Lineality Removal for Copositive–Plus Normal Maps [1]

Menglin Cao and Michael C. Ferris

University of Wisconsin,
Computer Sciences Department,
1210 West Dayton Street, Madison, Wisconsin 53706
email: ferris@cs.wisc.edu

Abstract. We are concerned with solving affine variational inequalities defined by a linear map A and a polyhedral set C. Most of the existing pivotal methods for such inequalities or mixed linear complementarity problems depend on the existence of extreme points in C or a certain non–singularity property of A with respect to the lineality of C. In this paper, we prove that if A is copositive–plus with respect to the recession cone of C, then the lineality space can be removed without any further assumptions. The reductions given here extend the currently known pivotal methods to solve affine variational inequalities or prove that no solution exists, whenever A is copositive–plus withe respect to the recession cone of C.

Keywords: *Copositive–plus matrices, normal maps, lineality space, variational inequalities, mixed complementarity problems*
AMS Subject Classification: 90C33, 90C30, 52A25

1. Introduction

We are interested in the affine variational inequality problem which can be described as follows. Let $C \in \mathbb{R}^n$ be a polyhedral convex set and A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. We wish to find a point $z \in C$ such that

$$\langle A(z) - a, c - z \rangle \geq 0, \qquad \forall c \in C. \tag{AVI}$$

The problem can be equivalently formulated as

$$0 \in A(z) - a + \partial \psi_C(z), \tag{GE}$$

where $\psi_C(\cdot)$ is the indicator function of the set C. The solutions of such problems arise for example in the determination of Newton–type methods for variational inequalities and mixed complementarity problems.

In [1], (AVI) was treated as the piecewise linear equation

$$A_C(x) = a, \tag{NE}$$

where A_C is the normal map

$$A_C(x) := A(\pi_C(x)) + x - \pi_C(x).$$

[1] This material is based on research supported by the National Science Foundation Grant CCR–9157632 and the Air Force Office of Scientific Research Grant F49620–94–1–0036

Here $\pi_C(x)$ denotes the projection (with respect to the Euclidean norm) of x onto the set C. The equivalence of these formulations arises from the observation [4] that if x solves (NE) then $\pi_C(x)$ solves (AVI) and if z solves (AVI), then $z + a - Az$ solves (NE). A path following method was used in [1] to find a solution of (NE), based on properties of the normal manifold [11]. The algorithm is a realization of a more general scheme due to Eaves [5], which can be thought of as a generalization of the pivotal method due to Lemke [9]. Termination properties of the algorithm were studied on two matrix classes. One of these was the class of copositive–plus matrices, defined as follows.

Definition 1.1. *Let K be a given cone. A matrix A is said to be* copositive *with respect to K if*

$$\langle z, Az\rangle \geq 0, \qquad \forall z \in K.$$

A matrix A is said to be copositive–plus *with respect to K if it is copositive with respect to K and*

$$\langle z, Az\rangle = 0, z \in K \implies (A + A^\top)z = 0.$$

We point out that the property of being copositive–plus is defined with respect to a cone. The bigger the cone, the stronger is the assumption of being copositive–plus. For example, a positive semi–definite matrix is copositive–plus with respect to $\mathbb{R}^n$, on the other hand, any matrix in $\mathbb{R}^{n\times n}$ is copositive–plus with respect to $\{0\}$. The analysis of this paper requires that the matrix A be copositive–plus with respect to the recession cone of C.

