z_{i_0}(x) - x_{i_0})[\nabla F(x)^T(z(x) - x)]_{i_0} \geq \mu\|z(x) - x\|^2. \tag{22}$$

Combining (21) with (22), we get $z(x) = x$. $\square$

3 A Hybrid Newton Method

In this section, we consider the hybrid Newton method proposed in [22] for solving the VIP (1) with general convex constraints. Our aim is to refine the convergence results obtained in [22] by restricting ourselves to the special case where the VIP (1) is box constrained.

The algorithm is stated as follows:

Algorithm.
Step 0: Choose $x^0 \in \Re^n, \omega \in (0,1), \zeta \in (0,1), \delta \in (0,1), \sigma \in (0,1)$, and sufficiently small $\epsilon \geq 0$. Let $k := 0$.

Step 1: If $g_{\alpha\beta}(x^k) \le \epsilon$ or $\|\nabla g_{\alpha\beta}(x^k)\| \le \epsilon$, stop.
Step 2: Find $z^k \in X$ such that

$$\langle F(x^k) + \nabla F(x^k)^T(z^k - x^k), x - z^k \rangle \ge 0, \quad \forall\, x \in X, \tag{23}$$

and let $d^k := z^k - x^k$. If

$$g_{\alpha\beta}(x^k + d^k) \le \zeta g_{\alpha\beta}(x^k), \tag{24}$$

then let $\lambda_k := 1$ and go to Step 4. If the linearized VIP (23) is not solvable or if d^k does not satisfy the condition

$$\langle \nabla g_{\alpha\beta}(x^k), d^k \rangle \le -\sigma \max\{\|\nabla g_{\alpha\beta}(x^k)\|^2, \|d^k\|^2\}, \tag{25}$$

then set $d^k := -\nabla g_{\alpha\beta}(x^k)$.
Step 3: Find the smallest nonnegative integer m_k satisfying

$$g_{\alpha\beta}(x^k + \omega^{m_k} d^k) - g_{\alpha\beta}(x^k) \le \delta \omega^{m_k} \langle \nabla g_{\alpha\beta}(x^k), d^k \rangle, \tag{26}$$

and let $\lambda_k := \omega^{m_k}$.
Step 4: Set $x^{k+1} := x^k + \lambda_k d^k$ and $k := k + 1$. Go to Step 1.

Global convergence of the algorithm has been established in [22]. For completeness, we quote the global convergence theorem without proof.

Theorem 3.1 *Suppose that the mapping F is continuously differentiable. Let $\epsilon = 0$ and suppose that the algorithm generates an infinite sequence $\{x^k\}$. Then any accumulation point x^* of the sequence $\{x^k\}$ is a stationary point of the D-gap function $g_{\alpha\beta}$.*

If F is a uniform P-function, then by [15, Theorem 4.1], the level sets of D-gap function $g_{\alpha\beta}$ are bounded. Since $\{g_{\alpha\beta}(x^k)\}$ is nonincreasing, the boundedness of level sets guarantees the boundedness of the generated sequence $\{x^k\}$ and hence the existence of at least one accumulation point of $\{x^k\}$. On the other hand, by [15, Theorem 3.1], any stationary point $\tilde{x}$ of $g_{\alpha\beta}$ such that $\nabla F(\tilde{x})$ is a P-matrix is a solution of the VIP (1). Therefore, if F is a uniform P-function, then it follows from Theorem 3.1 that any accumulation point of the generated sequence $\{x^k\}$ solves the VIP (1). Because the box constrained VIP with a uniform P-function has a unique solution, we obtain the next corollary to Theorem 3.1.

Corollary 3.2 *Suppose that F is a continuously differentiable uniform P-function and X is a box. Then for any starting point $x^0 \in \Re^n$, the sequence $\{x^k\}$ generated by the algorithm converges to the unique solution of the VIP (1).*

Now we turn our attention to the convergence rate of the algorithm. A solution x^* of the VIP (1) is said to be *regular* in the sense of Robinson [24] (see also [13]) if there exist a neighborhood Ω of x^* and a neighborhood V of $0 \in \Re^n$ such that, for every $v \in V$, the perturbed VIP of finding a vector $x \in X$ such that

$$\langle \tilde{F}(x; v), y - x \rangle \ge 0, \quad \forall\, y \in X, \tag{27}$$

where $\tilde{F}(x; v) := F(x^*) + v + \nabla F(x^*)^T(x - x^*)$, has a unique solution $x(v) \in \Omega$ that is Lipschitz continuous as a function of v, i.e.,

$$\|x(v) - x(v')\| \le \rho\|v - v'\|, \quad \forall\, v, v' \in V$$

for some $\rho > 0$.

