P_c–Matrices and the Linear Complementarity Problem

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ABSTRACT

We introduce a new matrix class P_c, which consists of those matrices M for which the solution set of the corresponding linear complementarity problem is connected for every $q \in R^n$. We consider Lemke's pivotal method from the perspective of piecewise linear homotopies and normal maps and show that Lemke's method processes all matrices in $P_c \cap Q_0$. We further investigate the relationship of the class P_c to other known matrix classes and show that column sufficient matrices are a subclass of P_c, as are 2×2 P_0–matrices.

1. INTRODUCTION

The linear complementarity problem is a classical problem from optimization theory of finding $x \in \mathbf{R}^n$ with

$$z \geq 0, \quad Mz + q \geq 0, \quad z^\top(Mz + q) = 0.$$

Here $M \in \mathbf{R}^{n \times n}$ and $q \in \mathbf{R}^n$ are given data, and the resulting problem will be denoted by LCP(q, M). We also define the set of feasible points of LCP(q, M) by

$$\text{FEA}(q, M) := \{z \mid z \geq 0, \quad Mz + q \geq 0\}.$$

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In this paper, we investigate a new class of matrices, P_c, which is defined by

$$M \in \mathrm{P}_c \iff \mathrm{SOL}(q, M) \text{ is connected for all } q \in \mathbf{R}^n,$$

where $\mathrm{SOL}(q, M)$ is the set of solutions of $\mathrm{LCP}(q, M)$. The most widely used algorithm for solving $\mathrm{LCP}(q, M)$ is the pivotal algorithm of Lemke[10]. In [1], it is shown that Lemke's method processes all matrices $M \in \mathrm{P}_0 \cap \mathrm{Q}_0$, that is, it either finds a solution, or determines that $\mathrm{FEA}(q, M) = \emptyset$. Here P_0 is the class of matrices having non–negative principal minors. The principal result of this paper, given in Section 2, is that if $M \in \mathrm{P}_c \cap \mathrm{Q}_0$, then Lemke's method processes $\mathrm{LCP}(q, M)$. Note that Q_0 is the set of matrices for which feasibility of $\mathrm{LCP}(q, M)$ implies its solvability, that is

$$M \in \mathrm{Q}_0 \qquad \iff \qquad (\mathrm{FEA}(q, M) \neq \emptyset \quad \Rightarrow \quad \mathrm{SOL}(q, M) \neq \emptyset)\,.$$

Before proving this result, let us explain our motivation. An $n \times n$ matrix is a member of the matrix class P if all its principal minors are positive. It is well known that an equivalent definition is that $\mathrm{SOL}(q, M)$ is a singleton for every $q \in \mathbf{R}^n$. Therefore, a natural extension of the class P is the class of column sufficient matrices S_c, characterized by

$$M \in \mathrm{S}_c \iff \mathrm{SOL}(q, M) \text{ is convex for all } q \in \mathbf{R}^n.$$

Although there are other extensions of the class S_c, the most natural geometric extension would seem to be to P_c. Note that it is clear that

$$\mathrm{P} \subset \mathrm{S}_c \subset \mathrm{P}_c.$$

In order to relate our result to others found in the literature, we explore the class P_c further in Section 3. It is known [4, Theorem 3.3.4, 3.4.2] that

$$\mathrm{P} \subset \mathrm{S}_c \subset \mathrm{P}_0 \subset \mathrm{E}_0, \tag{1}$$

where E_0 is the class of matrices for which $\mathrm{SOL}(q, M)$ is a singleton for all $q > 0$. We know of no geometric properties of $\mathrm{LCP}(q, M)$ that characterize P_0 or E_0. In this paper, the geometrically defined class of matrices P_c is shown to be closely related to the algebraically defined class P_0. The interplay between algebraic and geometric characterizations of matrix classes is of paramount importance to a complete understanding of such classes. We show that within the class of 2×2 matrices

$$\mathrm{P}_0 \subset \mathrm{P}_c \subset \mathrm{E}_0, \tag{2}$$

and that these inclusions are strict. We conjecture that (2) holds for $n \times n$ matrices and hence that our main result extends the class of matrices that Lemke's method is known to process. However, in the 2×2 case, we also show that