Most of the existing pivotal methods for solving (AVI) depend on the existence of extreme points in C or certain non–singularity property of A with respect to the lineality of C. This condition usually amounts to $W^\top AW$ being invertible where W is a basis for $\operatorname{lin} C$. For example, when

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad C = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y \geq 0 \right\},$$

then a basis for the lineality of C is $W = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $W^\top AW = 0$, which is not invertible. In this paper, we prove that if A is copositive–plus with respect to $\operatorname{rec} C$, then we can remove the lineality space in the absence of such a non–singularity assumption. For convenience, we refer to $A_C(x)$ as a *copositive–plus normal map* when the matrix A is copositive–plus with respect to $\operatorname{rec} C$. Our reductions are constructive and hence can be used in an algorithm that will construct a solution of an equation determined by a copositive–plus normal map. In particular, the algorithm given in [1] will construct a solution or determine that no solution exists, in this case. The reductions given here could also be used to extend the algorithms for solving affine variational inequalities using path following methods found in [2, 3, 6, 7, 13, 14].

A word about our notation. For any vectors x and y in $\mathbb{R}^n$, $\langle x, y\rangle$ or $x^\top y$ denotes the inner product of x and y, and in this paper, these two notations are freely interchangeable. Each $m \times n$ matrix A represents a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, the symbol A refers to either the matrix or the linear map as determined by the context. For any vector or matrix, a superscript $\top$ indicates the transpose. Index sets are represented by lower case Greek letters. In particular, for the index set α, $|\alpha|$ denotes the cardinality of α. Given any vector v and an index set α, v_α denotes the set of components of v with indices in α. Given any matrix M and index sets α and β, $M_{\alpha\cdot}$ denotes the submatrix formed by those rows of M with indices in α, $M_{\cdot\beta}$ denotes the submatrix formed by those columns of M with indices in β, and $M_{\alpha\beta}$ denotes the submatrix formed by those elements of M with row indices in α and column indices in β. If C is non–empty, closed, convex, $\operatorname{rec} C := \{d \in \mathbb{R}^n : c + \lambda d \in C, \forall c \in C, \forall \lambda \geq 0\}$ is called the recession cone of C, and $\operatorname{lin} C := \operatorname{rec} C \cap -\operatorname{rec} C$. If F is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, then F_C represents the normal map

$$F_C(x) = F(\pi_C(x)) + x - \pi_C(x).$$

The indicator function of a set C is defined by

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{otherwise,} \end{cases}$$

and ∂f is the convex subdifferential of the convex function f. Further details of our notation can be found in [12].

2. Basic Reduction Techniques

In [1] we gave a pivotal algorithm that will solve any affine variational inequality (AVI) determined by a copositive–plus matrix, provided that $\operatorname{lin} C = \emptyset$. This algorithm was extended to find solutions of equations determined by copositive–plus normal maps for which $W^\top AW$ was invertible, where W represented a basis for $\operatorname{lin} C$ [1, Corollary 4.5]. The purpose of this paper is to remove the last assumption. We first outline some of the reductions that were used in [1] since they will be used again here but in a more careful manner.

The first idea is to change bases so that $\operatorname{lin} C = \mathbb{R}^{|\gamma|}$. Suppose that $C = \{z : Bz \geq b\}$. Then $\operatorname{lin} C = \ker B$, and it is easy to construct an orthonormal basis for $\ker B$ using a QR factorization (see [1]). Let this basis be extended through $(\operatorname{lin} C)^\perp$ to an orthonormal basis of $\mathbb{R}^n$, say P. It is easy to see that

$$A_C(x) = a \iff (P^\top AP)_{P^\top C}(P^\top x) = P^\top a, \tag{1}$$

and that A is copositive–plus with respect to a cone K if and only if $P^\top AP$ is copositive–plus with respect to $P^\top K$. Furthermore, $P^\top C = \mathbb{R}^{|\gamma|} \times \bar{C}$ with $\operatorname{lin} \bar{C} = \{0\}$. Thus we assume for the remainder of this paper that the above change of variables has occurred and that

$$A = \begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix}, \qquad a = \begin{bmatrix} a_\gamma \\ a_\delta \end{bmatrix}, \qquad C = \left\{ z = \begin{bmatrix} z_\gamma \\ z_\delta \end{bmatrix} : Bz_\delta \geq b \right\}, \tag{2}$$

with $\ker B = \{0\}$. The variables in the lineality are represented by z_γ and we assume that $\gamma \neq \emptyset$.