Regularity conditions have been widely used in the study of variational inequality problems, particularly in the analysis of local convergence properties of iterative methods for VIPs. In [22], a strong but simple sufficient condition for a solution x^* of the VIP to be regular was given. Here we give a different condition pertaining to the box constrained VIP.

Lemma 3.3 *Assume that x^* is a solution of the VIP (1) with X being a box. If the matrix $\nabla F(x^*)$ is a P-matrix, then x^* is a regular solution.*

Proof The theorem follows from Lemma 2.2. Alternatively, the statement can be directly derived from a characterization of strong regularity given in [6, Theorem 3.4]. □

To study the convergence rate of the algorithm, we need the following results concerning an error bound property of the D-gap function. We denote by $y_\alpha(x)$ the unique maximizer on the right-hand side in the defining equation (3) of the regularized gap function f_α. Note that $y_\alpha(x) = \Pi_X(x - \alpha^{-1}F(x))$, where Π_X denotes the projection operator on X. We define $R_\alpha(x) := x - y_\alpha(x)$. Moreover $y_\beta(x)$ and $R_\beta(x)$ are defined similarly. Let $B(\Delta)$ denote the closed sphere centered at x^* with radius $\Delta > 0$, i.e.,

$$B(\Delta) := \{x \in \Re^n \mid \|x - x^*\| \le \Delta\}.$$

Following the proof of Lemma 5.1 in [15], we have the next lemma.

Lemma 3.4 *Let x^* be a solution of the VIP (1) with X being a box. Suppose that F is a uniform P-function with modulus μ. Suppose also that F is Lipschitz continuous with constant $\kappa > 0$ on $B(\Delta)$ for some $\Delta > 0$. Then there exists a constant $\eta > 0$ such that*

$$\|x - x^*\| \le \eta\|R_\beta(x)\|, \quad \forall\, x \in B(\Delta), \tag{28}$$

where $\eta = (\kappa + \beta)/\mu$.

The next lemma shows that the D-gap function provides a local error bound for the VIP (1) under suitable assumptions. This result will be useful in establishing the quadratic convergence of the proposed algorithm.

Lemma 3.5 *Let x^* be a solution of the VIP (1) with X being a box. Suppose that F is a uniform P-function with modulus μ. Suppose also that F is Lipschitz continuous with constant $\kappa > 0$ on $B(\Delta)$ for some $\Delta > 0$. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\|x - x^*\|^2 \;\le\; g_{\alpha\beta}(x) \;\le\; c_2\|x - x^*\|^2, \quad \forall\, x \in B(\Delta). \tag{29}$$

Proof By [27, Proposition 3.1], we have

$$\frac{\beta-\alpha}{2}\|R_\beta(x)\|^2 \;\le\; g_{\alpha\beta}(x) \;\le\; \frac{\beta-\alpha}{2}\|R_\alpha(x)\|^2, \quad \forall\, x\in\Re^n. \tag{30}$$

It follows from Lemma 3.4 and the left part of the above inequality that

$$g_{\alpha\beta}(x) \ge \frac{\beta-\alpha}{2\eta^2}\|x-x^*\|^2, \quad \forall\, x\in\Re^n,$$

where $\eta = (\kappa+\beta)/\mu$, which shows that the left inequality in (29) is true.