$$\mathrm{P}_0 \cap \mathrm{Q}_0 = \mathrm{P}_c \cap \mathrm{Q}_0.$$

2. TERMINATION OF LEMKE'S METHOD

Although the basic step of Lemke's method is a pivot (as in the simplex method for linear programming), the choice of pivot step is fundamentally different and is motivated by a path following or homotopy approach. An equivalent formulation of LCP(q, M) is to find a zero of the nonsmooth mapping

$$Mx_+ + q + x - x_+,$$

where $(x_+)_i := \max\{x_i, 0\}$ is the projection of x onto the nonnegative orthant. This map is sometimes referred to as the "normal map" [12]; the earliest known reference is [7]. The equivalence is established by noting that if z solves LCP(q, M), then $x = z - Mz - q$ is a zero of the normal map, and if x is a zero of the normal map, then $z = x_+$ is a solution of LCP(q, M). It is easy to see that the normal map is an affine map on each of the orthants of $\mathbf{R}^n$ and is continuous on $\mathbf{R}^n$. The normal mapping is thus an example of a piecewise affine map and is intimately related to the manifold defined by the collection of the faces of the set $\mathbf{R}^n_+$, called the normal manifold [12]. Lemke's method can be viewed as a clever way of traversing this manifold as each pivot step corresponds to changing the affine map that currently represents the normal map. In fact, Lemke's method is an instance of a more general algorithm for solving equations with piecewise linear homotopies due to Eaves [8]. The analysis in this paper uses many of the ideas contained in [8] without further proof. We will also use the fact that the general algorithm applied to LCP(q, M) is in fact Lemke's algorithm; this is shown elsewhere [2, 8].

Let $\mathcal{N}$ be the piecewise-linear manifold in $\mathbf{R}^{n+1}$ constructed by forming the Cartesian product of each orthant of $\mathbf{R}^n$ with $\mathbf{R}_+$, the non-negative half-line in $\mathbf{R}$. We abuse notation slightly and let $\mathcal{N}$ represent both the collection of cells of the manifold and the union of this collection. $\mathcal{N}$ is a piecewise linear $(n+1)$-manifold in $\mathbf{R}^{n+1}$, as can easily be verified (see [8, Example 4.3]). Now let $e > 0$ and consider the piecewise linear map $F: \mathcal{N} \to \mathbf{R}^n$ defined by

$$F(x, \mu) := Mx_+ + q + x - x_+ + \mu e.$$

Clearly any x satisfying $F(x,0)=0$ solves LCP(q,M). Let $w(\mu):=-q-\mu e$, and note that since

$$w(\mu)=-\mu[e+\mu^{-1}q], \tag{3}$$

$w(\mu)$ lies interior to the orthant $\mathbf{R}^n_-$, for large positive μ. Therefore $(w(\mu),\mu)$ lies interior to the cell $\mathbf{R}^n_- \times \mathbf{R}_+$ of $\mathcal{N}$, and so it is a regular point of $\mathcal{N}$ (see the proof of Theorem 2). Further, for such μ we have $(w(\mu))_+ = 0$, so that

$$F(w(\mu),\mu)=-q-\mu e-(q+\mu e)=0.$$

Therefore for some $\mu_0 \geq 0$, $F^{-1}(0)$ contains the ray $\{(w(\mu),\mu) \mid \mu \geq \mu_0\}$.

Now the algorithm of [8] is applied to the PL equation $F(x,\mu)=0$, using a ray start at $(w(\mu_1),\mu_1)$ for some $\mu_1 > \mu_0$ and proceeding in the direction $(-e,-1)$. As the manifold $\mathcal{N}$ is finite, according to [8, Th. 15.13] the algorithm generates, in finitely many steps, either a point (x_*,μ_*) in the boundary of $\mathcal{N}$ with $F(x_*,\mu_*)=0$, or a secondary ray in $F^{-1}(0)$ different from the starting ray. In the first case $\mu_*=0$ and, by our earlier remarks, x_* solves LCP(q,M). Many of the results pertaining to Lemke's method processing different classes of matrices just show that secondary ray termination cannot occur, or that ray termination guarantees FEA$(q,M)=\emptyset$.

