# On Lambert's $W$ Function

*by*

Robert M. Corless[1], Gaston H. Gonnet[2], D. E. G. Hare[3], and David J. Jeffrey[1]

[1] *Department of Applied Mathematics*
*University of Western Ontario*
*London, CANADA, N6A 5B7*

[2] *Informatik ETH Zürich*
*Switzerland*

[3] *Symbolic Computation Group*
*University of Waterloo*
*Waterloo, Canada*

**Abstract**

Lambert's $W$ function is the inverse of the function $w \mapsto we^w$. It has been applied in combinatorics, number theory, fluid mechanics, quantum mechanics, and biochemistry. Some of these applications are briefly described here. This paper presents a new discussion of the complex branches of $W$, asymptotic expansions valid for all branches, an efficient numerical procedure for evaluating the function to arbitrary precision, and a method for symbolic integration of expressions containing $W$.

## 1. Introduction

In [14], Euler studied Lambert's transcendental equation [21]

$$x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta} . \tag{1.1}$$

For the solution in a special case, he introduced a function of $x$ satisfying $\omega e^{-\omega} = x$. Since then, this function, or the related function $W(x)$ which satisfies $we^w = x$, has been used to solve many different problems.

Cayley [8] discusses the function in the context of enumeration of trees. Skovgaard *et al.* give a fluid-dynamical application of $W$ in [31]. $W$ is considered by Pólya and Szegö [24, Problem 209, p. 146]. Fritsch, Shafer and Crowley [15] present an algorithm for the fixed-precision computation of one branch of $W(x)$ for $x > 0$. The computer algebra language Maple [9] has had an arbitrary precision implementation of this same real-valued branch of $W$ for many years now, and has an arbitrary precision implementation of all branches in Maple V Release 2.

The purpose of this paper is to collect existing results on this function for convenient reference, and to present some new results on the asymptotics, complex analysis, numerical analysis, and symbolic calculus of the $W$ function. We have carried out a study of the complex analytic properties of $W$ and, building on the work of de Bruijn [7, pp. 27–28], determined the complex asymptotic expansions at both complex infinity and 0. This work led to an efficient and accurate implementation (in Maple V Release 2) of the arbitrary-precision complex floating-point evaluation of all branches of $W$ over the complex plane.

Finally, some results of K. B. Ranger given at the Tenth Canadian Fluid Mechanics Symposium (also discussed in [26]) led us to rediscover a very old method of integration of inverse functions, which allows the symbolic integration of a large class of functions containing $W$. This very elementary technique is suitable for inclusion in first year calculus classes and computer algebra systems.

**Notation.** We use the letter $W$ for this function, following early Maple notation. We propose to call this *Lambert's W function*, because Lambert posed the first problem which required $W$ for its solution, and because Euler attributed the series with which he solved the problem to Lambert. In [9], it is also called the Omega function.

If $x$ is real, then for $-1/e \le x < 0$ there are two possible real values of $W(x)$ (see Figure 1.1). We denote the branch satisfying $-1 \le W(x)$ by $W_0(x)$ or just $W(x)$ when there is no possibility for confusion, and the branch satisfying $W(x) \le -1$ by $W_{-1}(x)$. $W_0(x)$ is referred to as the *principal branch* of the $W$ function. This notation will be explained and extended in section 4.

## 2. Applications

Since $W$ is such a simple function, we would expect by Pareto's principle (eighty percent of your work is accomplished with twenty percent of your tools) that $W$ would have many applications. In fact this is the case, although, paradoxically, as the $W$ function is not widely known, this often goes unrecognized. We present below a selection of applications.

**Solution of a Transcendental Equation by Euler.**

The first use of $W$ (actually of a trivially related function, $z \mapsto -W(-z)$) was by Euler [14]. He gave a series solution for Lambert's trinomial equation

$$x^\alpha - x^\beta = (\alpha - \beta) v x^{\alpha+\beta} , \tag{2.1}$$

in the limiting case as $\alpha \to \beta$, when the equation becomes

$$\log(x) = v x^\beta . \tag{2.2}$$

The solution is, in our notation,

$$x = \exp\left(\frac{-1}{\beta} W(-\beta v)\right) . \tag{2.3}$$

**Iterated Exponentiation.**
Euler also investigated the problem of iterated exponentiation [20,5], that is, evaluating

$$h(z) = z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}} , \tag{2.4}$$

whenever this makes sense. Euler was the first to prove that this iteration converges for real $z$ between $e^{-e}$ and $e^{1/e}$. We direct the reader to the references and discussion in [20] for more detail.

However, it is not widely known that this function has a closed-form expression in terms of $W$. When this iteration converges, it converges to

$$h(z) = \frac{W(-\log(z))}{-\log(z)} , \tag{2.5}$$

as can be seen by solving $h(z) = z^{h(z)}$ for $h(z)$ by taking logarithms. This immediately answers the question posed in [20] about the analytic continuation of $h(z)$.

Moreover, the region of convergence of the iteration for $h(z)$ was proved in [5] to be the set $z = te^{-t}$ where $|t| < 1$, that is, the inverse image of the open unit disk under the map $x \mapsto -W(-x)$, and convergence also occurs for images of the roots of unity (but not elsewhere on the boundary).

**Solution of a Jet Fuel Problem.**

Consider the following equations, which describe the endurance $E_t$ and range $R$ of a jet airplane. We wish to find the thrust specific fuel consumption $c_t$ and the weight of the fuel $w_0 - w_1$, given the physical constants describing the plane and its environment. The equations are [3, pp. 312–323]

$$E_t = \frac{C_L}{c_t C_D} \log\left(\frac{w_0}{w_1}\right) \tag{2.6}$$

$$R = \frac{2}{c_t C_D}\left(\frac{2C_L}{\rho S}\right)^{1/2}\left(w_0^{1/2} - w_1^{1/2}\right) \tag{2.7}$$

where $E_t$ is the endurance, $C_L$ and $C_D$ are the lift and drag coefficients, $w_0$ is the initial weight of the plane, $w_1$ is the final weight of the plane, $R$ is the range, $S$ is the planform area of the plane's wings, and $\rho$ is the ambient air density. We simplify these equations by first grouping the physical parameters and nondimensionalizing. We put $w = w_1/w_0$ and $c = E_t C_D c_t / C_L$ and introduce the new parameter

$$A = -\frac{\sqrt{2}E_t}{R}\left(\frac{w_0}{\rho S C_L}\right)^{1/2} , \tag{2.8}$$

and the equations then become $c = -\log(w)$ and

$$2A\frac{1-\sqrt{w}}{\log(w)} = 1 . \tag{2.9}$$

This equation has exactly one real solution if $A < 0$, since the left-hand side is a strictly increasing function of $w$. It can be solved in terms of $W$, the solution being

$$w = \begin{cases} A^{-2} W_0^2\left(Ae^A\right) , & \text{if } A \le -1, \\ \\ A^{-2} W_{-1}^2\left(Ae^A\right) , & \text{if } -1 \le A < 0. \end{cases} \tag{2.10}$$

Once the weight parameter is identified by simply evaluating this function, the thrust specific fuel consumption follows from $c = -\log(w)$.

**Solution of A Combustion Problem.**

The model problem

$$\frac{dy}{dt} = y^2(1-y), \quad y(0) = \varepsilon > 0 \tag{2.11}$$

is used in [23,27] to explore perturbation methods for its solution. We show here that an explicit analytic solution is possible, in terms of $W$, and thus all the perturbation results in [23] can be simply tested by comparison with the exact solution. The model problem is separable and integration leads to an implicit form for the solution $y(t)$ (as given in [23]):

$$\frac{1}{y} + \log\left(\frac{1}{y} - 1\right) = \frac{1}{\varepsilon} + \log\left(\frac{1}{\varepsilon} - 1\right) - t\,, \tag{2.12}$$

which can be solved to get

$$y = \frac{1}{1 + W\left(ue^{u-t}\right)}\,. \tag{2.13}$$

where $u = 1/\varepsilon - 1$. Inspection of the differential equation shows that $0 < \varepsilon \le y < 1$ and this implies that the principal branch of $W$ should be used. From this explicit solution, all the series results for this function in [23] can easily be verified.

**Solution of an Enzyme Kinetics Problem.**

In contrast to the last section, where $W$ provided an exact solution to compare with perturbation expansions, this section gives an example where the perturbation expansion itself is done in terms of the $W$ function.

In [23] the Michaelis-Menton model of enzyme kinetics is solved with a perturbation technique. A similar model, with a better scaling, is examined in [29]. The outer solution is taken to be of the form $s(T) = s_0(T) + \varepsilon s_1(T) + \ldots$ and $c(T) = c_0(T) + \varepsilon c_1(T) + \ldots$, and the leading order terms $s_0$ and $c_0$ are found to satisfy

$$c_0 = \frac{(\sigma+1)s_0}{\sigma s_0 + 1} \tag{2.14}$$

$$\frac{ds_0}{dT} = -\frac{(\sigma+1)s_0}{\sigma s_0 + 1}\,, \tag{2.15}$$

The equation for $s_0$ can be solved explicitly in terms of $W$.

$$s_0 = \frac{1}{\sigma} W\left(\sigma e^{\sigma - (\sigma+1)T}\right)\,. \tag{2.16}$$

This solution satisfies $s_0(0) = 1$ and uses the principal branch of $W$. The first order correction terms can also be expressed in terms of $s_0$, and hence $W$.

**Solution of Certain Nonlinear Equations.**

Nonlinear algebraic equations often occur in the determination of equilibrium or turning points in differential equations. For example, in [23], p. 89, we find the condition

$$U_0^2 - e^{U_0} = 0 \tag{2.17}$$

arising from such a problem. The solution is given approximately as $U_0 \approx -0.7$, but it is exactly $U_0 = -2W(1/2) \approx -0.703467\ldots$.

If we consider as another example the following overdetermined boundary-value problem, where the right end-point $t_0$ is unknown,

$$(D-1)^n y = 0, \quad y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0, \quad y^{(n-1)}(0) = 1, \quad y(t_0) = \alpha \tag{2.18}$$

for $n \geq 2$, then since $y(t) = t^n e^t$ and the solution to $w^k e^w = x$ is $w = kW(x^{1/k}/k)$ we have that the unknown time at which $y$ achieves the prescribed value is $t_0 = nW(\alpha^{1/n}/n)$.

Finally, if we have the equation

$$\alpha + \beta v = \gamma e^{-v} \tag{2.19}$$

then we can multiply both sides by by $e^{v+\alpha/\beta}/\beta$ if $\beta \neq 0$ to get $v = -\alpha/\beta + W(\gamma e^{\alpha/\beta})$. This solution is used in a later section of this paper.