Next observe that

$$\begin{aligned}
\|R_\alpha(x)\| &= \|x - y_\alpha(x) - x^* + y_\alpha(x^*)\| \\
&\le \|x-x^*\| + \|\Pi_X(x-\alpha^{-1}F(x)) - \Pi_X(x^*-\alpha^{-1}F(x^*))\| \\
&\le \|x-x^*\| + \|x-\alpha^{-1}F(x) - x^* + \alpha^{-1}F(x^*)\| \\
&\le \left(2+\frac{\kappa}{\alpha}\right)\|x-x^*\|
\end{aligned}$$

for all $x\in B(\Delta)$, where the equality follows from the definition of R_α and the fact that $R_\alpha(x^*)=0$, the first inequality follows from the triangle inequality and the definition of y_α, the second inequality follows from the nonexpansiveness of the projection operator Π_X, and the last inequality follows from the Lipschitz continuity of F. The right inequality in (29) then follows from the right inequality of (30). The proof is complete. □

We are ready to prove quadratic convergence of the proposed algorithm.

Theorem 3.6 *Suppose that F is continuously differentiable and X is a box. Let x^* be an accumulation point of the sequence $\{x^k\}$ generated by the algorithm. If F is a uniform P-function and ∇F is locally Lipschitzian, then x^* is a solution of the VIP (1) and the sequence $\{x^k\}$ converges quadratically to x^*.*

Proof By Theorem 3.1, x^* is a stationary point of the D-gap function $g_{\alpha\beta}$. Since $\nabla F(x^*)$ is a P-matrix by Lemma 2.1, it follows from Theorem 3.1 in [15] that x^* is already a solution of the VIP (1). Moreover Lemma 3.3 shows that x^* is a regular solution. Since F is differentiable, it is locally Lipschitzian; hence there exists a $\Delta_1>0$ such that F is is Lipschitz continuous on $B(\Delta_1)$. Hence by Lemma 3.5, we have

$$c_1\|x-x^*\|^2 \;\le\; g_{\alpha\beta}(x) \;\le\; c_2\|x-x^*\|^2, \quad \forall\, x\in B(\Delta_1) \tag{31}$$

for some $c_1, c_2>0$. Moreover, by choosing a smaller Δ_1 if necessary, we may assume that ∇F is Lipschitz continuous on $B(\Delta_1)$. Then by the basic result on Josephy-Newton's method for the VIP [14, 13], there exists a $\Delta_2>0$ such that for any initial point chosen from $B(\Delta_2)$, the Newton iteration is well-defined and

$$\|z(x)-x^*\| \le c_3\|x-x^*\|^2, \quad \forall\, x\in B(\Delta_2) \tag{32}$$

holds for some constant $c_3 > 0$. Let $\Delta_3 := \min(\Delta_1, \Delta_2)$. Then it follows from (31) and (32) that

$$g_{\alpha\beta}(z(x)) \leq c_2 c_3 \|x - x^*\|^4, \quad \forall\, x \in B(\Delta_3). \tag{33}$$

Let

$$\Delta_4 := \min\left\{\Delta_3, \sqrt{\frac{\zeta c_1}{c_2 c_3}}\right\}.$$

Then it follows from (33) that for any $x \in B(\Delta_4)$

$$g_{\alpha\beta}(z(x)) \leq \zeta c_1 \|x - x^*\|^2 \leq \zeta g_{\alpha\beta}(x),$$

where the first inequality follows from the choice of Δ_4 and the second inequality follows from the left inequality in (31). This implies that, when $x^k \in B(\Delta_4)$, we have $d^k = z(x^k) - x^k$ and the step size $\lambda_k = 1$ is accepted, i.e., $x^{k+1} = z(x^k)$. Consequently it follows from (32) that the sequence $\{x^k\}$ converges to x^* quadratically. $\square$

Similarly we may prove superlinear convergence of the algorithm under slightly weaker assumptions. The proof is omitted.

Theorem 3.7 *Suppose that F is continuously differentiable and X is a box. Let x^* be an accumulation point of the sequence $\{x^k\}$ generated by the algorithm. If F is a uniform P-function, then x^* is a solution of the VIP (1) and the sequence $\{x^k\}$ converges superlinearly to x^*.*

