POLYHEDRAL BOUNDARY PROJECTION *

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Abstract. We consider the problem of projecting a point in a polyhedral set onto the boundary of the set using an arbitrary norm for the projection. Two types of polyhedral sets, one defined by a convex combination of k points in R^n and the second by the intersection of m closed halfspaces in R^n, lead to disparate optimization problems for finding such a projection. The first case leads to a mathematical program with a linear objective function and constraints that are linear inequalities except for a single nonconvex cylindrical constraint. Interestingly, for the 1-norm, this nonconvex problem can be solved by solving $2n$ linear programs. The second polyhedral set leads to a much simpler problem of determining the minimum of m easily evaluated numbers. These disparate mathematical complexities parallel known ones for the related problem of finding the largest ball, with radius measured by an arbitrary norm, that can be inscribed in the polyhedral set. For a polyhedral set of the first type this problem is NP-hard for the 2-norm and the ∞-norm [4] and solvable by a single linear program for the 1-norm [7], while for the second type this problem leads to a single linear program even for a general norm [6].

Key words. polyhedral set, boundary projection, largest inscribed ball

AMS subject classifications. 15A39, 90C05, 90C30

1. Introduction. We consider a polytope in the n-dimensional real space R^n defined as a convex combination of k points and represented as follows:

$$S := \{y \mid y = Bz,\ z \geq 0,\ e^T z = 1\}, \tag{1.1}$$

where B is an $n \times k$ real matrix and e is a vector of ones. Given a point $s \in S$ we want to find a projection p onto the boundary $bd(S)$ of S using an arbitrary norm on R^n. [1] Similar problems arise in Data Envelopment Analysis (DEA) [2] where the distance to the boundary is an efficiency measure of a Decision Making Unit (DMU) represented by that point. Each DMU is represented by an n-dimensional column of B and each of the n dimensions measures components required or products generated by that DMU. The set of efficient points for the DMUs is that part of the boundary of the convex hull of S that cannot be improved upon by any other DMU consuming less or equal amounts of components or generating more or equal amounts of products. Projecting a DMU onto the boundary of S gives an indication of how close to an efficient point that DMU is. We will show in Section 2, by using an arbitrary-norm projection onto a hyperplane [9], that this basically is a nonconvex problem, which however can be reduced (Theorem 2.2) to a mathematical program with a linear objective function and linear constraints except for a single nonlinear equality constraint that restricts a solution to be on the surface of a cylinder determined by the dual norm to that used in measuring the distance to the boundary of S. If the 1-norm is used to measure the distance, then because its dual is the ∞-norm, remaining on the ∞-norm cylinder and solving the problem can be achieved by solving $2n$ linear programs. A related problem to the boundary projection problem is that of determining the largest ball

*This work was supported by National Science Foundation Grant CCR-9322479 and Air Force Office of Scientific Research Grant F49620-97-1-0326 as Mathematical Programming Technical Report 97-10, October 1997. Revised May and July 1998.

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[1] The boundary projection problem for the set S using the 1-norm was suggested to the author by Holger Scheel of Dortmund University.

in R^n, with radius measured by an arbitrary norm, that is contained in S. This is a problem that has been studied by a number of authors [3, 4, 6, 7] and is NP-hard except for the 1-norm which can be solved by a single linear program [7, Theorem 3.3]. By using our polyhedral boundary projection result we formulate this problem as a maxmin problem (Theorem 2.4); an upper bound obtained by interchanging the max and min is however twice as large as the upper bound of Gritzmann and Klee [6, (1.3)] (Corollaries 2.5 and 2.6).

In contrast to the nonconvex problems arising from the boundary projection and largest-inscribed-ball problems associated with a polyhedral set described by (1.1), the corresponding optimization problems are much simpler when the polyhedral set is described as the intersection of m closed halfspaces as follows:

$$T := \{x \mid Ax \geq b\}, \tag{1.2}$$

where A is an $m \times n$ real matrix and b is a vector in R^m. The largest-inscribed-ball problem in T has again been studied in [6, 7] and in Section 3 we cite these results and state that for any norm on R^n the projection problem reduces to the rather trivial problem of finding the minimum of m easily calculated numbers, while the largest ball problem for any norm can be solved by a single linear program.