Such results are plentiful and Lemke's algorithm is known to process many classes of matrices, see, for example [3, 4, 11]. There are, in fact, two large but distinct classes that contain most of these classes of matrices, namely L–matrices [6] and the class $\mathrm{P}_0 \cap \mathrm{Q}_0$ [1]. This paper is concerned with extending the algebraically defined class $\mathrm{P}_0 \cap \mathrm{Q}_0$ using geometric ideas. To this purpose, we introduce the class P_c. In the remainder of this section we will show that Lemke's method processes matrices from $\mathrm{P}_c \cap \mathrm{Q}_0$. We shall explore the class P_c more fully in the following section.

The set

$$K(M):=\{q\in\mathbf{R}^n \mid \mathrm{SOL}(q,M)\neq\emptyset\},$$

is the set of all right hand side vectors for which LCP(q,M) is solvable. This set is intimately related to the class Q_0 as the following theorem shows.

THEOREM 1 ([6]) *For an $n\times n$ matrix M, the following are equivalent:*

1. *$M\in\mathrm{Q}_0$.*
2. *$K(M)$ is convex.*
3. *$K(M)=\mathrm{pos}(I,-M)$.*

Here $\mathrm{pos}(I,-M)$ *represents the cone generated by the columns of the matrix $(I,-M)$ and the origin.*

Note that $\mathrm{pos}(I, -M)$ is a polyhedral convex cone. Our main result is summarized in the following theorem. The two key geometric facts that we use in the proof are

1. connectedness of $\mathrm{SOL}(q, M)$ for all q
2. convexity of $K(M)$

THEOREM 2. *Suppose M is in $\mathrm{P}_c \cap \mathrm{Q}_0$. Then Lemke's algorithm terminates at a solution of* $\mathrm{LCP}(q, M)$ *or determines that* $\mathrm{FEA}(q, M) = \emptyset$. *Furthermore, the parameter μ in Lemke's algorithm is nonincreasing.*

Proof. Since 0 may not be a regular value of F, we use the pivotal algorithm from [8] which generates a solution of the original problem by solving the perturbed system

$$F(x, \mu) = -[\epsilon],$$

where $[\epsilon] = (\epsilon, \epsilon^2, \ldots, \epsilon^n)^\top$, with $\epsilon > 0$.

Let $w(\mu) := -q - \mu e = -\mu[e + \mu^{-1} q]$, so that $w(\mu)$ lies interior to the orthant $\mathbf{R}^n_-$, for large positive μ. Therefore $(w(\mu) - [\epsilon], \mu)$ lies interior to the cell $\mathbf{R}^n_- \times \mathbf{R}_+$ of $\mathcal{N}$ for μ sufficiently large and ϵ sufficiently small. It is a regular point of $\mathcal{N}$ since $F(\mathbf{R}^n_- \times \mathbf{R}_+)$ has a nonempty interior. Further, for large μ we have $(w(\mu) - [\epsilon])_+ = 0$, so that

$$F(w(\mu)-[\epsilon], \mu) = M(w(\mu)-[\epsilon])_+ + q + w(\mu) - [\epsilon] - (w(\mu)-[\epsilon])_+ + \mu e = -[\epsilon].$$

Hence, $F^{-1}(-[\epsilon])$ contains the ray $\{(w(\mu) - [\epsilon], \mu) \mid \mu \geq \mu_0\}$, for some $\mu_0 \geq 0$.

Now the algorithm of [8] is applied to the PL equation $F(x, \mu) = -[\epsilon]$, using a ray start at $(w(\mu_1), \mu_1)$ for some $\mu_1 > \mu_0$ and proceeding in the direction $(-e, -1)$.