4 Numerical Results

We implemented the hybrid Newton-type method suggested in this paper in MATLAB and run it on a SUN SPARC 10 station. We first give a brief description of the implementation: Let

$$r(x) := x - \mathrm{Proj}_{[l,u]}(x - F(x))$$

denote the natural residual of the box constrained variational inequality problem. We terminate our method if

$$\|r(x^k)\| \leq \epsilon_1 \quad \text{or} \quad g_{\alpha\beta}(x^k) \leq \epsilon_2 \tag{34}$$

for some iterate x^k, where

$$\epsilon_1 := 10^{-6} \quad \text{and} \quad \epsilon_2 := 10^{-11}.$$

In addition, the iteration was stopped if

$$k > k_{\max}$$

with

$$k_{\max} = 100.$$

For the D-gap function $g_{\alpha\beta}$, we used the parameters

$$\alpha = 0.9 \quad \text{and} \quad \beta = 1.1.$$

In the line search rule (26), we used

$$\omega = 0.5 \quad \text{and} \quad \delta = 10^{-4}.$$

However, we replaced the standard (monotone) Armijo-rule by a nonmonotone variant, see Grippo, Lampariello and Lucidi [12] for details.

As a solver for the linearized variational inequality problems, we used the semismooth Newton-type method from [6]. In contrast to what is said in the description of our algorithm, however, we always accept the corresponding search direction d^k whenever it satisfies the descent test

$$\nabla g_{\alpha\beta}(x^k)^T d^k < 0;$$

note that this guarantees that the Armijo line search is well-defined. In particular, we accept this search direction d^k even if we were not able to solve the corresponding linearized variational inequality problem. In this way, we try to overcome the problem that we have to take too many gradient steps in a row which is obviously not very desirable.

In order to improve the efficiency of our algorithm, however, we also used a preprocessor; more precisely, we first try to solve our test examples by using the recently proposed method from Kanzow and Fukushima [16]. This is a nonsmooth Newton-type method applied to the residual equation

$$r(x) = 0$$

and globalized by the D-gap function $g_{\alpha\beta}$, see [16] for details. The motivation for doing this is quite simple: The method from [16] works extremely well whenever it solves a problem successfully. Unfortunately, it does not seem to be very robust unless relatively strong assumptions are satisfied.

So we first apply the nonsmooth Newton-type method from [16] in order to solve a test example, but we stop this preprocessing iteration if either the termination criterion (34) is satisfied or if a certain test indicates that the preprocessor runs into difficulties. In the latter case, we switch to the hybrid Newton method introduced in this paper which is not as efficient as the method from [16], but which seems to be considerably more reliable.

Basically, our criterion for switching from the preprocessor to the hybrid Newton method is as follows: If

$$t_k \le t_{\min} \quad \text{or} \quad \|\nabla g_{\alpha\beta}(x^k)\| \le c g_{\alpha\beta}(x^k), \tag{35}$$

then terminate the preprocessing iteration and go to the hybrid Newton method using the previous iterate x^{k-1} as a starting point. The actual parameters used in (35) are

$$t_{\min} = 10^{-4} \quad \text{and} \quad c = 10^{-2}.$$

If the preprocessor is successful and converges to a solution of the box constrained variational inequality problem which satisfies the standard regularity conditions used in [16] for the local convergence theory, then $t_k = 1$ for all k sufficiently large and $g_{\alpha\beta}(x^k) = O(\|\nabla g_{\alpha\beta}(x^k)\|^2)$, so none of the tests in (35) will be satisfied.

We applied the method just described to all test problems from the MCPLIB and GAMSLIB libraries, see [4, 8], using all the different starting points which are available within the MATLAB environment.

We report the numerical results in Table 1 for the MCPLIB test problems and in Table 2 for the GAMSLIB test problems. The columns in these tables have the following meanings:

problem :	name of the test problem in MCPLIB
n :	number of variables
m :	number of (finite) bounds on the variables x_i
SP :	starting point
P-steps :	number of iterations used in the preprocessing phase
N-steps :	number of Newton steps used in the hybrid Newton phase
G-steps :	number of gradient steps used in the hybrid Newton phase
F-eval. :	number of function evaluations
$g_{\alpha\beta}(x^f)$:	value of $g_{\alpha\beta}(x)$ at the final iterate $x = x^f$
$\|r(x^f)\|$:	value of $\|r(x)\|$ at the final iterate $x = x^f$.

Looking at Tables 1 and 2, we see that we have just a few failures on some difficult test problems, whereas the overall behaviour of our method is quite good. Although many of the simple problems were solved by the preprocessor (i.e., there are no N- and no G-steps), the hybrid Newton-type method introduced in this paper was necessary in order to solve a number of other test examples.