**Resolution of a Paradox in Physics.**

We give here a brief overview of the recent use in [30] of $W$ to explain an anomaly in the calculation of exchange forces between two hydrogen nuclei. In considering a one-dimensional model problem, using the double-well Dirac delta function model, the wave equation in atomic units

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} - q\left[\delta(x) + \lambda\delta(x-R)\right]\psi = E(\lambda)\psi\,, \tag{2.20}$$

was used. The solution of this equation is expressed as a linear combination of *atomic orbital* solutions:

$$\psi = Ae^{-c|x|} + Be^{-c|x-R|}\,. \tag{2.21}$$

Making this solution continuous at each *well* ($x = 0$ and $x = R$) leads to the following transcendental equations for $c$:

$$c_\pm = q\left[1 \pm e^{-c_\pm R}\right]\,. \tag{2.22}$$

The solution of these equations can be expressed as

$$c_\pm = q + W(\pm qRe^{-qR})/R\,. \tag{2.23}$$

Scott *et al.* then go on to use this exact solution to explain how exponentially subdominant terms in the true asymptotic expansion of this solution account for differences between the predictions of previous numerical and asymptotic solutions for the model equations. In brief, the exponentially subdominant terms were missing from the previous asymptotic developments, but are still significant for small enough $R$.

**A similarity solution for Richards' equation.**

Recent work [6] uses both real branches of $W$ to give a new exact solution for Richards' equation for water movement in soil. By a similarity transformation, Richards' equation

$$\frac{d\theta}{d\Psi}\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial z}\left[K(\Psi)\frac{\partial \Psi}{\partial z} - K(\Psi)\right] \tag{2.24}$$

for the *moisture tension* $\Psi$ is reduced in one important special case to the ordinary differential equation

$$\alpha^* A \frac{dA}{dt} = 1 - A \tag{2.25}$$

which can be solved in terms of $W$ as

$$A(t) = 1 + W\left[(-1 + A(0))\exp\left\{\frac{(A(0)-1)\alpha^* - t}{1 - A(0)}\right\}\right] . \tag{2.26}$$

Both real branches of $W$ give rise to physically meaningful solutions. If we use $W_0$ the solution corresponds to *capillary rise*, while if we instead use $W_{-1}$ the solution can be interpreted as *infiltration*.

**The Volterra Equations for Population Growth.**

In [13], pages 102–109, we find an implicit analytic phase plane solution of the Volterra equations

$$\frac{dx}{dt} = ax(1-y), \quad \frac{dy}{dt} = -cy(1-x) , \tag{2.27}$$

essentially in terms of the $W$ function (see equations (11) and (12) on page 104 of [13]). These equations, with $a = 2$ and $c = 1$, appear as problem B1 of the DETEST test suite [18] for numerical methods for integration of ordinary differential equations. The analytic solution is a closed loop in the phase plane. If the upper branch is $y^+$ and the lower $y^-$, then

$$\begin{aligned} y^+ &= -W_{-1}\left(-Cx^{-c/a}e^{cx/a}\right) , \\ y^- &= -W_0\left(-Cx^{-c/a}e^{cx/a}\right) , \end{aligned} \tag{2.28}$$

where $C$ is an arbitrary constant.

The remaining problem is to find an expression for $t$, and we can find this in terms of quadrature:

$$t = \int_{x_0}^{x} \frac{d\xi}{a\xi(1 - y(\xi))} = \int_{y_0}^{y} \frac{d\eta}{-c\eta(1 - x(\eta))} . \tag{2.29}$$

There are square-root singularities and branches in these integrals, but these can be handled with standard changes of variables.

**Combinatorial Applications.**

In combinatorial analyses of trees [8,28] and their subsequent application to areas such as chemistry [25], the $W$ function has been used extensively, although not under that name. Let $t_n$ be the number of $n$-element rooted trees. Then the generating function is $T(x) = \sum t_n x^n/n!$. It can be shown that $T(x) = x\exp(T(x))$, and therefore $T(x) = -W(-x)$. From the series expansion for $W$ given below in equation (3.1), we conclude that $t_n = n^{n-1}$.

**Pedagogical Applications.**

The $W$ function provides a useful exercise for first-year calculus students in defining implicit functions, in differentiating such functions, in computing Taylor series, and, as we shall see, computing antiderivatives. For introductory numerical analysis classes, it provides a nice rootfinding problem (suitable for Newton's method or Halley's method), an example of a well-conditioned function for evaluation (away from the branch point $x = -1/e$) and an ill-conditioned function for evaluation (near the branch point). For complex variable courses, it provides a useful example, similar to the logarithm, of a multivalued function. It also provides one of the only simple examples of the application of the Lagrange Inversion Theorem [32]. The asymptotics of the $W$ function might profitably be used in a later course.

## 3. Calculus

The principal branch of $W$ is analytic at 0. This follows from the Lagrange Inversion Theorem [32] which gives the series expansion below for $W_0(z)$:

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n \tag{3.1}$$

It is an interesting exercise to calculate this series directly by reverting the series for $we^w$.

Because of the derivative singularity of $W_0$ at $z = -1/e$ (which follows from the derivative of $w \mapsto we^w$ being 0 at $w = -1$), the radius of convergence of this series must be less than or equal to $1/e$. That it is equal to $1/e$ is easily seen using the ratio test.

Differentiating the defining equation $x = W(x)e^{W(x)}$ for $W$ and solving for $W'$, we obtain the following expressions for the derivative of $W$:

$$\begin{aligned} W'(x) &= \frac{1}{(1+W(x))\exp(W(x))} \\ &= \frac{W(x)}{x(1+W(x))} \qquad \text{if } x \neq 0 \end{aligned} \tag{3.2}$$

Historically, it has been the case that symbolic algebra systems have been quite cavalier about the handling of exceptional points. The equation above is a typical example, for Maple V Release 2 will return the last expression given when asked to differentiate $W$, and hence is able to compute $W_0'(0)$ only as a limit. See [11] for further discussion of the handling of special cases (the so-called specialization problem) by computer algebra systems.

Taking logarithms of both sides of $W(z)e^{W(z)} = z$ and rearranging terms, we obtain the simplification transformation

$$\log(W(z)) = \log(z) - W(z)\,, \tag{3.3}$$

at least for the principal branch when $z > 0$.

We turn now to the question of integrating expressions containing $W$. At the Tenth Canadian Fluid Mechanics Symposium (1992) in St. John, New Brunswick, K. B. Ranger used the following example to illustrate some attempts to integrate the Navier-Stokes equations in parametric form:

$$x = pe^p \tag{3.4}$$

$$\frac{dy}{dx} = p\,. \tag{3.5}$$

Differentiating equation (3.4) with respect to $y$ gives

$$\frac{dx}{dy} = \frac{dp}{dy}e^p + pe^p\frac{dp}{dy} \tag{3.6}$$

and simplifying using equation (3.5) we then obtain

$$\frac{dy}{dp} = p(p+1)e^p \tag{3.7}$$

which is easily integrated to give

$$y = (p^2 - p + 1)e^p + C\,. \tag{3.8}$$

This example is pursued further in [26].

Since $y$ is clearly an antiderivative of $W(x)$, Ranger had discovered a simple technique for symbolic integration of $W(x)$. In our notation,

$$\begin{aligned} \int W(x)\,dx &= (W^2(x) - W(x) + 1)e^{W(x)} + C \\ &= x(W(x) - 1 + 1/W(x)) + C \end{aligned} \tag{3.9}$$

When one tries this technique on other functions containing $W$, one sees that the technique is really a special change of variable: for example, on putting $w = W(x)$, so $x = we^w$ and $dx = (w+1)e^w$, we see that

$$\begin{aligned} \int xW(x)\,dx &= \int we^w \cdot w \cdot (1+w)e^w\,dw \\ &= \frac{1}{8}(2w-1)(2w^2+1)e^{2w} + C \\ &= \tfrac{1}{2}\left(W(x) - \tfrac{1}{2}\right)\left(W^2(x) + \tfrac{1}{2}\right)e^{2W(x)} + C\,. \end{aligned} \tag{3.10}$$

This technique is valid for all branches of $W$, as by definition $\frac{d}{dw}we^w \neq 0$ at any interior point of any branch.

The problem of integrating functions containing $W$ is a special case of integrating expressions containing inverse functions. In the simplest form, one wants to find $\int f^{-1}(x)\,dx$ and from Ranger's work above one can see that the method is simply to substitute $x = f(y)$, obtaining

$$\begin{aligned}\int f^{-1}(x)\,dx &= \int y f'(y)\,dy \\ &= y f(y) - \int f(y)\,dy\end{aligned} \tag{3.11}$$

upon integrating by parts.

In the more general form, one can use the same substitution to integrate

$$\begin{aligned}\int G(x,\ f^{-1}(x))\,dx &= \int G(f(y),\ y) f'(y)\,dy \\ &= G(f(y),\ y) f(y) - \int f(y) \frac{d}{dy} G(f(y),\ y)\ \ dy\end{aligned} \tag{3.12}$$

This technique is not part of the standard calculus curriculum, even though it is not new (it can be shown that equation (3.11) is related to the Legendre Transform [4]). This technique allows a simpler derivation of $\int \sin^{-1}(x)\,dx$ than the standard method, and also allows for straightforward integration of more complicated expressions containing inverse functions, such as $\int (\sin^{-1}(x))^3\,dx$ and $\int 1/(1+\sin^{-1}(x))\,dx$. We believe that it would be useful to include this technique in a typical first-year calculus course.

Finally, note that this technique allows the Risch decision procedure [16] to be applied to determine if integrals containing $W$ are elementary or not.

## 4. Branches and Asymptotics

We have seen that $W$ has two branches on the real line. We have also seen that complex values for $W$ are required. Thus, to extend $W$ to the complex plane, we must define all of the branches of $W$. For the sake of clarity, the following discussion is quite detailed. Partly to establish our notation and partly because we use the complex logarithm later, we begin by recalling the definition of its branches.

If $p = \log z$, then $z = e^p$. We follow standard usage and say that $z$ is in the $z$-plane and $p$ is in the $p$-plane. One set of branches for the logarithm is obtained by partitioning the $p$-plane with horizontal boundary lines at $p = (2k+1)\pi i$, as shown in Figure (4.1).

Each of the regions so defined then maps precisely onto the $z$-plane minus $(-\infty, 0]$. Further, it is nearly universal to consider the points on the boundary between two regions as belonging to the region below them, or in other words, the boundary is *attached* to the region below it. This is shown on the figure by drawing two lines for each boundary: a solid line showing the points on the boundary attached to the region below them, and a dashed line showing points next to the boundary but belonging to the region above them. The region straddling the origin ($-\pi < \mathrm{Im}(p) \le \pi$) defines the range of the principal branch of the logarithm.