Since $-[\epsilon] \in F(\mathcal{N})$ for all sufficiently small ϵ, it follows from [8, Lemma 14.2], that $-[\epsilon]$ is a regular value of F for each small positive ϵ. It then follows by [8, Theorem 9.1] that for such ϵ, $F^{-1}(-[\epsilon])$ is a 1-manifold neat in $\mathcal{N}$. This means that $F^{-1}(-[\epsilon])$ is closed in $\mathcal{N}$ and its boundary is its intersection with the boundary of $\mathcal{N}$. It is subdivided by $\sigma \cap F^{-1}(-[\epsilon])$, where σ is an n–cell of $\mathcal{N}$. Furthermore, we have $(w(\mu)-[\epsilon], \mu) \in F^{-1}(-[\epsilon])$, for sufficiently large μ.

Now, assume that the algorithm generates a sequence of points (x_1, μ_1), (x_2, μ_2), $\cdots$, (x_k, μ_k) with $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k$ and either terminates at step k with a ray different from the starting one or generates a point (x_{k+1}, μ_{k+1}) with $\mu_{k+1} > \mu_k$. Let $W(\epsilon)$ be the set of chords traversed by the algorithm up to this point. Then, due to the ray start, $W(\epsilon)$ cannot be PA homeomorphic to a circle, and therefore it is homeomorphic to an interval [8, Lemma 5.1].

Upon ray termination, μ is non–decreasing on the terminating ray. Thus, the set

$$\Xi = \{\mu \mid (x,\mu) \in W(\epsilon)\}$$

admits a minimum $\bar{\mu} = \inf\{\mu \in \Xi\} > 0$, which is achieved on (x_j, μ_j) for some $1 \le j \le k$. Let

$$S = \{x \mid (x,\bar{\mu}) \in W(\epsilon)\},$$

then $F(x,\bar{\mu}) = -[\epsilon]$ for $x \in S$. Hence

$$S \subset \mathrm{SOL}(q + [\epsilon] + \bar{\mu}e, M).$$

But $\mathrm{SOL}(q + [\epsilon] + \bar{\mu}e, M)$ cannot contain any other point z_1 such that $(z_1,\bar{\mu}) \notin W(\epsilon)$, otherwise, by our hypothesis on the connectedness of the solution set, there is a continuous path $z\colon [0,1] \to \mathrm{SOL}(q+[\epsilon]+\bar{\mu}e, M)$ with $z(1) = z_1$ and $z(0) = z_0$ for any $z_0 \in S$. Thus

$$\{(z(t),\bar{\mu}) \mid 0 \le t \le 1\} \subset F^{-1}(-[\epsilon]).$$

But this contradicts the fact that $F^{-1}(-[\epsilon])$ is a 1–manifold, since $(z_0,\bar{\mu})$ contains a neighborhood not homeomorphic to an interval (see Figure 1).

Thus $S = \mathrm{SOL}(q + [\epsilon] + \bar{\mu}e, M)$ is a connected set. It is either a single point, or the union of finite number of consecutive chords in $W(\epsilon)$. In particular, S is closed.

We now show that if $q \in K(M)$ then $\bar{\mu} = 0$. The cone $K(M)$ contains the positive orthant in its interior and it is convex because $M \in \mathrm{Q}_0$. Since $q + [\epsilon] \in K(M)$ and $q + [\epsilon] + \bar{\mu}e \in K(M)$ it follows that

$$q + [\epsilon] + \lambda\mu e \in K(M)$$

for every $\mu \in [0,\bar{\mu}]$. Hence

$$\mathrm{SOL}(q + [\epsilon] + \mu e, M) \neq \emptyset$$

for all $\mu \in [0,\bar{\mu}]$. Consider a strictly increasing sequence $\{\mu_j \mid j = 1, 2, \cdots\}$ with $\mu_1 < \bar{\mu}$ and $\lim_{j\to\infty} \mu_j = \bar{\mu}$. Assume that $x(\mu_j) \in \mathrm{SOL}(q + [\epsilon] + \mu_j e, M)$. Then, $(x(\mu_j),\mu_j) \in F^{-1}(-[\epsilon])$, hence each $(x(\mu_j),\mu_j) \in F^{-1}(-[\epsilon])$ is contained in a 1–chord of $F^{-1}(-[\epsilon])$. Since the 1–manifold $F^{-1}(-[\epsilon])$ is finite, there exists a chord l (which is a line segment) such that $(x(\mu_j),\mu_j) \in l$