Section 4 gives a brief summary and conclusion.

We note that given a polytope set in the form of (1.1) it is not easy to derive the equivalent form (1.2) for that specific set, for example the 1-norm unit ball has $2n$ vertices but 2^n faces. Going backward from (1.2) to (1.1) is also difficult, for example ∞-norm unit ball has $2n$ faces but 2^n vertices, and may not even be possible because T may be unbounded. In fact by Motzkin's Polyhedral Decomposition Theorem [5, Theorem 1] the polyhedral set T is equivalent to the algebraic sum of a convex combination of points in R^n (a set similar to S) plus a convex polyhedral cone, neither of which are easy to find.

1.1. Notation and Background. All vectors will be column vectors unless transposed to a row vector by a superscript ^{T}. The scalar product of two vectors x and y in the n-dimensional real space R^n will be denoted by x^Ty. For a mathematical program $\min_{x\in X} f(x)$, where $f : R^n \longrightarrow R$, the notation $\arg\min_{x\in X} f(x)$, will denote the set of solutions of the mathematical program $\min_{x\in X} f(x)$. For $x \in R^n$ and $p \in [1,\infty)$, the norm $\|x\|_p$ will denote the p-norm: $(\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ and $\|x\|_\infty$ will denote $\max_{1\leq i\leq n} |x_i|$. For an $m \times n$ matrix A, A_i will denote row i of A and $A_{\cdot j}$ will denote column j of A. The identity matrix in a real space of arbitrary dimension will be denoted by I, while a column vector of ones of arbitrary dimension will be denoted by e, and a column vector of arbitrary dimension with zeros in every row except 1 in row i will be denoted by e^i. The symbol $:=$ will denote a definition.

A boundary point of a set $X \subseteq R^n$ is any point in R^n such that any open set containing the point contains points in X and points not in X. The closed set of all boundary points of X, denoted by $\mathrm{bd}(X)$, is contained in X if and only if X is closed. Hence the closed polyhedral sets S and T contain their boundaries. By a projection of a point $s \in R^n$ onto a closed set $X \subseteq R^n$ we mean an element of $P := \arg\min_{x\in X} \|x - s\|$, where $\|\cdot\|$ is some specified norm on R^n. Because P may not be a singleton as a consequence of the nonconvexity of X or the norm being the 1-norm or ∞-norm, we shall mean by "projection of s onto X", any element of P and similarly for "its projection". For a general norm $\|\cdot\|$ on R^n, the dual norm $\|\cdot\|'$ on R^n and the resulting Cauchy-Schwarz inequality are:

$$\|y\|' := \max_{\|x\|=1} y^T x, \quad \pm y^T x \leq |y^T x| \leq \|y\|' \|x\|. \tag{1.3}$$

For $p, q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, the p-norm and q-norm are dual norms.

We need, in Theorem 2.2 below, an explicit form for a projection of an arbitrary point onto a given hyperplane using a general norm. Gritzmann and Klee [6, Proof of (1.14)] give the distance between a point and its projection on the hyperplane, but do not give a projection explicitly. For later use we state the following result.

PROPOSITION 1.1. *[9]* **Arbitrary-Norm Projection onto a Hyperplane.** *Let $q \in R^n$ be any point in R^n not on the hyperplane:*

$$P := \{x \mid w^T x = \gamma\},\ 0 \neq w \in R^n,\ \gamma \in R. \tag{1.4}$$

A projection $p(q) \in P$ using a general norm $\|\cdot\|$ on R^n is given by:

$$p(q) = q - \frac{w^T q - \gamma}{\|w\|'} r(w), \tag{1.5}$$

where $\|\cdot\|'$ is the dual norm to $\|\cdot\|$ and:

$$r(w) \in \arg \max_{\|y\|=1} w^T y. \tag{1.6}$$

Consequently, the distance between q and its projection $p(q)$ is given by:

$$\|q - p(q)\| = \frac{|w^T q - \gamma|}{\|w\|'}. \tag{1.7}$$

Explicit expressions for the 1-norm, 2-norm and ∞-norm for (1.5)-(1.7) are given in [9, Corollaries 2.3-2.5].