The $z$-plane corresponding to Figure (4.1) is shown in Figure (4.2). All of the solid boundary lines in Figure (4.1) map onto the solid line running along the negative real axis, and all of the dashed near-boundary lines map onto the dashed line just below the axis. The letters $A, B, C, D$ further indicate the geometry of the map.

The negative real axis is called the branch cut for the logarithm, and the limiting value $z = 0$ is called the branch point. The branch choices shown in Figures (4.1) and (4.2) conform to the rule of *counter-clockwise continuity* (CCC) around the branch point, which is a mnemonic principle that gives some uniformity to the choices for branch cuts of all the elementary functions [19]. This convention distinguishes here between two possibilities, namely the choice of attaching the image of the boundary in the $p$-plane to the top or to the bottom of the branch cut in the $z$-plane. The phrase *counter-clockwise continuous* is intended to convey the idea that the image of a curve circling the branch point $z = 0$ in the counter-clockwise direction is continuous in the limit as we approach the branch cut from above. In Figure (4.2), if we traverse the circle from $C$ to $B$, then CCC ensures that the image of this circle is closed at the image of $B$ in the $p$-plane.

Turning to the $W$ function, we put $w = W(z)$ and $z = we^w$, so we must now specify the boundary curves that maximally partition the $w$-plane and find the images of these boundary curves, which are the branch cuts in the $z$-plane. If $w = \xi + i\eta$ and $z = x + iy$, then

$$x = e^{\xi}\left(\xi \cos\eta - \eta \sin\eta\right) \tag{4.1}$$
$$y = e^{\xi}\left(\eta \cos\eta + \xi \sin\eta\right) . \tag{4.2}$$

We would like the branch cut(s) for $W$ to be similar to that for the logarithm, and therefore we consider the images of the negative real $z$-axis in the $w$-plane. If $y = 0$ in (4.2) then

$$\eta = 0 \quad \text{or} \quad \xi = -\eta \cot\eta . \tag{4.3}$$

If further $\xi \cos\eta - \eta \sin\eta < 0$, then the images are precisely the *negative* real $z$-axis. See Figure (4.3). We number the resulting regions in the $w$-plane as indicated in the figure, and denote the branch of $W$ taking values in region $k$ by $W_k(z)$.

The curve which separates the principal branch, $W_0$, from the branches $W_1$ and $W_{-1}$ is

$$\{-\eta \cot(\eta) + \eta i : -\pi < \eta < \pi\} \tag{4.4}$$

together with $-1$ (which is the limiting value at $\eta = 0$). The curve separating $W_1$ and $W_{-1}$ is simply $(-\infty, -1]$. Finally, the curves separating the remaining branches are

$$\{-\eta \cot(\eta) + \eta i : 2k\pi < \pm\eta < (2k+1)\pi\} \quad \text{for } k = 1, 2, \ldots \tag{4.5}$$

**Remark.** The curves (4.4) and (4.5) together form a subset of the so-called *Quadratrix of Hippias* [12], the missing parts being the curves corresponding to $(2k-1)\pi < \pm\eta < 2k\pi$.

That is, the Quadratrix of Hippias is the union of the images of the real axis under the various branches of $W$, excluding $(-\infty, -1)$.

$W_0(z)$ is special, as it is the only branch which contains any part of the positive real axis, and as noted previously we call this the principal branch of $W$. It has a second-order branch point at $z = -1/e$ corresponding to $w = -1$, which it shares with both $W_{-1}(z)$ and $W_1(z)$. $W_0(z)$ is analytic at $z = 0$, with value $w = 0$. Its branch cut is $\{z : -\infty < z \le -1/e\}$. We choose to close the branch cut on the top, so $W_0(z)$ has counter-clockwise continuity (CCC) around the branch point $z = -1/e$. Thus image of the curve $z = -1/e + \varepsilon e^{it}$ around the branch point in the $z$-plane, for $-\pi < t \le \pi$, is a continuous curve in the 0-region of the $w$-plane. See Figure (4.4) and Figure (4.5).

Because of the extra branch point, $W_{-1}$ and $W_1$ each have a double branch cut, $\{z : -\infty < z \le -1/e\}$ and $\{z : -\infty < z \le 0\}$. We close the branch cut as before on the top. The cut is thus CCC around the branch point at $z = 0$. This choice of closure implies that $W_{-1}(z)$ is real for $z \in [-1/e, 0)$. Thus $W_0(z)$ and $W_{-1}(z)$ are the only branches of $W$ that take on real values. See Figure (4.6), Figure (4.7), and Figure (4.8).

All other branches of $W$ have only the branch cut $\{z : -\infty < z \le 0\}$, closed on the top for counter-clockwise continuity, and thus are similar to the branches of the logarithm.

In Maple V Release 3[1], the branch cuts for $W$ are as described above.

For all multivalued functions, the division of the complex plane into branches is somewhat arbitrary, and even the elementary functions do not have universally accepted branches [19]. The benefits of our choices for Lambert's $W$ function are as follows. Most importantly, the coincidence of the branch cut for the branch point at 0 with the corresponding branch cut for the logarithm and the fact that both functions are CCC yield nice asymptotic expansions of the branches of $W$ at both 0 and (complex) infinity. Secondly, the placement of the branch cuts in the real axis implies that $W$ has *near conjugate symmetry*, meaning that except for points in the branch cut we have $W_k(\overline{z}) = \overline{W_{-k}(z)}$ for all integers $k$. Systems which have a signed 0 (for example, systems which implement *IEEE Standard 754* [2]) can exploit this symmetry on the branch cut as well.

The main drawback with this choice of branches is that one particular expansion below, namely equation (5.8), is more complicated than it would be if the branch cuts were chosen as $\{z : -\infty < z \le 1/e\}$ and $\{z : 0 \le z < \infty\}$.

**Asymptotic Expansions.**

To develop the asymptotic expansions of the branches of $W$ at both 0 and (complex) infinity, note that in both cases it is the exponential character of $w \mapsto we^w$ which dominates, and hence the leading term of such an expansion will be some form of logarithm. We write

$$w = W(z) = \log z + u\,, \tag{4.6}$$

[1] Prior to Maple V Release 2, only the real-valued inverse of the real function $\xi \mapsto \xi e^\xi$ on $[-1, \infty)$ was implemented, so the question of branches did not arise. In Maple V Release 2, the branch cuts were chosen to be $(-\infty, -1/e]$ and $(0, \infty)$, but the simplification of the asymptotic expansions resulting from moving the latter branch cut to the negative real axis was deemed significant and thus included in Maple V Release 3.

where for $z$ either close to 0 or large we expect $u$ to be small relative to $\log z$, and where we are not specifying at this point which branch of the logarithm we intend to use. Substituting this expression into the defining relation for $W$, namely $we^w = z$, we obtain

$$(\log z + u)ze^u = z \tag{4.7}$$

and hence

$$(1 + \frac{u}{\log z})e^u = \frac{1}{\log z}\,. \tag{4.8}$$

Under the assumption that $|u| \ll |\log z|$ we have $e^u \sim 1/\log z$, hence

$$u \sim \log \frac{1}{\log z} \tag{4.9}$$

where it must be stressed that the two logarithms in (4.9) need not be the same. However, the size assumption on $u$ is best satisfied by choosing the principal branch for the outer logarithm in (4.9), so we will make that choice at this point. To emphasize that the branch of the inner logarithm is not yet chosen, we will rewrite (4.9) using Log to denote the inner logarithm:

$$u \sim \log \frac{1}{\operatorname{Log} z}\,. \tag{4.10}$$

One further observation to make here is that so long as $\operatorname{Log} z$ is not a negative real number we can replace $\log \frac{1}{\operatorname{Log} z}$ with $-\log \operatorname{Log} z$. $\operatorname{Log} z$ can only be a negative real number if Log is chosen to be the principal branch of the logarithm and $z$ is a positive real number less than 1. Keeping this in mind, then, we now adapt de Bruijn's argument [7, pp. 27–28] establishing the asymptotic expansions for the branches of $W$ at 0 and infinity. (de Bruijn computes the expansion only for real $z \to \infty$, but we will see below that the method is valid for all branches and for $z \to 0$ as well. de Bruijn comments that the proof is modeled on the usual proof of the Lagrange Inversion Theorem.) It is interesting that the asymptotic series are, in fact, convergent.

Continuing from (4.6) and (4.10), write

$$w = \operatorname{Log} z - \log \operatorname{Log} z + v \tag{4.11}$$

On substituting (4.11) into $we^w = z$ we get

$$(\operatorname{Log} z - \log \operatorname{Log} z + v)e^v z \frac{1}{\operatorname{Log} z} = z\,. \tag{4.12}$$

For convenience denote $1/\operatorname{Log} z$ by $\sigma$ and $\log \operatorname{Log} z / \operatorname{Log} z$ by $\tau$, as de Bruijn does. Then, assuming $z \neq 0$ and $\operatorname{Log} z \neq 0$,

$$e^{-v} - 1 - \sigma v + \tau = 0\,. \tag{4.13}$$

This is equation (2.4.6) in de Bruijn, except that we are interpreting the logarithms as being possibly any branch, and have performed only simplifications valid for all branches of the logarithm. We now ignore the relation that exists between $\sigma$ and $\tau$, and consider them as small independent complex parameters.

We will show that there exist positive numbers $a$ and $b$ such that if $|\sigma| < a$ and $|\tau| < a$ then equation (4.13) has exactly one solution in the domain $|v| < b$, and that this solution is an analytic function of both $\sigma$ and $\tau$ in the region $|\sigma| < a$, $|\tau| < a$.

Let $\delta$ be the lower bound of $|e^{-\zeta} - 1|$ on the circle $|\zeta| = \pi$. Then $\delta$ is positive, and $e^{-\zeta} - 1$ has just one root inside that circle, that is, at $\zeta = 0$. If we now choose the positive number $a = \delta/(2(\pi+1))$, then we have

$$|\sigma\zeta - \tau| \le |\sigma||\zeta| + |\tau| < \tfrac{1}{2}\delta \tag{4.14}$$

for $|\sigma| < a, |\tau| < a$ and $|\zeta| = \pi$.

This means that $|e^{-\zeta} - 1| > |\sigma\zeta - \tau|$ on the circle $|\zeta| = \pi$. Thus, by Rouché's Theorem [32, §3.42], equation (4.13) has exactly one root, $v$, inside the circle. By Cauchy's Theorem,

$$v = \frac{1}{2\pi i}\int_{|\zeta|=\pi} \frac{-e^{-\zeta} - \sigma}{e^{-\zeta} - 1 - \sigma\zeta + \tau}\zeta\, d\zeta\,, \tag{4.15}$$

where the integration contour is taken in the counterclockwise direction.