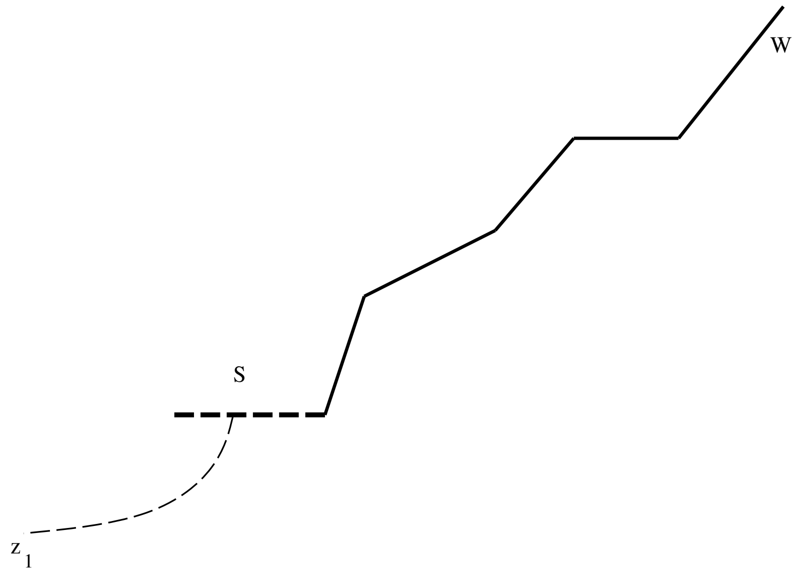

Figure 1: THE PATH CONNECTING z_1 TO S FORMS A BRANCH OF W

for infinitely many j, and without loss of generality we can assume that $(x(\mu_j), \mu_j) \in l$ for all j. Therefore l contains the set

$$\left\{ (x(\mu), \mu) \in F^{-1}(-[\epsilon]) \mid \bar{\mu} - \delta \leq \mu < \bar{\mu} \right\},$$

for some $\delta > 0$. Thus l contains a point $(w(\bar{\mu}), \bar{\mu})$ with $w(\bar{\mu}) \in S$.

On the other hand, by definition of $\bar{\mu}$

$$(w(\mu), \mu) \notin W(\epsilon),$$

for any $\mu < \bar{\mu}$. Hence l is not a subset of $W(\epsilon)$, and l forms a branch from $S \times \{\bar{\mu}\}$ (see Figure 2). This is in contradiction to the fact that $F^{-1}(-[\epsilon])$ is a 1–manifold.

So if $q \in K(M)$ the algorithm terminates at a point at the boundary, that is, a solution of $F(x, 0) = -[\epsilon]$.

If the algorithm also terminates in the boundary when $q \notin K(M)$, this leads immediately to a contradiction. Thus in this case, ray termination must occur. ■

In practice the algorithm does not actually use a positive ϵ, but only maintains the information necessary to compute $W(\epsilon)$ for all small positive ϵ, employing the lexicographic ordering to resolve possible ambiguities when

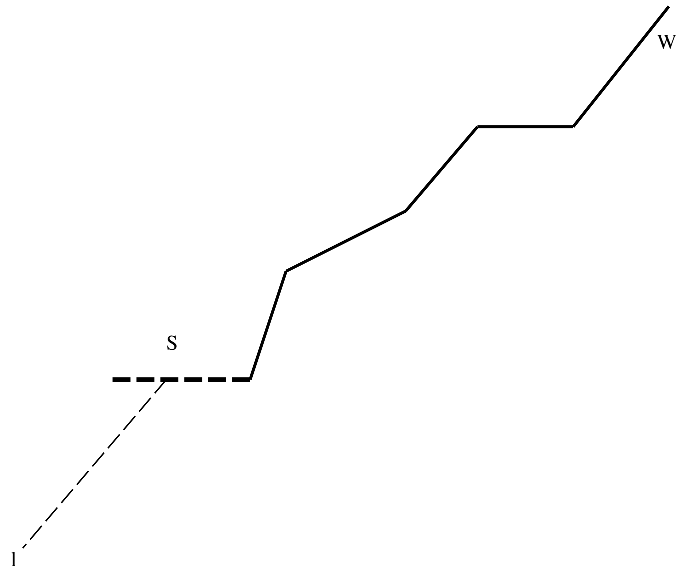

Figure 2: The chord l forms a branch of W

$\epsilon = 0$. Therefore after finitely many steps it will actually have computed x_0 with $M_{\mathbf{R}^n_+}(x_0) + q = 0$, or prove that $\mathrm{FEA}(q, M) = \emptyset$.