In fact, we made the following observation during the testing phase for our algorithm: Both the preprocessor from [16] and the hybrid Newton-type method discussed in this paper try to minimize the D-gap function $g_{\alpha\beta}$. Now, the D-gap function might have a local minimum which does not correspond to a solution of the box constrained variational inequality problem. In that case, we would expect both algorithms to run into difficulties by converging to one of these local minima, in particular, since the search directions computed by both methods are based on some local information of the variational inequality problem. In fact, this difficulty arises, e.g., for the `billups` example. In general, however, our observation is that the method from [16] tends to converge to a local minimum of $g_{\alpha\beta}$ much more often than the method discussed here. This seems to indicate that, from a global point of view, the search direction computed by our hybrid Newton-type method is a much better search direction than the one computed by the nonsmooth Newton-type method in [16].

It is therefore our feeling that the robustness of many existing solvers can be improved by using the search direction from our hybrid Newton-type method whenever the underlying solver does not seem to converge.

5 Concluding Remarks

The variational inequality problem is reformulated as an unconstrained minimization problem by using the D-gap function $g_{\alpha\beta}$. A hybrid Newton-type method is then proposed to minimize the function $g_{\alpha\beta}$. Under mild conditions, the proposed method is shown to be globally convergent. If some additional assumptions are satisfied, then the sequence converges quadratically or superlinearly to a solution of the original variational inequality problem. A sufficient condition is given for a solution x^* of the VIP to be regular. This condition is only concerned with the mapping F, unlike the conditions in [24, 17, 6].

Table 1: Numerical results for MCPLIB test problems

problem	n	m	SP	P-steps	N-steps	G-steps	F-eval.	$g_{\alpha\beta}(x^f)$	$\|r(x^f)\|$
bertsekas	15	15	1	0	4	0	11	7.5e-15	2.7e-7
bertsekas	15	15	2	1	4	0	12	7.5e-15	2.7e-7
bertsekas	15	15	3	1	4	0	12	7.5e-15	2.7e-7
bert_oc	5000	2000	1	4	0	0	6	3.8e-28	6.1e-14
billups	1	1	1	—	—	—	—	—	—
bratu	5625	11250	1	13	0	0	29	1.0e-20	1.9e-10
choi	13	26	1	4	0	0	5	4.4e-15	2.1e-7
colvdual	20	20	1	—	—	—	—	—	—
colvdual	20	20	2	—	—	—	—	—	—
colvnlp	15	15	1	0	3	0	10	2.0e-13	1.4e-6
colvnlp	15	15	2	1	3	0	20	3.5e-12	5.9e-6
cycle	1	1	1	3	0	0	5	2.3e-21	1.5e-10
ehl_k40	41	40	1	12	8	0	115	9.6e-23	3.1e-11
ehl_k60	61	60	1	22	8	0	221	1.7e-17	1.3e-8
ehl_k80	81	80	1	24	8	0	233	3.1e-17	1.7e-8
ehl_kost	101	100	1	28	8	0	273	6.6e-16	8.1e-8
ehl_kost	101	100	2	28	8	0	273	6.6e-16	8.1e-8
ehl_kost	101	100	3	28	8	0	273	6.6e-16	8.1e-8
explcp	16	16	1	15	0	0	31	0	0
freebert	15	10	1	0	3	0	10	2.4e-16	4.8e-8
freebert	15	10	2	1	3	0	16	6.6e-23	2.6e-11
freebert	15	10	3	0	3	0	10	2.4e-16	4.8e-8
freebert	15	10	4	0	4	0	11	4.4e-21	2.1e-10
freebert	15	10	5	1	3	0	9	3.2e-19	1.8e-9
gafni	5	10	1	13	0	0	46	7.5e-19	2.7e-9
gafni	5	10	2	12	0	0	44	7.5e-19	2.7e-9
gafni	5	10	3	13	0	0	46	7.5e-19	2.7e-9
hanskoop	14	14	1	0	4	0	13	2.3e-14	4.8e-7
hanskoop	14	14	2	0	6	0	17	2.1e-14	4.5e-7
hanskoop	14	14	3	0	5	0	14	5.3e-14	7.3e-7
hanskoop	14	14	4	4	4	0	22	3.2e-14	5.7e-7
hanskoop	14	14	5	0	7	0	24	6.0e-15	2.4e-7
hydroc06	29	11	1	5	0	0	7	1.1e-25	1.1e-12
hydroc20	99	39	1	8	0	0	10	3.7e-14	6.0e-7
jel	6	6	1	8	0	0	16	1.4e-15	1.2e-7
josephy	4	4	1	10	0	0	23	2.9e-17	1.7e-8
josephy	4	4	2	7	0	0	15	4.5e-22	6.7e-11
josephy	4	4	3	11	0	0	24	2.9e-17	1.7e-8
josephy	4	4	4	4	0	0	5	4.2e-15	2.1e-7
josephy	4	4	5	3	0	0	4	4.3e-15	2.1e-7
josephy	4	4	6	6	0	0	12	2.2e-15	1.5e-7