2. Boundary Projection and Largest Ball for Polytope S. We begin with the problem of finding a projection of a point $s \in S$ onto the boundary $\mathrm{bd}(S)$ of S for an arbitrary norm on R^n. For that purpose we need to characterize the set $\bar{S}$ of points not in S by means of a separating hyperplane argument as follows.

LEMMA 2.1. **Characterization of the Complement of** S *The set of points $\bar{S}$ in R^n not in S can be characterized as follows:*

$$\bar{S} = \{p \mid (p, y, \zeta) \in R^{2n+1},\ B^T y + e\zeta \geq 0,\ p^T y + \zeta < 0,\ \| y \|' = 1\}, \tag{2.1}$$

where $\|\cdot\|'$ is the dual of an arbitrary norm $\|\cdot\|$ on R^n and $p^T y + \zeta = 0$ is a supporting hyperplane of S separating S from a point in $\bar{S}$.

Proof By the strict separation theorem for convex sets [8, Theorem 3.2.6], $p \in \bar{S}$ if and only if there exists a hyperplane in R^n: $\{x \mid v^T x + \xi = 0\}$, for some $v \neq 0$ and $\xi \in R$, which strictly separates p from S, that is:

$$v^T p + \xi < 0,\ v^T B z + \xi \geq 0,\ \forall z \geq 0,\ e^T z = 1,$$

or equivalently: $p^T v + \xi < 0$, and $B^T v + e\xi \geq 0$. Hence:

$$\bar{S} = \{p \mid (p, v, \xi) \in R^{2n+1},\ B^T v + e\xi \geq 0,\ p^T v + \xi < 0\}.$$

Since v cannot equal zero in the definition of the last set, it follows upon normalization by dividing by $\| v \|'$ and defining $y := v/ \| v \|'$ and $\zeta := \xi/ \| v \|'$ that (2.1) holds. □

By using this lemma and Proposition 1.1 we are able now to state a mathematical program that characterizes the boundary projection problem.

THEOREM 2.2. **Boundary Projection** $p(s)$ **for** $s \in S$ *The distance between* $s \in S$ *and its projection* $p(s)$ *onto the boundary* $bd(S)$ *of* S *using a general norm* $\|\cdot\|$ *on* R^n *can be determined as follows:*

$$(2.2)\qquad \begin{aligned} \|s-p(s)\| &= \min_{y,\zeta}\{s^Ty+\zeta \;|B^Ty+e\zeta \geq 0,\ \|y\|'=1\} \quad (a)\\ &= \min_{\|y\|'=1}\{s^Ty - \min_{1\leq j\leq n} y^TB_{\cdot j}\}. \quad (b)\end{aligned}$$

Furthermore, if $(\bar{y},\bar{\zeta})$ *is a solution of (2.2(a)), then a projection* $p(s)$ *of* s *onto the boundary bd(S) of* S *is given by:*

$$(2.3)\qquad p(s) = s-(s^T\bar{y}+\bar{\zeta})r(\bar{y}),\ \textit{where}\ r(\bar{y}) \in \arg\max_{\|y\|=1} \bar{y}^Ty.$$

Proof Denote the feasible region of (2.2(a)) as

$$(2.4)\qquad Z = \{(y,\zeta) \mid (y,\zeta) \in R^{n+1},\ B^Ty+e\zeta \geq 0,\ \|y\|'=1\}.$$

The equality of (2.2(a)) and (2.2(b)) is obvious once we define

$$(2.5)\qquad \zeta := -\min_{1\leq j\leq n} y^TB_{\cdot j}.$$

Since the objective function of (2.2(b)) is piecewise-linear convex and hence continuous, it attains a minimum on the compact unit sphere at some $\bar{y}$. The corresponding $\bar{\zeta}$ computed by (2.5) gives an optimal solution $(\bar{y},\bar{\zeta})$ to (2.2(a)).