3. RELATIONSHIP TO OTHER CLASSES

The aim of this section is to explore the relationship of P_c to other known classes of matrices. In particular, we consider its relationship to column sufficient matrices, P_0 and E_0.

We first show how column sufficient matrices are related to P_0, and also to P_c. A matrix M is said to be *column sufficient* if, given $z \in \mathbf{R}^n$

$$z_i(Mz)_i \leq 0 \quad \text{for all } i \qquad \Rightarrow \qquad z_i(Mz)_i = 0 \quad \text{for all } i$$

The class of such matrices is denoted as S_c and it is shown in [4, Proposition 3.5.8] that this definition is equivalent to the one given in the introduction. M is *row sufficient* if its transpose is column sufficient and M is *sufficient* if it is both column and row sufficient.

Clearly, $\mathrm{P} \subset \mathrm{S}_c \subset \mathrm{P}_c$ since the solutions sets are: a singleton, convex and connected, respectively. The following corollary, which also follows from [5] is now immediate.

COROLLARY 3. *Suppose $M \in \mathrm{S}_c \cap \mathrm{Q}_0$. Then Lemke's algorithm terminates at a solution of* $\mathrm{LCP}(q, M)$ *or determines that* $\mathrm{FEA}(q, M) = \emptyset$.

Proof. Since M is column sufficient, $\mathrm{SOL}(q, M)$ is convex, and is hence connected for all q. The corollary now follows from Theorem 2. ∎

It is also known that Lemke's method processes row sufficient matrices since these are contained in $\mathrm{P}_0 \cap \mathrm{Q}_0$ [4, 3.5.3 and 3.5.5].

The following example shows that P_c is not a subclass of P_0. The matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

does not belong to P_0. But for any $q \in \mathbf{R}^n$, we have

$$\mathrm{SOL}(q, M) = \begin{cases} \{(0,0)\} & \text{if } q_1 > 0, q_2 > 0 \\ \{(0,y) \mid y \geq 0\} & \text{if } q_1 > 0, q_2 = 0 \\ \emptyset & \text{if } q_1 > 0, q_2 < 0 \\ \{(x,0) \mid x \geq 0\} & \text{if } q_1 = 0, q_2 > 0 \\ \{(x,0) \mid x \geq 0\} \cup \{(0,y) \mid y \geq 0\} & \text{if } q_1 = 0, q_2 = 0 \\ \{(x,0) \mid x \geq -q_2\} & \text{if } q_1 = 0, q_2 < 0 \\ \emptyset & \text{if } q_1 < 0, q_2 > 0 \\ \{(0,y) \mid y \geq -q_1\} & \text{if } q_1 < 0, q_2 = 0 \\ \{(-q_2, -q_1)\} & \text{if } q_1 < 0, q_2 < 0 \end{cases}.$$

We see that $\mathrm{SOL}(q, M)$ is connected for all q and hence $M \in \mathrm{P}_c$. Clearly $M \notin \mathrm{Q}_0$.

Note also that E_0 is not contained in P_c. The following example proves this fact:

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Here, $M \in \mathrm{E}_0$, but the solution set for the given q is

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \right\},$$

which is not connected.

Now that P_0 does not contain P_c, does P_c contain P_0? When an $n \times n$ matrix M is in P_0, we can prove that for all q except those in a set $\kappa(M)$ of measure zero, the solution set is connected.

THEOREM 4. *Let $M \in \mathrm{P}_0$ and $\kappa(M)$ denote the union of the facets of all the complementary cones of M. If $q \notin \kappa(M)$ or $q > 0$ the number of solutions of* $\mathrm{LCP}(q, M)$ *is zero or one, hence the solution set is connected.*

Proof. Since $\mathrm{P}_0 \subset \mathrm{E}_0$ ((1)), $\mathrm{SOL}(q,M)$ is a singleton for all $q > 0$ and hence connected.