Table 1 (continued): Numerical results for MCPLIB test problems

problem	n	m	SP	P-steps	N-steps	G-steps	F-eval.	$g_{\alpha\beta}(x^f)$	$\|r(x^f)\|$
kojshin	4	4	1	9	0	0	22	1.2e-20	3.5e-10
kojshin	4	4	2	7	0	0	14	1.3e-23	1.2e-11
kojshin	4	4	3	10	0	0	23	1.2e-20	3.5e-10
kojshin	4	4	4	1	0	0	2	0	0
kojshin	4	4	5	3	0	0	4	4.3e-15	2.1e-7
kojshin	4	4	6	5	0	0	7	6.3e-16	7.59e-8
mathinum	3	3	1	22	0	0	47	4.1e-14	6.4e-7
mathinum	3	3	2	4	0	0	5	3.4e-16	5.8e-8
mathinum	3	3	3	32	0	0	76	4.3e-14	6.5e-7
mathinum	3	3	4	5	0	0	6	2.7e-14	5.1e-7
mathisum	4	4	1	4	0	0	6	2.0e-22	4.4e-11
mathisum	4	4	2	5	0	0	6	5.4e-15	2.3e-7
mathisum	4	4	3	26	0	0	53	3.7e-14	6.1e-7
mathisum	4	4	4	5	0	0	6	8.2e-21	2.8e-10
methan08	31	15	1	4	0	0	5	6.3e-24	7.9e-12
nash	10	10	1	6	0	0	7	3.8e-17	1.9e-8
nash	10	10	2	12	0	0	39	6.1e-15	2.5e-7
obstacle	2500	5000	1	10	0	0	11	3.3e-31	2.3e-15
opt_cont31	1024	1024	1	5	0	0	12	3.6e-30	4.3e-15
opt_cont127	4096	4096	1	8	0	0	32	1.4e-29	9.0e-15
opt_cont255	8192	8192	1	11	0	0	51	2.9e-29	1.2e-14
opt_cont511	16384	16384	1	12	0	0	52	6.0e-29	1.6e-14
pgvon105	105	105	1	20	0	0	91	2.7e-12	5.2e-6
pgvon106	106	106	1	—	—	—	—	—	—
pies	42	52	1	20	6	0	153	2.1e-16	4.6e-8
powell	16	16	1	4	3	0	17	7.5e-20	8.6e-10
powell	16	16	2	6	4	0	22	1.8e-16	4.3e-8
powell	16	16	3	0	12	0	35	2.3e-21	1.5e-10
powell	16	16	4	0	7	0	21	1.7e-17	1.3e-8
powell_mcp	8	0	1	6	0	0	7	4.2e-24	6.5e-12
powell_mcp	8	0	2	7	0	0	8	4.7e-25	2.2e-12
powell_mcp	8	0	3	8	0	0	9	2.2e-16	4.7e-8
powell_mcp	8	0	4	7	0	0	8	1.7e-15	1.3e-7
scarfanum	13	13	1	0	4	0	20	8.2e-13	2.9e-6
scarfanum	13	13	2	0	5	0	20	1.3e-16	3.7e-8
scarfanum	13	13	3	9	0	0	12	5.2e-20	7.2e-10
scarfasum	14	14	1	4	0	0	6	3.2e-18	5.6e-9
scarfasum	14	14	2	0	3	0	16	8.8e-13	3.0e-6
scarfasum	14	14	3	9	0	0	12	5.2e-20	7.2e-10
scarfbnum	39	39	1	—	—	—	—	—	—
scarfbnum	39	39	2	—	—	—	—	—	—