**Remark** This already establishes that for any branch choice of the logarithm denoted by Log in (4.11) (which all differ by multiples of $2\pi i$) there is exactly one number $v \in |\zeta| < \pi$ so that $w$ defined by (4.11) satisfies $we^w = z$. This establishes the existence of the infinite number of roots $W_k(z)$. This point was not noted in de Bruijn.

Now for every $\zeta$ on the integration path, $|\sigma\zeta| + |\tau|$ is less than $\frac{1}{2}|e^{-\zeta} - 1|$, so we can expand the denominator of the integrand of (4.15) in an absolutely and uniformly convergent power series

$$\frac{1}{e^{-\zeta} - 1 - \sigma\zeta + \tau} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(e^{-\zeta} - 1)^{-k-m-1}\zeta^k\sigma^k\tau^m(-1)^m C_m^{m+k} \tag{4.16}$$

Substituting (4.16) into (4.15) and integrating term by term, we obtain $v$ as the sum of an absolutely convergent double power series in $\sigma$ and $\tau$. Notice that all terms not containing $\tau$ vanish, because the corresponding integrands are regular at $\zeta = 0$. Therefore, we can conclude that (4.13) has exactly one solution $v$ satisfying $|v| < \pi$, and this solution can be written as

$$v = \tau \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{km}\sigma^k\tau^m\,, \tag{4.17}$$

where the $c_{km}$ are constants not depending on $\sigma$ or $\tau$.

Now return to the special values of $\sigma$ and $\tau$ as functions of $z$. For $z$ sufficiently large, we have that $|\sigma| < a$ and $|\tau| < a$, and likewise for $z$ sufficiently small (this point was not observed by de Bruijn). Thus we have established that the following formula gives the asymptotics for *all* branches of $W$ both at infinity and at 0:

$$W(z) = \operatorname{Log} z - \log\operatorname{Log} z + \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{km}(\log\operatorname{Log} z)^{m+1}(\operatorname{Log} z)^{-k-m-1} \tag{4.18}$$

which can be rearranged into the form

$$
\begin{aligned}
W(z) =& L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2\left(-2+L_2\right)}{2\,{L_1}^2} + \frac{L_2\left(6-9\,L_2+2\,{L_2}^2\right)}{6\,{L_1}^3} \\
&+ \frac{L_2\left(-12+36\,L_2-22\,{L_2}^2+3\,{L_2}^3\right)}{12\,{L_1}^4} \\
&+ \frac{L_2\left(60-300\,L_2+350\,{L_2}^2-125\,{L_2}^3+12\,{L_2}^4\right)}{60\,{L_1}^5} + O\left(\left\{\frac{L_2}{L_1}\right\}^6\right) ,
\end{aligned}
\tag{4.19}
$$

where $L_1 = \operatorname{Log} z$ and $L_2 = \log \operatorname{Log} z$, as the series is absolutely convergent. This expansion corrects a typographical error in equation (2.4.4) in [7] and extends the expansion to higher-order terms.

**Remark** We have further shown that once the branch, Log, for the logarithm is chosen, $w$ is determined by $v$, and that $v$ is determined in terms of a known series in $\log \operatorname{Log} z$ and $\operatorname{Log} z$, whose coefficients do not depend on $\sigma$ or $\tau$, and hence not on the branch of the logarithm chosen. We have thus established that the *same series* works at infinity and at 0 for all branches of $W$, with the exception of $W_0$. Since $W_0$ is analytic at 0, this series of course cannot be used to approximate the principal branch for small $z$.

To complete this development, it only remains to determine the relation between the branch of the logarithm taken for Log and the branch of $W$ thus approximated asymptotically. A straightforward analysis of the imaginary parts of the first two terms of the series (4.18) yields the nice relation that if $\operatorname{Log}(z) = \log(z) + 2k\pi i$, where log denotes the principal branch of the logarithm, then the branch of $W$ approximated by (4.18) is $W_k$.

**Other Series Expansions**

A similar but purely real-valued series is useful for the branch $W_{-1}(x)$ for small $x < 0$. We can get a real-valued asymptotic formula from the above by using $\log(-x)$ in place of $\operatorname{Log}(z)$ and $\log(-\log(-x))$ in place of $\log(\operatorname{Log}(z))$. A similar argument to that above arrives at precisely equation (4.13) for $v$, and establishes existence and convergence. Again, the coefficients of the expansion are exactly the same. This series is not useful for complex $x$ as the branch cuts do not correspond.

When we actually compute the first few terms in the series (4.18), we notice that the terms containing only $\tau$ are $\tau + \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3 + \frac{1}{4}\tau^4 + \cdots$ and this sums exactly to $-\log(1-\tau)$. Investigation shows that other terms also have exact sums, and we are led to wonder if there is a more convenient form for this asymptotic development.

In fact this is the case. Starting from equation (4.13) and solving for $\sigma$, we have

$$
\begin{aligned}
\sigma =& \frac{e^{-v}+\tau-1}{v} \\
=& \frac{1-\tau}{\log(1-\tau)}(v-v_0) - \tfrac{1}{2}\frac{(1-\tau)(\log(1-\tau)-2)}{\log^2(1-\tau)}(v-v_0)^2 + \cdots
\end{aligned}
\tag{4.20}
$$

where $v_0 = -\log(1-\tau)$, and the principal branch of the logarithm is chosen to agree with the earlier series calculation. The series in (4.20) can easily be reverted to give

$$
v = -\log(1-\tau) + \frac{\log(1-\tau)}{1-\tau}\sigma + \tfrac{1}{2}\frac{(\log(1-\tau)-2)\log(1-\tau)}{(1-\tau)^2}\sigma^2 + \cdots \tag{4.21}
$$

Thus the first few terms in this new asymptotic development for $W(z)$ can be written

$$
\begin{aligned}
W(z) = {} & \operatorname{Log}(z) - \log(\operatorname{Log}(z)) - \log(1-\tau) + \frac{\log(1-\tau)}{1-\tau}\sigma \\
& + \tfrac{1}{2}\frac{(\log(1-\tau)-2)\log(1-\tau)}{(1-\tau)^2}\sigma^2 + \cdots ,
\end{aligned}
\tag{4.22}
$$

where $\sigma = 1/\operatorname{Log}(z)$ and $\tau = \log(\operatorname{Log}(z))/\operatorname{Log}(z)$, and $\operatorname{Log}(z) = \log(z) + 2\pi i k$ determines the branch of $W$. If further $k = -1$ and $z = x < 0$ is small, we may wish to use instead the purely real development that results if we use $\log(-x)$ and $\log(-\log(-x))$ in the above.

We remark that equation (4.20) can be solved explicitly for $v$ in terms of $\sigma$ and $\tau$, recursively but not circularly using the $W$ function (see equation (2.19)), and that Maple can use this explicit solution to compute the series in equation (4.21) directly. However, the series reversion technique discussed above is actually quite efficient, and we can calculate the series terms to any desired order using it.

Another series development is possible and somewhat simpler in appearance. Beginning with equation (4.13) again, we solve this for $\tau$ and expand in a series in $v$ about $v = 0$, and revert this series to get an expansion in powers of $\tau$:

$$
\begin{aligned}
v = {} & \frac{1}{\sigma+1}\tau + \frac{1}{(\sigma+1)^3}\frac{\tau^2}{2!} - \frac{\sigma-2}{(\sigma+1)^5}\frac{\tau^3}{3!} \\
& + \frac{\sigma^2-8\sigma+6}{(\sigma+1)^7}\frac{\tau^4}{4!} - \frac{\sigma^3-22\sigma^2+58\sigma-24}{(\sigma+1)^9}\frac{\tau^5}{5!} + O(\tau^6)
\end{aligned}
\tag{4.23}
$$

Numerical tests of these series shows that equation (4.23) gives much better agreement with $W(z)$ for small and moderate values of $z$ than the series from equation (4.22). In particular, for the $O(\tau^6)$ series above, we find that the relative error for equation (4.23) is initially oscillatory and has a maximum value (on $2 < x$) of $2.75 \cdot 10^{-6}$ occurring at $x \approx 163.2$, and the error for equation (4.22) is several orders of magnitude larger for small $x$, catching up only when $x \approx 12000$. Asymptotically, however, inspection shows that the error for equation (4.22) is $O(L_2/L_1^7)$ while that for equation (4.23) is $O\left((L_2/L_1)^6\right)$, and hence ultimately the series resulting from equation (4.22) is more accurate. However, the behaviour for small $x$ and its relative simplicity makes equation (4.23) attractive.

Finally, we consider expansions of $W_0(\rho e^{it})$ in terms of real-valued $W$ functions. If $\rho e^{it} = we^w$ and $\rho = \xi e^\xi$, and we assume that $t \ll \rho$, consideration of the various sizes of the terms suggests that we put

$$
w = \xi + it + v\,. \tag{4.24}
$$

Then, using $we^w = \rho e^{it}$ we get

$$
\left(1 + \frac{it}{\xi} + \frac{v}{\xi}\right) e^v = 1 \tag{4.25}
$$

or

$$
1 - \tau + \sigma v = e^{-v} \tag{4.26}
$$

where $\tau = -it/\xi$ and $\sigma = 1/\xi$. This equation is exactly (4.13) and all the expansions derived from that follow. This gives in particular from (4.23) that

$$\begin{aligned} W(\rho e^{it}) =& W(\rho) + it + \frac{1}{\sigma+1}\tau + \frac{1}{(\sigma+1)^3}\frac{\tau^2}{2!} - \frac{\sigma-2}{(\sigma+1)^5}\frac{\tau^3}{3!} \\ &+ \frac{\sigma^2-8\sigma+6}{(\sigma+1)^7}\frac{\tau^4}{4!} - \frac{\sigma^3-22\sigma^2+58\sigma-24}{(\sigma+1)^9}\frac{\tau^5}{5!} + O(\tau^6) \end{aligned} \tag{4.27}$$

and we notice that the even-powered terms are purely real and the odd-powered terms are purely imaginary, so we have separate real series for the real and imaginary parts of $W(\rho e^{it})$.

## 5. Numerical Analysis

We are interested in the numerical analysis of the $W$ function in two main cases: for fixed precision implementations, and for *arbitrary* precision implementations, such as are needed for computer algebra systems such as Maple. We first consider the conditioning of $W$, and then go on to examine methods for efficient approximation.