According to a result in [9, Theorem 2], originally due to Cottle and Guu, $\mathrm{SOL}(q,M)$ contains either 0, 1, or infinitely many points whenever $M \in \mathrm{P}_0$. However, by [4, Theorem 6.1.8], $q \notin \kappa(M)$ implies that the the local degree of q relative to M is well defined, which implies that $\mathrm{SOL}(q,M)$ is finite. Thus, $\mathrm{SOL}(q,M)$ has 0 or 1 elements for all $q \in \mathbf{R}^n$ except those that belong to a finite union of polyhedral convex cones of dimension less than n. ■

The question whether $\mathrm{SOL}(q,M)$ is connected when it has infinitely many elements remains open. However, in the 2×2 case we can show the following results.

THEOREM 5. *Suppose* $M \in \mathbf{R}^{2\times 2}$. *Then*

1. $\mathrm{P}_0 \subset \mathrm{P}_c$.
2. $\mathrm{P}_c \subset \mathrm{E}_0$.
3. $\mathrm{P}_0 \cap \mathrm{Q}_0 = \mathrm{P}_c \cap \mathrm{Q}_0$.

Proof. The following proofs assume that

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and essentially consider all cases. Some details are omitted.

1. From Theorem 4, we only need to consider $q \in \kappa(M)$, that is (without loss of generality)

   $$q = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \text{ or } q = -\lambda \begin{bmatrix} a \\ c \end{bmatrix}$$

   for $\lambda \geq 0$. Furthermore, since $M \in \mathrm{P}_0$, $a \geq 0$, $d \geq 0$ and $ad \geq bc$. When $\lambda = 0$, using the set valued inverse operator

   $$A^{-1}(S) := \{x \mid Ax \in S\}$$

   we see that $\mathrm{SOL}(0,M)$ is given by

   $$(M^{-1}(0) \cap \mathbf{R}^2_+) \cup (M^{-1}(0 \times \mathbf{R}_+) \cap \mathbf{R}_+ \times 0) \cup (M^{-1}(\mathbf{R}_+ \times 0) \cap 0 \times \mathbf{R}_+).$$

   Each of the three sets of the union is polyhedral and contains the origin, hence $\mathrm{SOL}(q,M)$ is connected.

For $\lambda > 0$ and $q = (\lambda, 0)^\top$,

$$\mathrm{SOL}(q, M) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup (0,0),$$

where

$$\begin{aligned}
\mathcal{A}_1 &:= M^{-1}(-q) \cap \mathbf{R}^2_+ \\
\mathcal{A}_2 &:= M_{1\cdot}^{-1}(-\lambda) \cap M_{2\cdot}^{-1}(\mathbf{R}_+) \cap \mathbf{R}_+ \times 0 \\
\mathcal{A}_3 &:= M_{1\cdot}^{-1}(-\lambda + \mathbf{R}_+) \cap M_{2\cdot}^{-1}(0) \cap 0 \times \mathbf{R}_+
\end{aligned}$$

Note that $\mathcal{A}_2 = \emptyset$. Furthermore, $(0,0) \in \mathcal{A}_3$. It remains to show that $\mathcal{A}_1$ and $\mathcal{A}_3$ have nontrivial intersection if $\mathcal{A}_1$ is nonempty. If $\mathcal{A}_1$ is nonempty, then it is easy to show that it has a point in common with $\mathcal{A}_3$ by considering the cases when M is invertible and when M is not invertible.
Now consider the case $\lambda > 0$ and $q = -\lambda(a,c)^\top$. We may assume without loss of generality that either a or c is nonzero. Then

$$\mathrm{SOL}(q, M) = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4,$$

where

$$\begin{aligned}
\mathcal{A}_1 &:= M^{-1}(-q) \cap \mathbf{R}^2_+ \\
\mathcal{A}_2 &:= M_{1\cdot}^{-1}(\lambda a) \cap M_{2\cdot}^{-1}(\lambda c + \mathbf{R}_+) \cap \mathbf{R}_+ \times 0 \\
\mathcal{A}_3 &:= M_{1\cdot}^{-1}(\lambda a + \mathbf{R}_+) \cap M_{2\cdot}^{-1}(\lambda c) \cap 0 \times \mathbf{R}_+ \\
\mathcal{A}_4 &:= M_{1\cdot}^{-1}(\lambda a + \mathbf{R}_+) \cap M_{2\cdot}^{-1}(\lambda c + \mathbf{R}_+) \cap (0,0)
\end{aligned}$$