Table 1 (continued): Numerical results for MCPLIB test problems

problem	n	m	SP	P-steps	N-steps	G-steps	F-eval.	$g_{\alpha\beta}(x^f)$	$\|r(x^f)\|$
scarfbsum	40	40	1	13	0	0	62	5.3e-14	7.3e-7
scarfbsum	40	40	2	26	0	0	66	9.6e-14	9.8e-7
sppe	27	27	1	23	3	0	60	1.4e-22	3.7e-11
sppe	27	27	2	21	4	0	59	3.1e-21	1.7e-10
tobin	42	42	1	15	0	0	46	9.6e-23	3.1e-11
tobin	42	42	2	22	0	0	83	4.0e-21	2.0e-10

Table 2: Numerical results for GAMSLIB test problems

problem	n	m	SP	P-steps	N-steps	G-steps	F-eval.	$g_{\alpha\beta}(x^f)$	$\|r(x^f)\|$
cafemge	101	101	1	10	0	0	29	3.3e-16	5.7e-8
cammcp	242	242	1	6	0	0	8	1.1e-16	3.3e-8
cirimge	9	6	1	6	0	0	53	2.0e-32	4.4e-16
co2mge	208	208	1	2	0	0	15	1.6e-15	1.3e-7
dmcmge	170	170	1	147	33	8	1658	1.5e-14	3.9e-7
ers82mcp	232	0	1	5	0	0	6	1.8e-24	4.2e-12
etamge	114	114	1	15	0	0	42	1.5e-14	3.8e-7
hansmcp	43	43	1	33	11	16	787	2.4e-13	1.6e-6
hansmge	43	43	1	5	5	0	62	2.8e-13	1.7e-6
harkmcp	32	32	1	26	0	0	60	1.0e-14	3.2e-7
harmge	11	9	1	1	6	0	18	1.4e-13	1.2e-6
kehomge	9	9	1	12	0	0	18	4.3e-23	2.1e-11
kormcp	78	0	1	3	0	0	5	1.8e-25	1.3e-12
mr5mcp	350	350	1	1	9	0	51	7.3e-22	8.5e-11
nsmge	212	212	1	5	14	0	59	2.3e-17	1.5e-8
oligomcp	6	6	1	6	0	0	9	1.0e-20	3.2e-10
scarfmcp	18	18	1	0	4	0	14	5.4e-12	7.3e-6
scarfmge	18	18	1	0	6	0	18	1.4e-15	1.2e-7
transmcp	11	11	1	0	1	0	3	2.0e-17	1.4e-8
two3mcp	6	6	1	8	0	0	16	1.4e-15	1.2e-7
unstmge	5	5	1	8	0	0	11	2.3e-18	4.7e-9
vonthmcp	125	125	1	—	—	—	—	—	—
vonthmge	80	80	1	200	8	1	1263	1.4e-13	1.2e-6
wallmcp	6	0	1	2	0	0	3	3.5e-21	1.9e-10

Acknowledgments: The first author would like to thank Prof. Y. Yuan for his constant help and encouragement.

References

[1] G. Auchmuty, "Variational principles for variational inequalities", *Numerical Functional Analysis and Application* 10 (1989), 863–874.

[2] A. Auslender, *Optimisation: Méthodes Numériques*, Masson, Paris, France, 1976.

[3] D.P. Bertsekas and J.N. Tsitsiklis, *Parallel and Distributed Computation – Numerical Methods*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[4] S.P. Dirkse and M.C. Ferris, "MCPLIB: A collection of nonlinear mixed complementarity problems", *Optimization Methods and Software* 5 (1995), 319–345.

[5] F. Facchinei, A. Fischer, and C. Kanzow, "Regularity properties of a semismooth reformulation of variational inequalities," *SIAM Journal on Optimization*, to appear.

[6] F. Facchinei, A. Fischer, and C. Kanzow, "A semismooth Newton method for variational inequalities: The case of box constraints", *Complementarity and Variational Problems: State of the Art*, M.C. Ferris and J.S. Pang (eds.), SIAM, Philadelphia, PA, 1997, 76–90.