The following elementary results concerning projections onto the lineality of convex sets are easily verified.

Lemma 2.1. *Let C be a nonempty closed convex set and let L be a subspace of* $\operatorname{lin} C$. *Then for any $x \in \mathbb{R}^n$*

$$\pi_{C \cap L^\perp}(x) = \pi_C(\pi_{L^\perp}(x)) = \pi_{L^\perp}(\pi_C(x))$$

and

$$\pi_L(\pi_C(x)) = \pi_L(x).$$

If $Q_{L^\perp}$ is an orthonormal basis of $L^\perp$ then

$$Q_{L^\perp}^\top \pi_C(x) = \pi_{Q_{L^\perp}^\top C}(Q_{L^\perp}^\top x).$$

Our second reduction is a simple extension to that given in [11, Proposition 4.1]. It uses the notion of the Schur complement of a matrix, by which we mean

$$A = \begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix}, \qquad (A/A_{\gamma\gamma}) := A_{\delta\delta} - A_{\delta\gamma}A_{\gamma\gamma}^{-1}A_{\gamma\delta}.$$

Proposition 2.2. *Let C be a polyhedral set and let L be a subspace of* $\operatorname{lin} C$. *Let Q_L be an orthonormal basis of L, $Q_{L^\perp}$ be an orthonormal basis of $L^\perp$ and $Q = [Q_L \quad Q_{L^\perp}]$. Suppose that $Q_L^\top A Q_L$ is invertible. If x solves (NE) then $Q_{L^\perp} x$ solves*

$$\left(Q^\top AQ/Q_L^\top AQ_L\right)_{Q_{L^\perp}^\top C}(y) = Q_{L^\perp}^\top a - Q_{L^\perp}^\top AQ_L(Q_L^\top AQ_L)^{-1}Q_L^\top a. \tag{3}$$

If y solves (3) then

$$Q\begin{bmatrix}(Q_L^\top AQ_L)^{-1}(Q_L^\top a - Q_L^\top AQ_{L^\perp}y) \\ y\end{bmatrix}$$

solves (NE).

Proof. Using Lemma 2.1, we see that

$$\begin{aligned} A_C(x) = a \quad &\iff \quad A\pi_C(x) + x - \pi_C(x) = a \\ &\iff \quad A\pi_L(x) + A\pi_{C\cap L^\perp}(x) + x - \pi_L(x) - \pi_{C\cap L^\perp}(x) = a \\ &\iff \quad AQ_{L^\perp}Q_{L^\perp}^\top \pi_C(x) + x - Q_{L^\perp}Q_{L^\perp}^\top \pi_C(x) = a + (I-A)Q_LQ_L^\top x. \end{aligned}$$

First premultiply the above equation by $Q_L^\top$. This gives

$$Q_L^\top x = (Q_L^\top AQ_L)^{-1}Q_L^\top(a - AQ_{L^\perp}Q_{L^\perp}^\top \pi_C(x)).$$

Substituting for $Q_L^\top x$ in the above, and premultiplying by $Q_{L^\perp}^\top$ gives

$$\begin{aligned}\left(Q_{L^\perp}^\top AQ_{L^\perp} - Q_{L^\perp}^\top AQ_L(Q_L^\top AQ_L)^{-1}Q_L^\top AQ_{L^\perp}\right) Q_{L^\perp}^\top \pi_C(x) + Q_{L^\perp}^\top x - Q_{L^\perp}^\top \pi_C(x) \\ = Q_{L^\perp}^\top a - Q_{L^\perp}^\top AQ_L(Q_L^\top AQ_L)^{-1}Q_L^\top a\end{aligned}$$

from which the first statement of the proposition follows using Lemma 2.1.