If $\mathcal{A}_4 \neq \emptyset$ then $a \leq 0$, hence $a = 0$ and thus $(0,0) \in \mathcal{A}_2$. If $\mathcal{A}_3 \neq \emptyset$, let $x \in \mathcal{A}_3$. If $x_2 = 0$, then $a = c = 0$, which is a contradiction. Thus $x_2 > 0$ and it then follows that $bc = ad$. It is easy to see that this implies $x \in \mathcal{A}_1$. The proof is completed by noting that $(\lambda, 0) \in \mathcal{A}_1 \cup \mathcal{A}_2$.

2. It is easy to see that the 2×2 matrix $M \in \mathrm{E}_0$ if and only if $a \geq 0$, $d \geq 0$ and either $ad \geq bc$, $b \geq 0$ or $c \geq 0$. Thus suppose that $M \in \mathrm{P}_c$ but $M \notin \mathrm{E}_0$. Then $M \notin \mathrm{P}_0$ and one of the above inequalities must be violated. It is easy to see that if $a < 0$ or $d < 0$, then for $q > 0$, $\mathrm{SOL}(q, M)$ is not connected, which is a contradiction. To complete the proof, we derive a contradiction in the case where $a \geq 0$, $d \geq 0$, $b < 0$, $c < 0$ and $bc > ad$. Again, let $q > 0$. Note that if $x_1 = 0$,then $x_2 = 0$ and conversely. Since M is invertible, it now follows that the only solutions are $(0,0)$ and $-M^{-1}q$. Note that $-M^{-1}q > 0$, contradicting the connectedness of the solution set.

3. From above, it is known that $\mathrm{P}_0 \cap \mathrm{Q}_0 \subset \mathrm{P}_c \cap \mathrm{Q}_0$. We now show the reverse inclusion. Let $M \in \mathrm{P}_c \cap \mathrm{Q}_0$. First note that from the above it follows that $M \in \mathrm{E}_0 \cap \mathrm{Q}_0$ and hence that $a \geq 0$, $d \geq 0$ and either $ad \geq bc$, $b \geq 0$ or $c \geq 0$. Suppose that $bc > ad$, so that $b \geq 0$ or $c \geq 0$ and thus both are strictly positive. It now follows that $K(M) = \mathbf{R}^2$. Taking $q_1 > 0$, $q_2 < 0$ implies that $d > 0$, similarly $a > 0$. Now let $q < 0$. It then follows from the connectedness of $\mathrm{SOL}(q, M)$ that exactly one of the following must hold:
   (a) $aq_2 - cq_1 \geq 0$
   (b) $dq_1 - bq_2 \geq 0$
   (c) $aq_2 - cq_1 > 0$ and $dq_1 - bq_2 > 0$

   A contradiction now follows by letting $q_1 = -a$, $q_2 = -c$.

■

Note that $\mathrm{E}_0 \cap \mathrm{Q}_0$ is strictly bigger than $\mathrm{P}_c \cap \mathrm{Q}_0$ as the example given above shows. In fact,

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

is in $\mathrm{E}_0 \cap \mathrm{Q}_0$.

4. CONCLUSIONS

This paper has introduced a new class of matrices P_c and exhibited some of its properties. Some outstanding questions remain, which include determining an effective test for inclusion in the class P_c. An effective test of this sort will allow the conjecture relating P_0 and P_c to be verified or proven false and establish whether the solution set of $\mathrm{LCP}(q, M)$ is in fact connected when $M \in \mathrm{P}_0 \cap \mathrm{Q}_0$. Essentially, a key open question is to establish Theorem 5 or exhibit counter examples in the general $n \times n$ case.

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