The standard theory of conditioning of function evaluation [10, p. 14] gives the expression $C = xf'(x)/f(x)$ which estimates the relative effect on $y = f(x)$ of a small relative change in $x$. Here it is easy to see that for $y = W(z)$, the condition number $C$ is

$$C = \frac{1}{1+W(z)}\,. \tag{5.1}$$

So, when $z$ is very large (or very small) and known to good relative accuracy, this number tells us the approximate relative effect of the uncertainty in $z$. We see immediately that $W$ is ill-conditioned near the branch point $z = 1/e$ (for $W_0$, $W_{-1}$ and $W_1$), when $W(z) \approx -1$. Interestingly, there appears to be no ill-conditioning near the singularity $z = 0$ for any branch; however, this is a consequence of considering *relative* error.

We can in fact say more than the standard theory gives. If we have some approximation to $W(z)$, say $\widetilde{w}$, then we can define the *residual*

$$r = \widetilde{w}e^{\widetilde{w}} - z\,, \tag{5.2}$$

and we note that this implies, since obviously $z + r = \widetilde{w}e^{\widetilde{w}}$,

$$\widetilde{w} = W(z+r) = W(z) + \int_z^{z+r} W'(\zeta)\,d\zeta \tag{5.3}$$

exactly, where the path of integration in the second equation does not cross any branch cut. We emphasize that we have exact equality here — our approximation $\widetilde{w}$ is the exact value of $W$ for a slightly different argument, where the backward error is just $r$. Then, use of the fundamental theorem of calculus (or, in the real-variable case, the Mean Value

Theorem) as above gives us an exact expression for the forward error, as well. If we can find a simple bound for $|W'(z)|$ we will then have a good forward error *bound*.

To find such a bound, we first look at the real case. It is obvious from the graph of $W(x)$ that $W'(x) < 1$ if $x > 0$. Hence we can say immediately that $W(x) - \widetilde{w} = W'(x + \theta r)r$ (for some $0 \le \theta \le 1$) is bounded in magnitude by $|r|$, which provides a very convenient and computable error bound (it is very pessimistic, though, for very large $z$). This suggests examining the region in the $z$-plane where $W'(z)$ has magnitude 1. We thus put

$$W'(z) = \frac{1}{(1+W)e^W} = e^{i\psi} \tag{5.4}$$

and now try to find the curve or curves in the $z$-plane which satisfy that equation.

To do this, we first modify and then solve equation (5.4) exactly, as follows. We consider the more general case where we wish to find the curves where $W'(z)$ has magnitude $\rho$:

$$W'(z) = \frac{1}{(W(z)+1)e^{W(z)}} = \rho e^{i\psi} . \tag{5.5}$$

Invert, and then multiply both sides by $e$ and notice that

$$1 + W(z) = W(\rho^{-1}e^{-i\psi}) \tag{5.6}$$

(for *some* branch of $W$) and use $z = We^W$ to get

$$z = \rho^{-1}e^{-i\psi}\left(1 - \frac{1}{W(\rho^{-1}e^{-i\psi})}\right) . \tag{5.7}$$

This gives us an exact and very interesting expression for the curves in the $z$-plane describing the location of the curves where $|W'(z)| = \rho$. One can see immediately that the curves are the same for $W_k(z)$ as they are for $W_{-k}(z)$ by using the near conjugate symmetry relation $\overline{W_k(z)} = W_{-k}(\overline{z})$ (see Section 4). Further, since the magnitude of $W_k(z)$ increases as $k$ increases, we see that for large $k$ the curve in the z-plane is really a small modification to the circle of radius $\rho^{-1}$ centred around the branch point $z = 0$. Thus for large $k$ one sees that essentially the forward error is less than 10 times the residual if $z$ is outside a circle of radius $1/10$ centred around the origin.

This gives us a complete error analysis, because away from the branch points we can bound $W'(z)$ and thus compute a bound on the forward error from the computable backward error.

However, the above analysis is not terribly practical if, for example, we are at all close to the branch point $z = -1/e$ and wish to compute $W_0(z)$, $W_{-1}(z)$ or $W_1(z)$. In these cases, though, we may use a series expansion of $W(z)$ valid near the branch point itself:

$$W(z) = -1 + p - \tfrac{1}{3}p^2 + \tfrac{11}{72}p^3 + \ldots \tag{5.8}$$

where $p = +\sqrt{2(ez+1)}$ for $W_0(z)$. For $W_{-1}$ we take $p = -\sqrt{2(ez+1)}$, provided $\mathrm{Im}(z) \ge 0$. For $\mathrm{Im}(z) < 0$, this series with the negative square root gives instead good approximations for $W_1(z)$. This is what we meant in Section 4 when we said that the expansion at the branch point $-1/e$ was complicated. The series (5.8) can be computed to any desired order by reverting the series for $w \mapsto we^w$ near $w = -1$.

The residual $r$ of approximations based on $n$ terms of (5.8) is $O(p^{n+1})$. Thus for $z$ close enough to the branch point, the smallness of the residual itself compensates somewhat for the amplification of the forward error due to ill-conditioning near that point. For simplicity we abandon *error bounds* for *error estimates*, and make this compensation precise.

Equation (5.8) implies that near the branch point $W_0'(z) = (1 - 2/3p + O(p^2))dp/dz$. Now $p^2 = 2(ez+1)$ so $dp/dz = e/p$. This implies that $W_0'(z) = O(1/p)$ near the branch point. Thus if we are attempting to calculate $W_0(z_0)$ and $p_0 = \sqrt{2(ez_0+1)}$ is small, and further we have an estimate with residual $O(p_0^k)$ by taking $k-1$ terms in the series about the branch point, then the forward error is clearly $O(p_0^{k-1})$.

For definiteness, suppose we take $w = -1 + p_0 - \frac{1}{3}p^2$. Then the residual is $r = -11p_0^4/(72e) + O(p_0^5)$. Thus the forward error is

$$W_0(p_0) - W_0(p_0 + r) = -\int_{p_0}^{p_0+r} W_0'(\zeta)\, d\zeta \tag{5.9}$$

$$= \frac{11}{72}p_0^3 + O(p_0^4)\,. \tag{5.10}$$

Note that this agrees, as it should, with simply comparing $w$ with a higher-order accurate initial approximation. Thus we lose one order of accuracy of the series approximation in going from the residual to the forward error.

We now turn to practical methods for computing all the branches of $W$. We here only consider arbitrary-precision computations, such as are needed in Maple. We find that a fixed order approximation scheme such as that given in [15] is not adequate. Rather, we suggest an iterative scheme with crude initial guesses and gradual refinement of the precision used as the residuals decrease. Consider iterative rootfinding methods applied to finding $w$ such that $f(w) = we^w - z = 0$. We note that $f'(w) = (w+1)e^w$ and $f''(w) = (w+2)e^w$ are available at the cost of only one (complex) flop each, given that $e^w$ must be computed for $f(w)$, and so we can use *Halley's method* [1]

$$w_{j+1} = w_j - \frac{f(w_j)}{f'(w_j) - \frac{1}{2}\frac{f''(w_j)f(w_j)}{f'(w_j)}}\,, \tag{5.11}$$

which is a third order method. We thus have:

$$w_{j+1} = w_j - \frac{w_j e^{w_j} - z}{e^{w_j}(w_j+1) - \dfrac{1/2(w_j+2)(w_j e^{w_j} - z)}{w_j + 1}}\,. \tag{5.12}$$

A very important feature of this method is that the first few iterations can be performed at low precision. Once the iterations converge to several digits, the precision can then be (almost) tripled for each successive iteration, so that only the final iteration needs to be computed at the full user-requested precision. Furthermore, the computation of the denominator, and the final division, in (5.12) need only be computed at two thirds of the working precision. It can be shown that the number of correct digits essentially triples with each iteration, so that we may stop iterating when the current precision of computation is more than the user requested.

For most arguments $z$, a sufficiently accurate value to start the Halley iteration is given by just the first two terms of the asymptotic expansion given in Section 4. When computing any of $W_0(z)$, $W_{-1}(z)$ or $W_1(z)$ for $z$ near the branch point at $-1/e$, the first two terms of the generalized series (5.8) can be used (with the appropriate changes of sign in the case of $W_{-1}$ and $W_1$). When computing $W_0(z)$ near 0, a Padé approximant can be used (Maple uses a (3,2)-Padé approximant). Finally, when computing $W_{-1}(z)$ for $z$ near but not too near either of the branch points 0 or $-1/e$ a simple rational approximation to $W_{-1}$ should be used, as for such $z$ neither the asymptotic expansion nor the series expansion at $-1/e$ is very accurate. A similar remark holds true for $W_1$.

For the implementation in Maple, the approximate boundaries of these regions were determined empirically.

## 6. Concluding Remarks

We have collected here all available results on Lambert's $W$ function, for convenient reference. We have presented some of the history and some sample applications of $W$, and proposed a standard notation for all its branches in the complex plane. We have also presented some new results on the numerical analysis, asymptotic analysis, and symbolic calculus of this function.

The original motivation of this work was to provide a convenient reference for users of the Maple computer algebra program, who might encounter Lambert's $W$ function, but the work evolved to encompass what we feel are the interesting results presented here. Many of the results of this paper will be included in future versions of Maple.

J. Borwein pointed out the connection of equation (3.11) with the Legendre Transform. The jet-fuel problem was communicated to us by Michael Kamprath, with commentary by Harald Hanche-Olsen. We are indebted to G.-C. Rota and D. Loeb for their explanation of the applications of $W$ to combinatorics. We are also indebted to D. E. Gerber for his work in translating Euler's paper [14].

This work was supported by the Natural Science and Engineering Research Council of Canada, and the Information Technology Research Council of Ontario.

## References

[1] G. Alefeld, "On the Convergence of Halley's method", American Mathematical Monthly, **88** (1981), 530–536.

[2] American National Standards Institute/Institute of Electrical and Electronic Engineers: *IEEE Standard for Binary Floating-Point Arithmetic*, ANSI/IEEE Std 754-1985, New York, 1985.