For the second implication, suppose y solves (3). Define

$$w := (Q_L^\top AQ_L)^{-1}Q_L^\top(a - AQ_{L^\perp}\pi_{Q_{L^\perp}^\top C}(y)).$$

and note that $Q_L w \in L$. Hence, from Lemma 2.1,

$$\pi_C(Q\begin{bmatrix} w \\ y\end{bmatrix}) = \begin{bmatrix} Q_L w \\ Q_{L^\perp}\pi_{Q_{L^\perp}^\top C}(y)\end{bmatrix}.$$

The result now follows from elementary algebra. □

The reduction given in [1] is now immediate from Proposition 2.2 by letting $L = \operatorname{lin} C$ and then solving (3) under the assumption that $A_{\gamma\gamma} = Q_L^\top AQ_L$ is invertible. We now exhibit a more careful reduction that also uses Proposition 2.2. To prove this reduction is valid, we first state some technical results.

Lemma 2.3 ([10, Result 1.6]). *Let M be a positive semi-definite matrix, and assume*

$$M = \begin{bmatrix} 0 & u^\top \\ 0 & M' \end{bmatrix},$$

then $u = 0$.

Consequently, we have the following corollary.

Corollary 2.4. *Let $M \in \mathbb{R}^{n \times n}$ be a positive semi-definite matrix, and let*

$$\gamma \subset \{1, 2, \cdots, n\}.$$

Assume $M_{\cdot\gamma} = 0$, then $M_{\gamma\cdot} = 0$.

Proof. Apply the previous Lemma to each index of γ. □

We now make a change of variables over $\operatorname{lin} C$ which transforms the submatrix $A_{\gamma\gamma}$ corresponding to the lineality space, into a matrix of the form

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \tag{4}$$

where D is a positive definite matrix.

Lemma 2.5. *Suppose that A, a and C are defined by (2) and that A is copositive with respect to* $\operatorname{rec} C$. *Then, there exists an orthonormal matrix Q such that*

$$Q^\top A Q = \begin{bmatrix} D & 0 & A_{\alpha\delta} \\ 0 & 0 & A_{\beta\delta} \\ A_{\delta\alpha} & A_{\delta\beta} & A_{\delta\delta} \end{bmatrix}$$

with D positive definite. Furthermore, x solves (NE) if and only if $Q^\top x$ solves

$$(Q^\top A Q)_{Q^\top C}(y) = Q^\top a.$$

Proof. Since A is copositive with respect to $\operatorname{rec} C$ it follows that

$$x_\gamma^\top A_{\gamma\gamma} x_\gamma = \begin{bmatrix} x_\gamma^\top & 0 \end{bmatrix} \begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix} \begin{bmatrix} x_\gamma \\ 0 \end{bmatrix} \geq 0,$$

for all $x_\gamma \in \mathbb{R}^{|\gamma|}$. That is, $A_{\gamma\gamma}$ is positive semi–definite. Consider a QR factorization of $A_{\gamma\gamma}$

$$A_{\gamma\gamma} = Q_{\gamma\gamma} R,$$

where

$$R = \begin{bmatrix} R_0 \\ 0 \end{bmatrix}.$$

Here, R_0 is an upper triangular matrix whose row rank equals the rank of $A_{\gamma\gamma}$. By orthonormality of $Q_{\gamma\gamma}$,

$$Q_{\gamma\gamma}^\top A_{\gamma\gamma} Q_{\gamma\gamma} = D^*,$$

where $D^* = R Q_{\gamma\gamma}$. Furthermore D^* is of the form

$$D^* = \begin{bmatrix} D' \\ 0 \end{bmatrix},$$

with D^* positive semi–definite, and

$$\operatorname{rank} D' = \operatorname{rank} R = \operatorname{rank} R_0.$$

Thus, D is a matrix in the form of (4) due to Corollary 2.4.