We now show that this minimum gives the distance between $s \in S$ and its projection onto the boundary of S. For each $p \in \bar{S}$ there exists, by Lemma 2.1, $(y,\zeta) \in R^{n+1}$ such that $(y,\zeta) \in Z$ and such that p lies in the open halfspace $\{q \mid q^Ty+\zeta < 0\}$ and such that S lies in the complementary closed halfspace $\{q \mid q^Ty+\zeta \geq 0\}$. Since $\|y\|'=1$ for $(y,\zeta) \in Z$ it follows by Proposition 1.1 that the distance between s and its projection onto the hyperplane $\{q \mid q^Ty+\zeta = 0\}$ separating points in S and $\bar{p} \in \bar{S}$ is $s^Ty+\zeta$. Furthermore, the minimum $s^T\bar{y}+\bar{\zeta}$ of $s^Ty+\zeta$ over all $(y,\zeta) \in Z$ is the desired distance between s and its projection onto the boundary bd(S) of S because of the following. Since $s^T\bar{y}+\bar{\zeta}$ is the distance from s to a projection of s onto a closest separating hyperplane that separates S from a point in its complement $\bar{S}$, any such projection is also a projection of s onto its boundary. For, it lies on a separating hyperplane and hence cannot lie in the interior of S, while if it were not in S, a point on the line segment joining the projection to s would intersect the boundary of S closer to s and the separating hyperplane at this boundary point would have a closer projection to s contradicting the minimality of $s^T\bar{y}+\bar{\zeta}$. Also, a projection $p(s)$ of S onto the boundary bd(S) of S is given by a projection of s onto the hyperplane $\{q \mid q^T\bar{y}+\bar{\zeta}=0\}$, which again by Proposition 1.1 is given by (2.3). □

For the 2-norm and the ∞-norm the boundary projection problem (2.2) is NP-hard (see [7, Theorem 5.1], taking s to be the origin). However for the 1-norm it can solved in polynomial time by solving $2n$ linear programs as follows.

COROLLARY 2.3. *For the case of $\|\cdot\| = \|\cdot\|_1$, the mathematical program (2.2) can be solved by solving the following* $2n$ *linear programs:*

$$\begin{aligned} \|s-p(s)\|_1 &= \min_{i=1,\dots,n,\sigma=\pm1} P_{i\sigma}, \text{ where :} \\ P_{i\sigma} &:= \min_{y,\zeta}\{s^Ty+\zeta \mid B^Ty+e\zeta \geq 0,\ -e\leq y\leq e,\ y_i=\sigma\}. \end{aligned} \tag{2.6}$$

Any $(\bar y, \bar\zeta)$ determined by solving (2.6) can be used in (2.3) to determine a projection $p(s)$ onto the boundary $bd(S)$ of S using the 1*-norm.*

We turn our attention to the problem of finding the radius, measured by an arbitrary norm, of the largest ball in R^n that is contained in S. By using Theorem 2.2 and maximizing over all s in S we can formulate this problem as follows.

THEOREM 2.4. **Largest Ball Inscribed in S** *The radius ρ of the largest ball, measured by an arbitrary norm $\|\cdot\|$ on R^n, that can be inscribed in S is given by:*

$$\rho = \|\bar s - p(\bar s)\| = \max_{s\in S}\ \min_{y,\zeta}\{s^Ty+\zeta\ |B^Ty+e\zeta\geq 0,\ \|y\|'=1\}. \tag{2.7}$$

Proof The distance function $\|s-p(s)\|$ of (2.2) is an upper semicontinuous function of s on S [1, p 115, Theorem 1]. Since S is compact it follows that the upper semicontinuous distance function $\|s-p(s)\|$ on S attains its maximum on S. □

Problem (2.7) is a difficult problem to solve for a general norm. Freund and Orlin [4] have shown that this is an NP-hard problem for the 2-norm and the ∞-norm, while Gritzmann and Klee [7, Theorem 3.3] have shown that for the 1-norm the problem can be formulated as a single linear program. For a general norm problem (2.7) is a minmax problem over the product of two sets one of which is nonconvex. The nonconvexity of the set $Z = \{(y,\zeta) \mid (y,\zeta)\in R^{n+1},\ B^Ty+e\zeta\geq 0,\ \|y\|'=1\}$ precludes the use of a minmax theorem to switch the maxmin to a minmax which would simplify the problem. However since the minmax is an upper bound to the maxmin, which is the case here because the various minima and maxima exist, we obtain the following corollary to the above theorem by using the minmax as an upper bound to the maxmin and then simplifying the resulting expression.