[3] John D. Anderson, *Introduction to Flight*, 3rd Ed., McGraw-Hill, New York, 1989.
[4] V. I. Arnold, *Mathematical Methods for Classical Mechanics*, Springer-Verlag, 1978.
[5] I. N. Baker and P. J. Rippon, "A Note on Complex Iteration", Amer. Math. Monthly, **92** (1985), 501–504.
[6] A. Barry, "A class of exact solutions for Richard's equation", J. Hydrology, (1993) *to appear.*
[7] N. G. de Bruijn, *Asymptotic Methods in Analysis*, North-Holland, 1961.
[8] A. Cayley, "A theorem on trees", Quarterly Journal of Mathematics, Oxford Series, **23** (1889), 376–378.
[9] Bruce W. Char, Keith O. Geddes, Gaston H. Gonnet, Benton L. Leong, Michael B. Monagan, and Stephen M. Watt, *The Maple V Language Reference Manual*, Springer-Verlag, 1991.
[10] S. D. Conte & C. de Boor, *Elementary Numerical Analysis*, 3rd Ed., McGraw-Hill, 1980.
[11] Robert M. Corless and D. J. Jeffrey, "Well, It Isn't Quite That Simple", *SIGSAM Bulletin*, **26** (1992), 2–6.
[12] *CRC Standard Mathematical Tables*, 28th Ed., W. H. Beyer (ed.), 1987.
[13] Harold T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, 1962.
[14] L. Euler, "De Serie Lambertina Plurismique Eius Insignibus Proprietatibus", Leonhardi Euleri Opera Omnia, Ser. 1. Opera Mathematica, Bd 6, 1921 (orig. date 1779).
[15] F. N. Fritsch, R. E. Shafer, & W. P. Crowley, "Algorithm 443: Solution of the Transcendental Equation $we^w = x$", Comm. ACM, **16** (1973), 123–124.
[16] K. O. Geddes, S. R. Czapor & G. Labahn, *Algorithms for Computer Algebra*, Kluwer Academic Publishers, 1992.
[17] Ch. C. Gillispie, ed., *Dictionary of Scientific Biography*, Scribners, N. Y., 1973.
[18] T. E. Hull, W. H. Enright, B. M. Fellen, and A. E. Sedgwick, "Comparing numerical methods for ordinary differential equations", SIAM J. Numer. Anal., **9** (1972), 603–637.
[19] W. Kahan,"Branch cuts for complex elementary functions", In *The state of the art in numerical analysis*, M. J. D. Powell and A. Iserles, Eds, Oxford University Press.
[20] R. A. Knoebel, "Exponentials reiterated", Amer. Math. Monthly, **88** (1981), 235–252.
[21] J. H. Lambert, "Observationes variae in Mathesin Puram" Acta Helvetica, physico-mathematico-anatomico-botanico-medica, **3**, Basel, (1758), 128–168
[22] J. H. Lambert, *Photometria sive de mensura et gradibus luminis, colorum et umbra,*, Augustae Vindelicorum, Basel, 1760.
[23] Robert E. O'Malley, Jr., *Singular Perturbation Methods for Ordinary Differential Equations*, Springer-Verlag Applied Mathematical Sciences, **89**, 1991.
[24] G. Pólya & G. Szegö, *Problems and Theorems in Analysis*, Springer-Verlag, 1972.
[25] G. Pólya and R. C. Read, *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*, Springer-Verlag, 1987.
[26] K. B. Ranger, "A complex variable integration technique for the 2-dimensional Navier–Stokes equations", Q. Applied Maths, **49** (1991), 555–562.

[27] E. L. Reiss, "A new asymptotic method for jump phenomena", SIAM J. Appl. Math, **39** (1980), 440–455.

[28] Gian-Carlo Rota, *Finite Operator Calculus*, Academic Press, 1975.

[29] Lee A. Segel and Marshall Slemrod, "The Quasi-Steady-State Assumption: A Case Study in Perturbation", SIAM Review, **31** (1989), 446–477.

[30] T. C. Scott, J. F. Babb, A. Dalgarno, and John D. Morgan III, "Resolution of a Paradox in the Calculation of Exchange Forces", *preprint*.

[31] Ove Skovgaard, Ivar G. Jonsson, and Jens A. Bertelsen, "Computation of Wave Heights Due to Refraction and Friction", J. Waterways Harbours and Coastal Engineering Division, February, 1975, 15–32.

[32] E. C. Titchmarsh, *Theory of Functions*, 2nd ed., Oxford, 1939.

## Appendix: Biographical Notes.

Johann Heinrich Lambert was probably born in Alsace in 1728, and died in Berlin in 1777 [17]. His scientific interests were remarkably broad. The self-educated son of a tailor, he produced fundamentally important work in number theory, geometry, statistics, astronomy, meteorology, hygrometry, pyrometry, optics, cosmology and philosophy. The title of Euler's paper [14] can be translated as *On a series of Lambert's and some of its significant properties*. Indeed, Euler refers to him in this paper as "acutissimi ingenii Lambertus", which may be freely translated as *the ingenious engineer Lambert*. Thus we feel it is appropriate to name the function after Lambert.

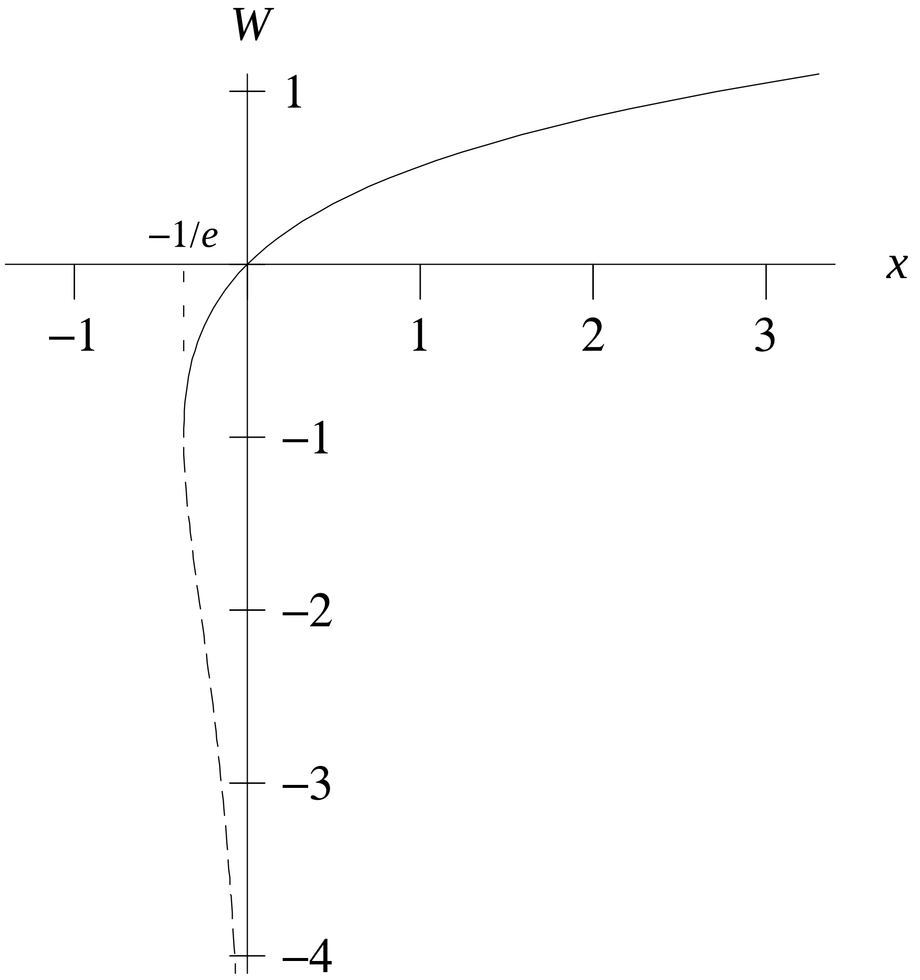


Figure 1.1. The two real branches of $W(x)$.
——, $W_0(x)$; − − −, $W_{-1}(x)$.

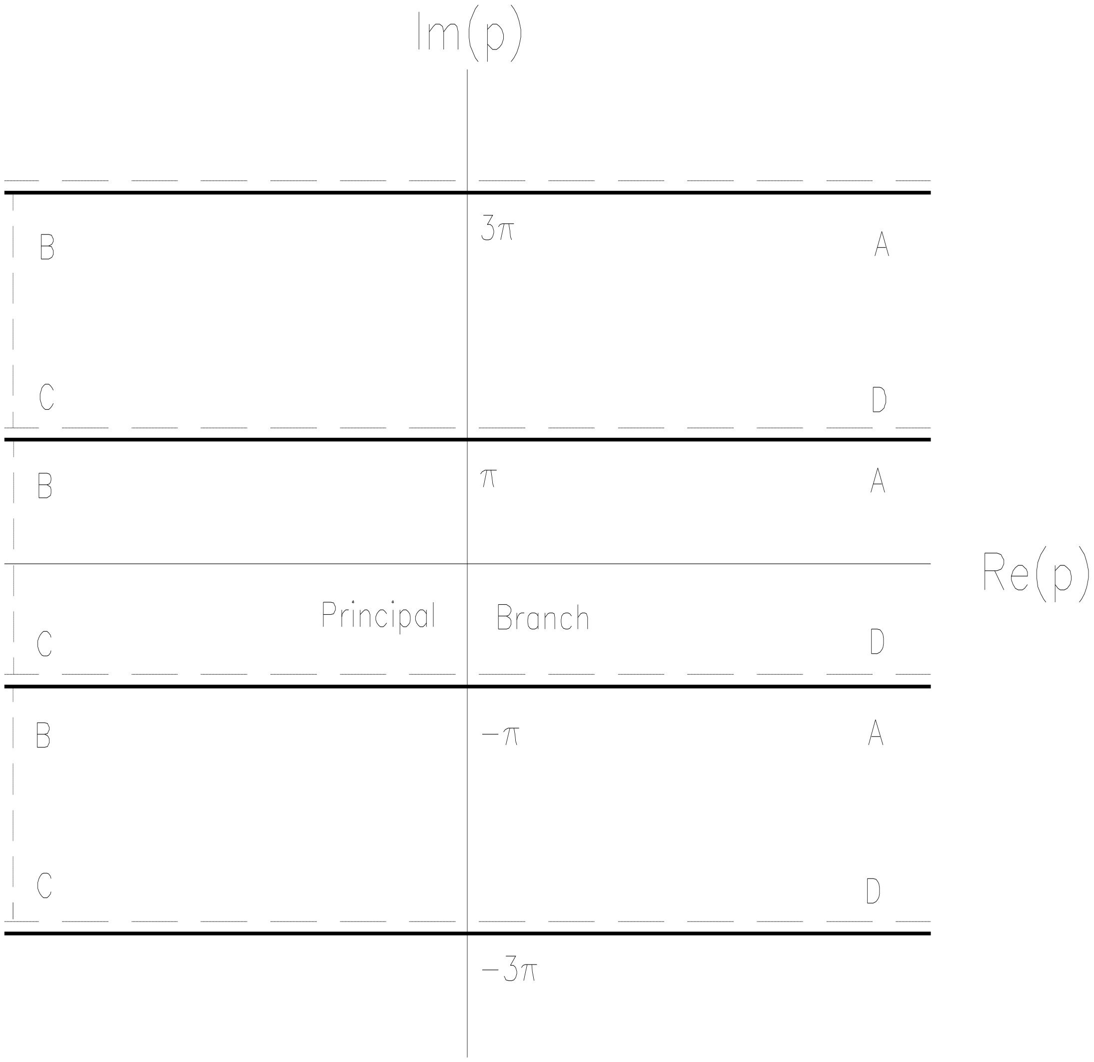


Figure 4.1. The ranges of the branches of $p = \log z$.