Let

$$Q = \begin{bmatrix} Q_{\gamma\gamma} & \\ & I \end{bmatrix},$$

then Q is orthonormal and the result follows from (1). $\square$

3. Reductions for Copositive–plus Normal Maps

In this section we show that any invertibility assumption over the lineality space is unnecessary in the case that A is copositive–plus with respect to $\operatorname{rec} C$. The proof of this result requires two separate reductions. The first, which we gave as Lemma 2.5, allows us to assume that

$$A = \begin{bmatrix} D & 0 & A_{\alpha\delta} \\ 0 & 0 & A_{\beta\delta} \\ A_{\delta\alpha} & A_{\delta\beta} & A_{\delta\delta} \end{bmatrix}, \qquad a = \begin{bmatrix} a_\alpha \\ a_\beta \\ a_\delta \end{bmatrix}, \qquad C = \left\{ z = \begin{bmatrix} z_\alpha \\ z_\beta \\ z_\delta \end{bmatrix} : Bz_\delta \geq b \right\}, \tag{5}$$

with $\ker B = \{0\}$. It is also clear that if the original matrix was copositive–plus with respect to the recession cone of the original C, then A is copositive–plus with respect to $\operatorname{rec} C$ as defined in (5).

It is a crucial part of our analysis to note that the reduction given in Proposition 2.2 maintains the copositive–plus property. We state this as the following lemma.

Lemma 3.1. *Let K be a cone and suppose that L is a linear space contained in K. Let Q_L be an orthonormal basis of L, $Q_{L^\perp}$ be an orthonormal basis of $L^\perp$ and $Q = [Q_L \quad Q_{L^\perp}]$. If A is copositive–plus with respect to K, then $\left(Q^\top AQ/Q_L^\top AQ_L\right)$ is copositive–plus with respect to $Q_{L^\perp}^\top K$.*

Proof. Note that by the remarks after (1), it is sufficient to prove the result for $Q = I$, with A being partitioned conformally as

$$A = \begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix}.$$

For any $z \in I_\delta.K$

$$\begin{aligned} z^\top (A/A_{\gamma\gamma})\, z &= z^\top(A_{\delta\delta} - A_{\delta\gamma}A_{\gamma\gamma}^{-1}A_{\gamma\delta})z \\ &= \begin{bmatrix} w^\top & z^\top \end{bmatrix} \begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}, \end{aligned}$$

where $w = -A_{\gamma\gamma}^{-1}A_{\gamma\delta}z$. Thus $(w, z) \in K$. Since A is copositive–plus with respect to K, we have

$$z^\top (A/A_{\gamma\gamma})\, z = \begin{bmatrix} w^\top & z^\top \end{bmatrix} \begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \geq 0,$$

so that $(A/A_{\gamma\gamma})$ is copositive with respect to $I_\delta.K$.

For any $z \in I_\delta.K$ such that

$$z^\top (A/A_{\gamma\gamma})\, z = 0,$$

we have

$$\begin{bmatrix} w^\top & z^\top \end{bmatrix} \begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = 0,$$

where $w = -A_{\gamma\gamma}^{-1} A_{\gamma\delta} z$. Hence

$$\begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} + \begin{bmatrix} A_{\gamma\gamma} & A_{\gamma\delta} \\ A_{\delta\gamma} & A_{\delta\delta} \end{bmatrix}^\top \begin{bmatrix} w \\ z \end{bmatrix} = 0, \tag{6}$$

due to A being copositive–plus with respect to K. In particular

$$\begin{aligned} A_{\gamma\gamma} w + A_{\gamma\delta} z &= 0 \\ A_{\gamma\gamma}^\top w + A_{\delta\gamma}^\top z &= 0 \\ A_{\delta\gamma} w + A_{\delta\delta} z + A_{\gamma\delta}^\top w + A_{\delta\delta}^\top z &= 0, \end{aligned}$$

where the first equation is due to the definition of w and the second equation follows from the first and (6). By using the first two equations in the third

$$(A_{\delta\delta} - A_{\delta\gamma} A_{\gamma\gamma}^{-1} A_{\gamma\delta}) z + (A_{\delta\delta} - A_{\delta\gamma} A_{\gamma\gamma}^{-1} A_{\gamma\delta})^\top z = 0.$$

