Department of Statistics
University of Wisconsin, Madison
PhD Qualifying Exam Part II
January 21, 2010
1:00-4:00pm, Room 133 SMI

- There are a total of FOUR (4) problems in this exam. Please do a total of TWO (2) problems.
- Each problem must be done in a separate exam book.
- Please turn in TWO (2) exam books.
- Please write your code name and **NOT** your real name on each exam book.

4. Suppose $(Z_1, T_1), \cdots, (Z_n, T_n)$ are i.i.d. as (Z, T), where $T > 0$ is a survival time and $Z \in \mathbb{R}$ with $0 < E[Z^2] < \infty$ and $P(Z = 0) = 0$ is a predictor random variable. Let $H(z, t; \beta)$, $\beta \in \mathbb{R}$, denote the distribution of (Z, T) and assume that the marginal distribution of Z does not involve β. Consider the estimating equation

$$\frac{1}{n}\sum_{i=1}^{n} \psi_0(z_i, t_i; \beta) = 0,$$

where $\psi_0(z, t; \beta) = \frac{\dot{r}(z;\beta)}{r(z;\beta)} - \dot{r}(z; \beta)t$, where $r(.; \beta) > 0$ is a known 1-1 function of β for almost all z with derivative $\dot{r}(z; \beta) = \partial r(z; \beta)/\partial\beta$.

Suppose H is such that T given $Z = z$ has the Weibull distribution with c.d.f.

$$1 - \exp\{-r(z; \beta_0)t^\alpha\} \quad \alpha > 0.$$

Hint: You may use the fact that if W has the Weibull distribution with c.d.f. $1 - \exp(-w^\alpha/\theta)$, then $E[W] = \theta^{1/\alpha}c_0(\alpha)$ and $\text{var}(W) = \theta^{2/\alpha}c(\alpha)$, where $c_0(\alpha) = \Gamma(\alpha^{-1} + 1)$ and $c(\alpha) = \Gamma(2\alpha^{-1} + 1) - [\Gamma(\alpha^{-1} + 1)]^2$.

(a) For what values of α, does $E_H[\psi_0(Z, T; b)] = 0$ have the solution $b = \beta_0$? Show your work.

(b) Let $r(z; \beta) = \exp(-\beta z)$ and let α be the answer(s) to part (a). Is the solution $b = \beta_0$ in part (a) unique? Justify your answer.

(c) Let $D(b) = \frac{1}{n}\sum_{i=1}^{n} \psi_0(Z_i, T_i; b)$. Suppose $r(z, \beta) = \exp(-\beta z)$. Show that $D(b) = 0$ has a unique solution with probability one.

(d) Let β_1 be the solution to $E_H[\psi_0(Z, T; b)] = 0$ where H is the distribution for which $(T \mid z)$ has the Weibull distribution given above. Let $r(z, \beta) = \exp(-\beta z)$ and let $\widehat{\beta}$ be the solution to $D(b) = 0$ as detailed in (c). Derive the asymptotic distribution of $\sqrt{n}(\widehat{\beta} - \beta_1)$ assuming that (Z, T) has the distribution H. Simplify your answer as much as possible using the given distributional assumptions.

(e) Suppose $P(Z = 1) = P(Z = -1) = 1/2$. Give the asymptotic distribution in (d) for this case. What is the value of the variance in the asymptotic distribution of $\sqrt{n}(\widehat{\beta} - \beta_1)$ when $\alpha = 1$?

1. Let X be a positive random variable with finite $E(X)$, t be a positive number, $Y_t = \min\{X, t\}$, and $Z_t = \max\{X, t\}$. Assume that $P(X \le t) > 0$ and $P(X \ge t) > 0$.

   (a) Show that $E(X|Y_t) = X$ a.s. if and only if X is a constant on the event $\{X \ge t\}$.

   (b) Find expressions for $E(X|Y_t)$ and $E(X|Z_t)$, and show that they are indeed the conditional expectations.

   (c) Show that $E(X|Y_t) = Y_t$ a.s. if and only if $P(X > t) = 0$.

   (d) Assume that the distribution function of X is continuous and strictly increasing over $(0, \infty)$. Show that both $P(E(X|Y_t) < E(X|Z_t))$ and $P(E(X|Y_t) > E(X|Z_t))$ are positive and the sum of them is one.