The range of the principal branch is $-\pi < \mathrm{Im}(p) \leq \pi$. The heavy solid lines indicate closure when approached from below. The points $A$, $B$, $C$, and $D$ are mapped by $z = e^p$ onto the corresponding points in Figure (4.2).

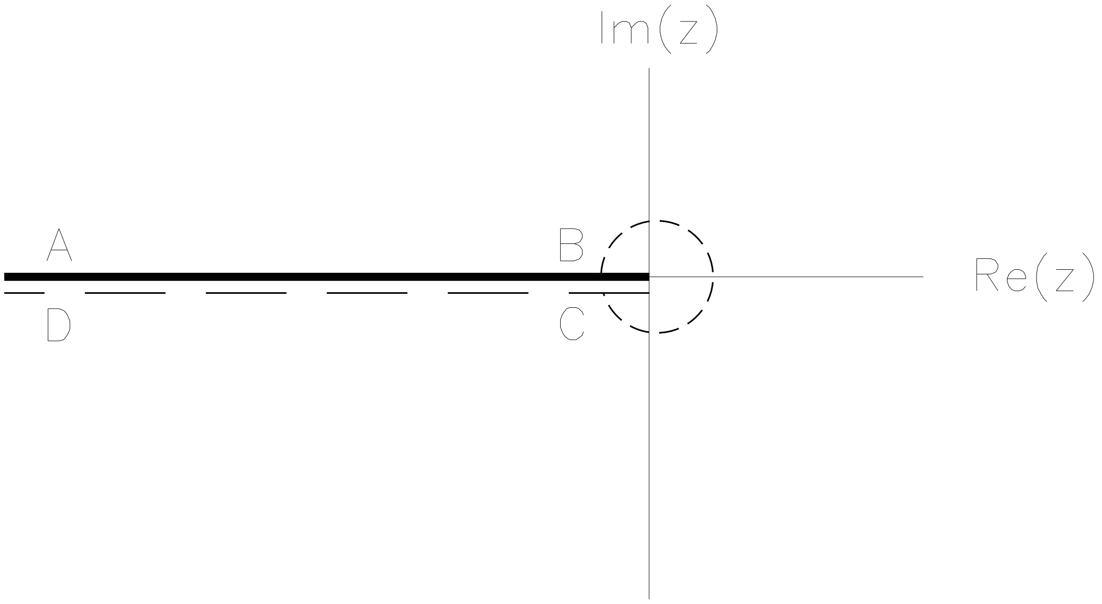


Figure 4.2. The branch cut for $p = \log z$.

The heavy solid line is the image under $z = e^p$ of the solid lines in Figure (4.1). The dashed line from $C$ to $D$ is the image of the similarly dashed lines in Figure (4.1), indicating in this Figure the open edge of the domain of $p = \log z$. The dashed circle running counterclockwise from $C$ to $B$ is the image of the similarly dashed lines in Figure (4.1), with closure at $B$.

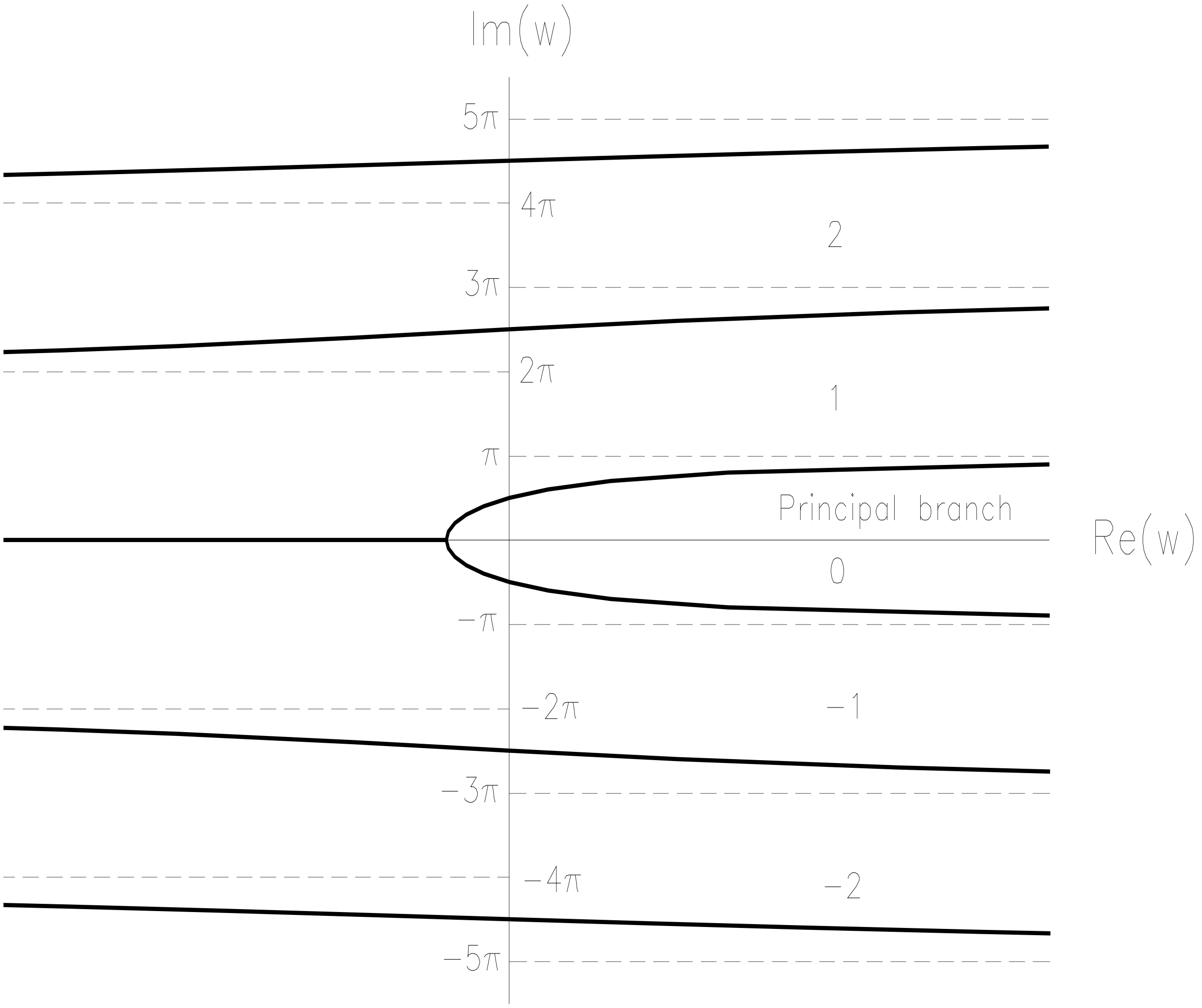


Figure 4.3. The ranges of the branches of $W$.

Closure is not indicated on this figure. See later figures for closure information. Each range is given a number, the principal branch being numbered 0. The boundaries of the ranges are asymptotic to the dashed lines.

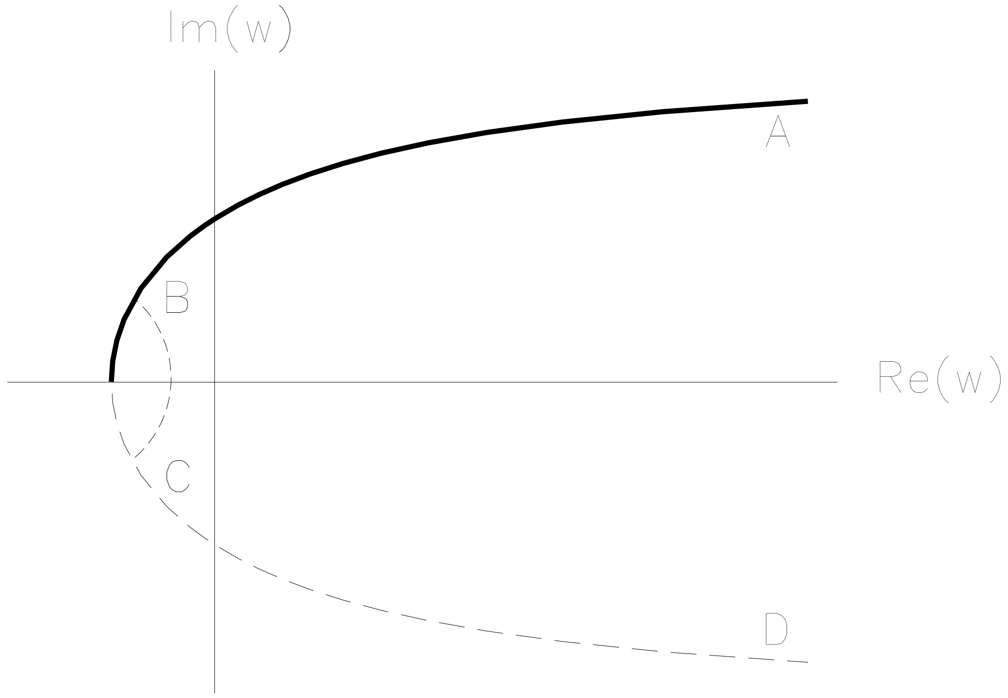


Figure 4.4. The range of $W_0(z)$.

The heavy solid line again indicates closure. The points $A$, $B$, $C$, and $D$ are the images of the corresponding points in Figure (4.5).