[7] M.C. Ferris and J.S. Pang, "Engineering and economic applications of complementarity problems", *SIAM Review*, to appear.

[8] M.C. Ferris and T.F. Rutherford, "Accessing realistic mixed complementarity problems within MATLAB", *Nonlinear Optimization and Applications*, G. Di Pillo and F. Giannessi (eds.), Plenum Press, New York, NY, 1996, 141–153.

[9] M. Fukushima, "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems", *Mathematical Programming* 53 (1992), 99–110.

[10] M. Fukushima, "Merit functions for variational inequality and complementarity problems", *Nonlinear Optimization and Applications*, G. Di Pillo and F. Giannessi (eds.), Plenum Press, New York, NY, 1996, 155–170.

[11] M. Fukushima and J.S. Pang, "Minimizing and stationary sequences of merit functions for complementarity problems and variational inequalities", *Complementarity and Variational Problems: State of the Art*, M.C. Ferris and J.S. Pang (eds.), SIAM, Philadelphia, PA, 1997, 91–104.

[12] L. Grippo, F. Lampariello and S. Lucidi, "A nonmonotone linesearch technique for Newton's method", *SIAM Journal on Numerical Analysis* 23 (1986), 707–716.

[13] P.T. Harker and J.S. Pang, "Finite dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications", *Mathematical Programming* 48 (1990), 161–220.

[14] N.H. Josephy, "Newton's method for generalized equations", Technical Report No. 1965, Mathematics Research Center, University of Wisconsin, Madison, WI, 1979.

[15] C. Kanzow and M. Fukushima, "Theoretical and numerical investigation of the D-gap function for box constrained variational inequalities", *Mathematical Programming*, to appear.

[16] C. Kanzow and M. Fukushima, "Solving box constrained variational inequality problems by using the natural residual with D-gap function globalization. Mathematical Programming Technical Report 97-14, Computer Sciences Department, University of Wisconsin – Madison, Madison, WI, November 1997.

[17] J.M. Liu, "Strong stability in variational inequalities", *SIAM Journal on Control and Optimization* 33 (1995), 725–749.

[18] P. Marcotte, "A new algorithm for solving variational inequalities with application to the traffic assignment problem", *Mathematical Programming* 33 (1985), 339–351.

[19] P. Marcotte and J.P. Dussault, "A note on a globally convergent Newton method for solving monotone variational inequalities", *Operations Research Letters* 6 (1987), 35–42.

[20] J.J. Moré and W.C. Rheinboldt, "On P- and S-functions and related classes of n-dimensional nonlinear mappings", *Linear Algebra and Its Applications* 6 (1973), 45–68.

[21] J.M. Peng, "Equivalence of variational inequality problems to unconstrained minimization", *Mathematical Programming* 78 (1997), 347–356.

[22] J.M. Peng and M. Fukushima, "A hybrid Newton method for solving the variational inequality problem via the D-gap function", Preprint, Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Science, Beijing, China, May 1997.

[23] L. Qi, "Regular pseudo-smooth NCP and BVIP functions and globally and quadratically convergent generalized Newton methods for complementarity and variational inequality problems", Applied Mathematics Report 97/14, School of Mathematics, The University of New South Wales, Sydney, Australia, July 1997 (revised September 1997).

[24] S.M. Robinson, "Strongly regular generalized equations", *Mathematics of Operations Research* 5 (1980), 43–62.

[25] D. Sun, M. Fukushima and L. Qi, "A computable generalized Hessian of the D-gap function and Newton-type methods for variational inequality problems", *Complementarity and Variational Problems: State of the Art*, M.C. Ferris and J.S. Pang (eds.), SIAM, Philadelphia, PA, 1997, 452–472.

[26] K. Taji, M. Fukushima and T. Ibaraki, "A globally convergent Newton method for solving strongly monotone variational inequalities", *Mathematical Programming* 58 (1993), 369–383.

[27] N. Yamashita, K. Taji and M. Fukushima, "Unconstrained optimization reformulations of variational inequality problems", *Journal of Optimization Theory and Applications* 92 (1997), 439–456.