COROLLARY 2.5. **Upper Bound for Largest Inscribed Ball** *An upper bound on the radius ρ of the largest ball, measured by an arbitrary norm $\|\cdot\|$ on R^n, that can be inscribed in S is given by:*

$$\rho = \|\bar s - p(\bar s)\| \leq \min_{\|y\|'=1}\ (\max_{1\leq j\leq k}\ y^TB_{.j} - \min_{1\leq j\leq k}\ y^TB_{.j}). \tag{2.8}$$

Proof The minmax upper bound to the maxmin of (2.7) is given by:

$$\rho = \|\bar s - p(\bar s)\| \leq \min_{y,\zeta}\ \{\max_{s=Bz,\, e^Tz=1,\, z\geq 0}\ \{s^Ty+\zeta\} \mid B^Ty+e\zeta\geq 0,\ \|y\|'=1\}. \tag{2.9}$$

Upon noting that for a fixed y, $y^Ts = y^TBz$ is maximized over $s\in S$ or equivalently over $z\geq 0$, $e^Tz=1$, by taking y^TBz equal to $\max\limits_{1\leq j\leq k} y^TB_{.j}$, and noting that:

$$\zeta = \max_{1\leq j\leq k}\ -y^TB_{.j} = -\min_{1\leq j\leq k}\ y^TB_{.j},$$

we find that the desired upper bound (2.8) follows from (2.9). □

M. J. Todd gave a geometric interpretation of the upper bound (2.8) that can cut it by a factor of two as follows. He noted that for a fixed $y\in R^n$ such that $\|y\|'=1$, the term in the parentheses of (2.8), e.g. by Proposition 1.1, gives the width of a slab parallel to the hyperplane $y^T x = 0$ and containing the set S. Hence the minimum, over all y such that $\|y\|' = 1$, of such slab widths bounds the diameter of the largest ball that can be inscribed in S and hence the upper bound (2.8) can be cut by a factor of 2 as stated in the following corollary. This result is a special case of (1.3) in Gritzmann and Klee [6], which states that the inner n-radius is bounded above by the outer 1-radius. The bound is tight for symmetric convex bodies, and always within a factor of about $\sqrt{n}$ [6].

COROLLARY 2.6. **Improved Upper Bound for Largest Inscribed Ball**

$$\rho = \|\bar{s} - p(\bar{s})\| \le \frac{1}{2} \min_{\|y\|'=1} \left(\max_{1 \le j \le k} y^T B_{\cdot j} - \min_{1 \le j \le k} y^T B_{\cdot j} \right). \tag{2.10}$$

In contrast to these rather nontrivial problems for the boundary projection and largest radius problems for a polyhedral set characterized by (1.1) we turn to the considerably simpler corresponding problems for a polyhedral set characterized by (1.2).

3. Boundary Projection and Largest Ball for Polyhedral Set T . We consider in this section the polyhedral set T defined by (1.2), and state results parallel to Theorems 2.2 and 2.4. The analysis below is essentially contained in [3] for the 2-norm and [6], but the argument is very simple from our earlier considerations. For a given point $s \in T$, the distance between s and an arbitrary-norm projection of s onto any of the hyperplane $A_i x = b_i$, $i = 1, \ldots, m$ defining T, is given by the proof of (1.14) of [6] or Proposition 1.1. Hence the closest point to s on the boundary bd(T) of T is given by the closest of these projections to s, which is the first boundary point that an expanding ball around s would touch. This leads then to the following straightforward result.

THEOREM 3.1. **Boundary Projection $p(s)$ for $s \in T$** *The distance between $s \in T$ and its projection $p(s)$ onto the boundary bd(T) of T using a general norm $\|\cdot\|$ on R^n is given by*

$$\|s - p(s)\| = \min_{i=1,\ldots,m} \frac{A_i s - b_i}{\|A_i\|'}. \tag{3.1}$$

A projection $p(s)$ of p onto boundary bd(T) is given by

$$p(s) = s - \frac{A_{\bar{i}} s - b_{\bar{i}}}{\|A_{\bar{i}}\|'} y(A_{\bar{i}}), \tag{3.2}$$

where $\|\cdot\|'$ is the dual norm to $\|\cdot\|$ and:

$$y(A_{\bar{i}}) \in \arg \max_{\|y\|=1} A_{\bar{i}} y, \tag{3.3}$$

and $\bar{i}$ is any index that solves (3.1).