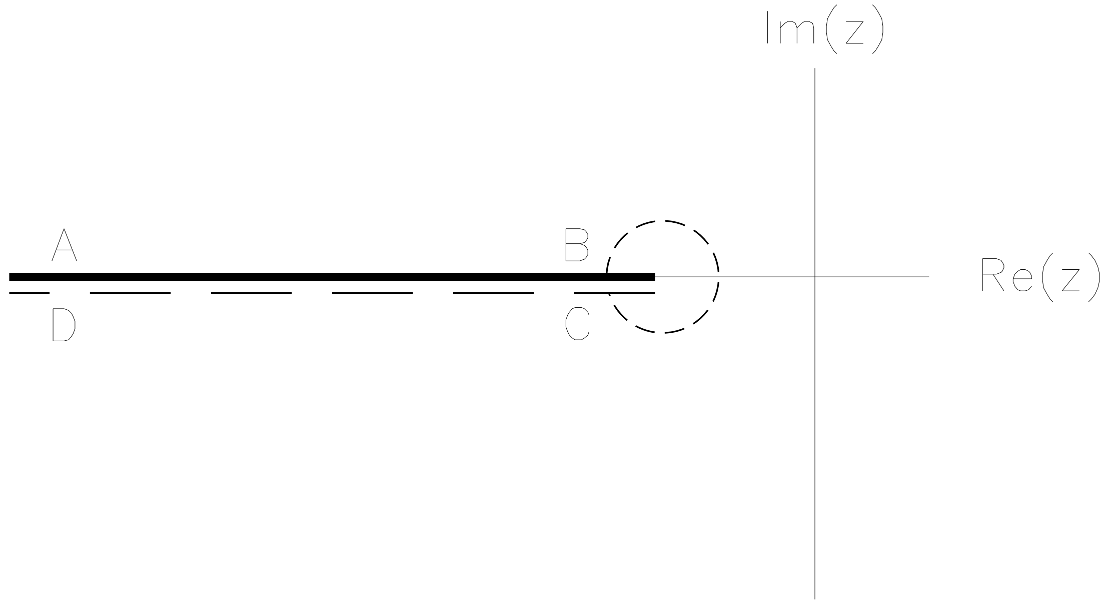


Figure 4.5. The branch cut for $W_0(z)$.

The heavy solid line is the image under $z = we^w$ of the heavy solid line in Figure (4.4), and similarly for the dashed line. The dashed circle running counterclockwise from $C$ to $B$ is the image of the similarly dashed line in Figure (4.4), and is closed at $B$. The branch cut is $\{z : -\infty < z \leq -1/e\}$.

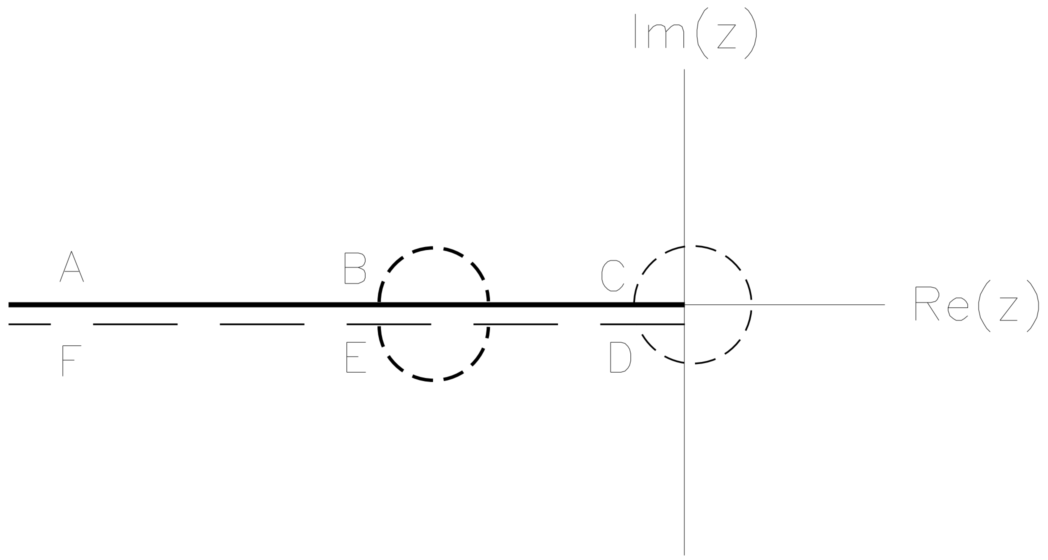


Figure 4.6. The branch cut for $W_k(z)$, $k \neq 0$.
Closure is indicated by a heavy solid line. For $W_1(z)$ and $W_{-1}(z)$, the dashed semicircles centred at $z = -1/e$ are the images under $z = we^w$ of the corresponding arcs in Figures (4.7) and (4.8). The dashed circle running counterclockwise from $D$ to $C$, closed at $C$, is the image of a line running from the corresponding points $D$ to $C$ in each of Figures (4.7) and (4.8).

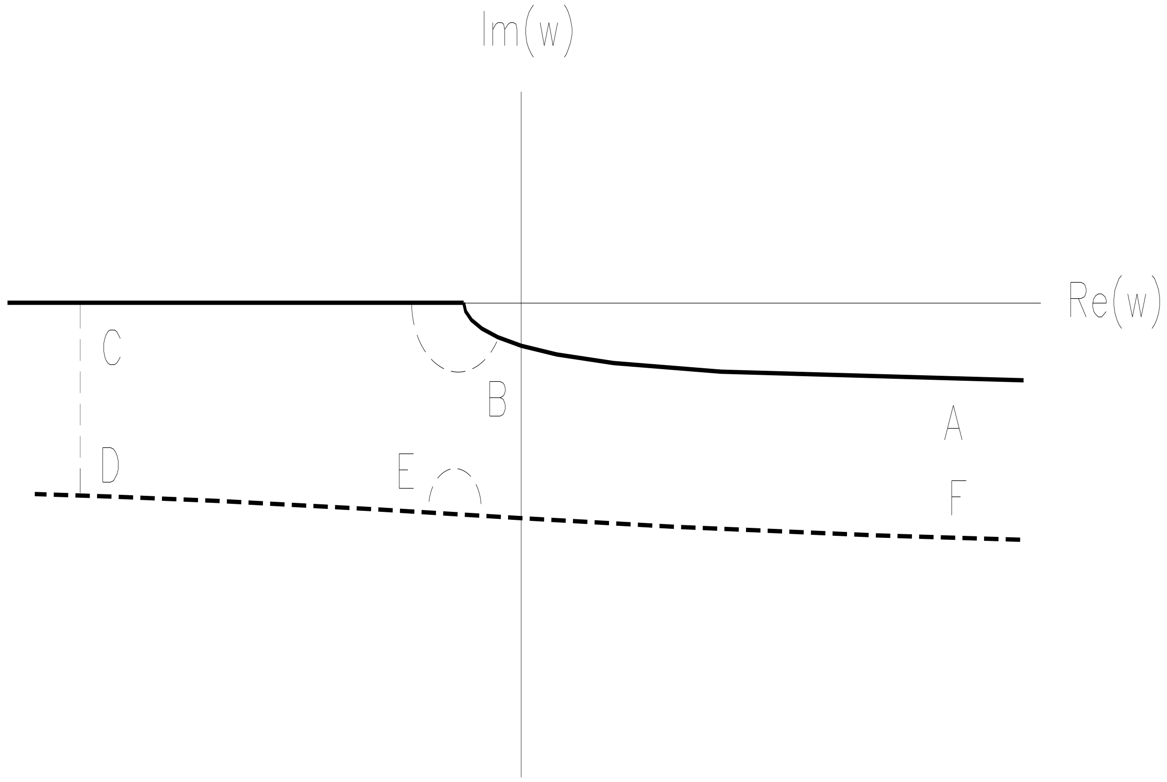


Figure 4.7. Details of the range of $W_{-1}(z)$.

The curve from $A$ to the 'corner' at $w = -1$ is the image of $z = (-\infty, -1/e]$. The curve from $w = -1$ to $C$ is the image of $z = [-1/e, 0)$. Thus, the range of $W_{-1}(z)$ includes part of the real line by this choice of closure.

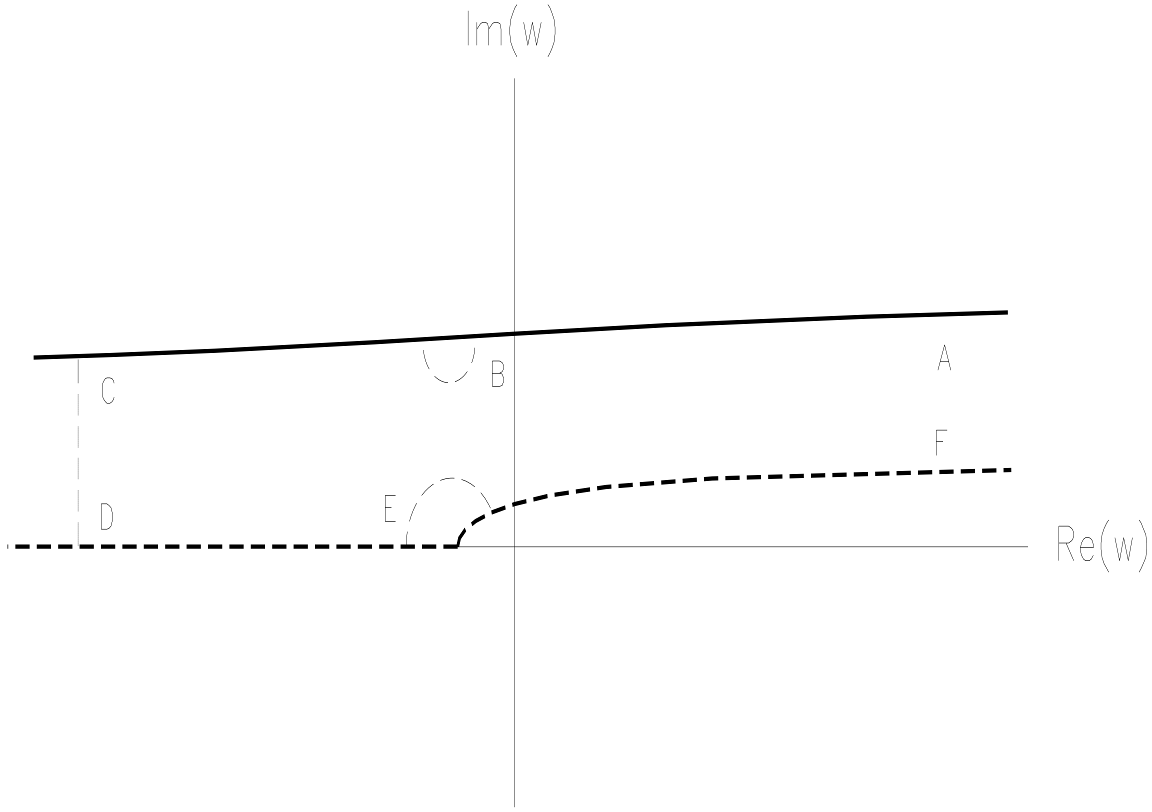


Figure 4.8. Details of the range of $W_1(z)$.

The curve from $A$ to $C$ is the image of the negative real $z$-axis. Note that $W_1(z)$ contains no part of the real axis in its range, by choice of closure.