That is

$$(A/A_{\gamma\gamma})\, z + (A/A_{\gamma\gamma})^\top z = 0.$$

Thus $(A/A_{\gamma\gamma})$ is copositive–plus with respect to I_δ.. □

In the following proposition, we reduce the problem resulting from Lemma 2.5 by eliminating the variables associated with the positive definite matrix D. The proof relies heavily on Proposition 2.2 and Lemma 3.1.

Proposition 3.2. *Suppose A, a and C are given by* (5) *and that A is copositive–plus with respect to* $\operatorname{rec} C$. *If x solves (NE), then (x_β, x_δ) solves*

$$\bar{A}_{\bar{C}}(y) = \begin{bmatrix} a_\beta \\ a_\delta - A_{\delta\alpha} D^{-1} a_\alpha \end{bmatrix}, \tag{7}$$

where

$$\bar{A} = \begin{bmatrix} 0 & A_{\beta\delta} \\ A_{\delta\beta} & A_{\delta\delta} - A_{\delta\alpha} D^{-1} A_{\alpha\delta} \end{bmatrix}, \qquad \bar{C} = \{z = (z_\beta, z_\gamma) \colon B z_\delta \geq b\}. \tag{8}$$

If $y = (y_\beta, y_\delta)$ solves (7), *then*

$$\begin{bmatrix} D^{-1}(a_\alpha - A_{\alpha\delta} y_\delta) \\ y \end{bmatrix}$$

solves (NE). Furthermore, $\bar{A}$ is copositive–plus with respect to $\operatorname{rec} \bar{C}$.

Proof. Let $L = \mathbb{R}^{|\alpha|}$ and apply Proposition 2.2. The final statement of the proposition follows from Lemma 3.1. □

Thus after this reduction, we may assume that the problem has the form

$$A = \begin{bmatrix} 0 & A_{\beta\delta} \\ A_{\delta\beta} & A_{\delta\delta} \end{bmatrix}, \qquad a = \begin{bmatrix} a_\beta \\ a_\delta \end{bmatrix}, \qquad C = \{z = (z_\beta, z_\gamma) \colon Bz_\delta \geq b\}, \tag{9}$$

with $\ker B = \{0\}$ and A copositive–plus with respect to $\operatorname{rec} C$. However, it follows from the copositive–plus property that $A_{\delta\beta} = -A_{\beta\delta}^\top$ as the following lemma shows.

Lemma 3.3. *Suppose*

$$A = \begin{bmatrix} 0 & A_{\beta\delta} \\ A_{\delta\beta} & A_{\delta\delta} \end{bmatrix}$$

is copositive–plus with respect to $\mathbb{R}^{|\beta|} \times K$ *where* K *is any cone containing the origin. Then* $A_{\delta\beta} + A_{\beta\delta}^\top = 0$.

Proof. Take $z_\beta \in \mathbb{R}^{|\beta|}$, $z_\delta = 0$ and apply the definition of copositive–plus to conclude that

$$(A_{\delta\beta} + A_{\beta\delta}^\top) z_\beta = 0.$$

However, z_β is arbitrary. $\square$

The following result shows that given an equation defined by a normal map of the form (9), we are able to reduce it to one whose feasible set has zero lineality.