It is interesting to contrast the simplicity of finding a minimum of m numbers specified by (3.1) in order to determine the distance between $s \in T$ and its projection $p(s)$ onto the boundary bd(T) of T with the nonconvex program (2.2) required for the corresponding problem for the set S.

Using the above result we can recover the linear program of Gritzmann and Klee [6, (1.14)] for the problem of determining the largest ball in R^n, with radius measured by an arbitrary norm $\|\cdot\|$, that is contained in T. For the 2-norm a different linear programming formulation is given by Eaves and Freund [3, p 143].

THEOREM 3.2. **Largest Ball Inscribed in T** *The radius ρ of the largest ball, measured by an arbitrary norm $\|\cdot\|$ on R^n, that can be inscribed in T is given by:*

$$\rho = \|\bar{s} - p(\bar{s})\| = \sup_{\rho, s} \{\rho \mid A_i s - \|A_i\|' \rho \geq b_i,\ i = 1, \ldots, m\}, \tag{3.4}$$

where $\|\cdot\|'$ denotes the dual norm to $\|\cdot\|$ on R^n.

4. Summary and Conclusion. We have formulated the problem of projecting a point in a polytope, defined by a convex combination of points in R^n, onto its boundary, as a mathematical program that has linear constraints and objective function and one nonconvex cylindrical constraint. For the 1-norm this problem can be solved by solving $2n$ linear programs. When the set is given as the intersection of a number of closed halfspaces, the projection problem is a straightforward problem of finding the minimum of m numbers where m is the number of halfspaces defining the set. We have also related our boundary projection problem to the largest-inscribed-ball problem considered by others [3, 4, 6, 7]. For the case of intersecting halfspaces, this problem can be solved by a single linear program. For the other case of a polyhedral set defined as a convex combination of given points, the problem is formulated as a maxmin problem of a bilinear function on the product of two sets, one of which contains a convex cylindrical constraint. Again, for the 1-norm, this problem can be solved by linear programming while for other norms it is NP-hard. It is interesting to note the disparate difficulty of the problems depending on the polyhedral set characterization.

Acknowledgements I am indebted to M. J. Todd for many important suggestions for revising this paper and to two referees for constructive criticisms that made me aware of the vast world of computational geometry.

REFERENCES

[1] C. Berge. *Topological Spaces*. McMillan, New York, 1963.
[2] A. Charnes, W. Cooper, A. Y. Lewin, and L. M. Seiford, editors. *Data Envelopment Analysis*. Kluwer Academic Publishers, Dordrecht, Netherlands, 1994.
[3] B. C. Eaves and R. M. Freund. Optimal scaling of balls and polyhedra. *Mathematical Programming*, 23:138–147, 1982.
[4] R. M. Freund and J. B. Orlin. On the complexity of four polyhedral set containment problems. *Mathematical Programming*, 33:139–145, 1985.
[5] A. J. Goldman and A. W. Tucker. Polyhedral convex cones. In H. W. Kuhn and A. W. Tucker, editors, *Linear Inequalities and Related Systems*, pages 19–40, Princeton, 1956. Princeton University Press.
[6] P. Gritzmann and V. Klee. Inner and outer j-radii of convex bodies in finite-dimensional normed spaces. *Discrete and Computational Geometry*, 7:255–280, 1992.
[7] P. Gritzmann and V. Klee. Computational complexity of inner and outer j-radii of of polytopes in finite-dimensional normed spaces. *Mathematical Programming*, 59:163–213, 1993.
[8] O. L. Mangasarian. *Nonlinear Programming*. McGraw–Hill, New York, 1969. Reprint: SIAM Classic in Applied Mathematics 10, 1994, Philadelphia.
[9] O. L. Mangasarian. Arbitrary-norm separating plane. Technical Report 97-07, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, May 1997. *Operations Research Letters*, submitted. ftp://ftp.cs.wisc.edu/math-prog/tech-reports/97-07.ps.Z.