2. A population consists of X_n individuals at times $n = 0, 1, 2, \cdots$. Between time n and time $n+1$ each of these X_n individuals dies with probability $q \in (0,1)$ independently of others, and the population at time $n+1$ is equal to the survivors from the X_n individuals plus an independent Poisson random number with mean λ. Let X_0 be Poisson with mean λ_0.

(a) Find the distribution of X_n

(b) Derive the limiting distribution of X_n as $n \to \infty$.

(c) For $n, k \geq 1$, compute $E[X_{n+k}|X_n]$.

(d) Show that as $n \to \infty$, $\sum_{k=1}^{n} X_k/n$ converges in probability to some constant c and identify c.

(e) Can you find constants a and b such that $\sum_{k=1}^{n}(X_k - a)/\sqrt{nb}$ has an asymptotic standard normal distribution?

3. Table 1 shows the yields from an experiment involving three two-level factors A, B and C in a balanced incomplete block design. Identify the effects confounded with block effects in each replicate and complete the ANOVA table below.

Table 1: Results from a balanced incomplete block experiment

Blk	A	B	C	Yield	Blk	A	B	C	Yield
Replicate 1					Replicate 2				
1	-1	-1	-1	10	3	-1	-1	-1	11
1	1	1	-1	17	3	-1	1	-1	9
1	-1	-1	1	9	3	1	-1	1	16
1	1	1	1	10	3	1	1	1	16
2	1	-1	-1	17	4	1	-1	-1	8
2	-1	1	-1	12	4	1	1	-1	9
2	1	-1	1	19	4	-1	-1	1	6
2	-1	1	1	11	4	-1	1	1	2
Replicate 3					Replicate 4				
5	-1	-1	-1	6	7	1	-1	-1	17
5	1	-1	-1	15	7	-1	1	-1	13
5	-1	1	1	8	7	-1	-1	1	9
5	1	1	1	1	7	1	1	1	16
6	-1	1	-1	9	8	-1	-1	-1	9
6	1	1	-1	14	8	1	1	-1	15
6	-1	-1	1	7	8	1	-1	1	17
6	1	-1	1	14	8	-1	1	1	14

Source	Df	Sum of squares	Mean square	F
Blocks		266.875		
A		180.500		
B		6.125		
C		8.000		
AB				
AC				
BC				
ABC				
Residuals				
Total (corr.)				

4. A scale has two pans. The measurements given by the scale is the difference between the weights in pan # 1 and pan # 2 plus a random error ϵ. Suppose that $E[\epsilon] = 0$ and $\text{Var}(\epsilon) = \sigma^2$, and that in repeated uses of the scale, observations are independent.

Suppose that three objects, #1, #2, and #3, have weights β_1, β_2, and β_3. Measurements are taken as follows:

(1) Object #1 and object #3 are put on pan #1, nothing on pan # 2.

(2) Object #2 and object #3 are put on pan #1, nothing on pan # 2.

(3) Object #2 and object #3 are put on pan #1, object #1 on pan # 2.

(4) Object # 3 is put on pan # 1, and object #1 and object # 2 on pan # 2.

(5) Object #1 and object #3 are put on pan #1, object # 2 on pan # 2.

(6) Object #1 and object #2 are put on pan #1, nothing on pan # 2.

(7) Object #2 is put on pan # 1 and nothing on pan # 2.

(8) Object #1 is put on pan # 1 and nothing on pan # 2.

Answer the following questions based on the above measurements.

(a) Let $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8)'$ be the vector of observations. Formulate this as a linear model, and find $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ such that $\hat{\beta}_1 = \mathbf{c}_1'\mathbf{Y}$, $\hat{\beta}_2 = \mathbf{c}_2'\mathbf{Y}$, and $\hat{\beta}_3 = \mathbf{c}_3'\mathbf{Y}$.

(b) Discuss the bias and variance properties of the estimators you found in part (a) by first proving and then using the Gauss-Markov theorem.

(c) Find the matrix $\mathbf{A}$ such that $s^2 = \mathbf{Y}'\mathbf{A}\mathbf{Y}$, where s^2 is an unbiased estimator of σ^2.

(d) For the observation $\mathbf{Y} = (9, 8, 5, 1, 7, 6, 3, 4)'$, find s^2, and estimate the covariance matrix of $\hat{\beta}$.

(e) Using the measurements from part (d), test whether the weights of the three objects are equal to each other at a significance level of 0.05.

(f) Show that eight such weighings can be made in such a way that the least squares estimators of β_1, β_2, and β_3 have smaller variances than the experiment above.