Theorem 3.4. *Suppose* A, a *and* C *are given by*

$$A = \begin{bmatrix} 0 & A_{\beta\delta} \\ -A_{\beta\delta}^\top & A_{\delta\delta} \end{bmatrix}, \qquad a = \begin{bmatrix} a_\beta \\ a_\delta \end{bmatrix}, \qquad C = \{z = (z_\beta, z_\delta) \colon Bz_\delta \geq b\}.$$

Suppose A *is copositive–plus with respect to* $\operatorname{rec} C$, *and define*

$$\bar{A} := A_{\delta\delta}, \qquad \bar{C} := \{z_\delta \colon Bz_\delta \geq b, A_{\beta\delta} z_\delta = a_\beta\}.$$

If $\bar{x}$ *solves (NE), then* $\bar{x}_\delta$ *solves*

$$\bar{A}_{\bar{C}}(y) = a_\delta. \tag{10}$$

If $\bar{x}_\delta$ *solves* (10), *then there exists* $\bar{x}_\beta$ *such that* $(\bar{x}_\beta, \bar{x}_\delta)$ *solves (NE). Moreover,* $\bar{A}$ *is copositive–plus with respect to* $\operatorname{rec} \bar{C}$.

Proof. The first implication is easy. For the second implication, notice that $\bar{x}_\delta$ satisfies $\bar{A}_{\bar{C}}(y) = a_\delta$ if and only if $\bar{z} = \pi_{\bar{C}}(\bar{x}_\delta)$ satisfies

$$-A_{\delta\delta}\bar{z} + a_\delta \in \partial\psi_{\bar{C}}(\bar{z}).$$

Let $\tilde{C} = \{z_\delta \colon Bz_\delta \geq b\}$ and note that $\bar{C} = \tilde{C} \cap \{z \colon A_{\beta\delta} z = a_\beta\}$. By reference to [12, Corollary 23.8.1] we have

$$-A_{\delta\delta}\bar{z} + a_\delta \in \partial\psi_{\tilde{C}}(\bar{z}) + \operatorname{im} A_{\beta\delta}^\top,$$

or

$$-A_{\delta\delta}\bar{z} + a_\delta + A_{\beta\delta}^\top \bar{y} \in \partial\psi_{\tilde{C}}(\bar{z}), \tag{11}$$

for some $\bar{y}$. Hence $\bar{z}$, together with $\bar{y}$, satisfies

$$\begin{aligned}
&\bar{z} \in \{z_\delta \colon Bz_\delta \geq b\} \\
&A_{\beta\delta}\bar{z} - a_\beta = 0 \\
&-A_{\delta\delta}\bar{z} + a_\delta + A_{\beta\delta}^\top \bar{y} \in \partial\psi_{\bar{C}}(\bar{z}),
\end{aligned}$$

that is $(\bar{y}, \bar{z}) \in C$ and

$$-\begin{bmatrix} 0 & A_{\beta\delta} \\ -A_{\beta\delta}^\top & A_{\delta\delta} \end{bmatrix} \begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} + a \in \partial\psi_{\mathbb{R}^{|\beta|} \times \bar{C}}(\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix}),$$

or

$$-A \begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} + a \in \partial\psi_C(\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix}).$$

Therefore

$$\bar{x} := \begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} + a - A \begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} \tag{12}$$

solves $A_C(x) = a$.

It is obvious that $\bar{A}$ is copositive–plus with respect to $\tilde{C}$, and $\bar{C} \subset \tilde{C}$. Hence, $\bar{A}$ is copositive–plus with respect to $\bar{C}$. □

Theorem 3.4 is actually a variant of the results regarding augmented LCP discussed by Eaves in [4] and by Gowda and Pang in [8].

Notice that δ can be determined easily from a single QR factorization (see (2)) so that $\bar{A}$ and $\bar{C}$ can be easily formed. Furthermore, the path following algorithm of [1] can be used to solve the reduced problem. The fact that $\bar{A}$ is copositive–plus with respect to $\operatorname{rec} \bar{C}$ guarantees that this algorithm will process the reduced problem. Given the projection $\bar{z}$ of a solution of $\bar{A}_{\bar{C}}(y) = a_\delta$, a solution of (NE) can be constructed from (11) by solving a linear program, since (11) is equivalent to

$$\begin{aligned}
-A_{\delta\delta}\bar{z} + a_\delta + A_{\beta\delta}^\top \bar{y} = B^\top u, &\qquad u \leq 0, \\
u_i = 0 &\qquad \text{if } B_{i\cdot}\bar{z} > b_i.
\end{aligned}$$

Thus, $\bar{x}$ can be constructed from $\bar{z}$ by solving a linear program to obtain $\bar{y}$ and using (12). Thus we have the following result.

Theorem 3.5. *Let $A \in \mathbb{R}^{n\times n}$ and C be a nonempty polyhedral set, such that A is copositive–plus with respect to $\operatorname{rec} C$. The algorithm given in [1], applied to (10), solves (NE) or determines that no solution exists.*

Proof. Let $C = \{z \colon Bz \geq b\}$. First find δ and γ by performing a QR factorization to determine a basis of $\ker B$. Reduce the original problem to form (2). Construct a QR factorization of the resulting $A_{\gamma\gamma}$ and reduce the problem again so that it has form (5). Factor out the contribution from D and reduce the problem to the form given as (8). By noting Lemma 3.3, reduce (8) to the form of (10). (10) is solvable (or demonstrably not solvable) by the algorithm of [1]. □

4. Conclusions

We have shown in the proof of Theorem 3.5 how to reduce any equation determined by a copositive–plus normal map to one with zero lineality. The resulting equation can be solved or proven infeasible using the variant Lemke's algorithm given in [1]. A corresponding solution of (AVI) can be reconstructed from the solution of the reduced problem. An outstanding research question is whether the analysis given in this paper can be extended to the case of normal maps determined L-matrices (with respect to $\operatorname{rec} C$).

References

1. M. Cao and M. C. Ferris, A pivotal method for affine variational inequalities, Tech. Rep. 1114, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, 1992, To appear in Mathematics of Operations Research.
2. Y. Dai and D. Talman, Linear stationary point problems on unbounded polyhedra, *Mathematics of Operations Research*, pp. 635–644, 1993.
3. Y. Dai, G. van der Laan, A. J. J. Talman and Y. Yamamoto, A simplicial algorithm for the nonlinear stationary problem on an unbounded polyhedron, *SIAM Journal on Optimization*, 1, 151–165, 1991.
4. B. C. Eaves, On the basic theorem of complementarity, *Mathematical Programming*, 1, 68–87, 1971.
5. B. C. Eaves, A short course in solving equations with PL homotopies, in R. W. Cottle and C. E. Lemke (eds.), *Nonlinear Programming*, pp. 73–143, Providence, RI, 1976. American Mathematical Society, SIAM–AMS Proceedings.
6. B. C. Eaves, Computing stationary points, *Mathematical Programming Study*, 7, 1–14, 1978.
7. B. C. Eaves, Computing stationary points, again, in O. L. Mangasarian, R. R. Meyer and S. M. Robinson (eds.), *Nonlinear Programming 3*, pp. 391–405. Academic Press Press, New York, 1978.
8. M. S. Gowda and J. S. Pang, The basic theorem of complementarity revisited, *Mathematical Programming*, 58, 161–178, 1993.
9. C. E. Lemke, Bimatrix equilibrium points and mathematical programming, *Management Science*, 11, 681–689, 1965.
10. K. G. Murty, *Linear Complementarity, Linear and Nonlinear Programming.* Helderman–Verlag, Berlin, 1988.
11. S. M. Robinson, Normal maps induced by linear transformations, *Mathematics of Operations Research*, 17, 691–714, 1992.
12. R. T. Rockafellar, *Convex Analysis.* Princeton University Press, Princeton, NJ, 1970.
13. A. J. J. Talman and Y. Yamamoto, A SImplicial algorithm for stationary point problems, *Mathematics of Operations Research*, 14, 383–399, 1989.
14. Y. Yamamoto, A path–following algorithm for stationary point problems, *Journal of Operations Society of Japan*, 30, 181–198, 1987.