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Department of Statistics
University of Wisconsin-Madison
PhD Qualifying Exam Part I
September 4, 2007
12:30–4:30pm, Room 133 SMI

- There are a total of FOUR (4) problems in this exam. Please do a total of THREE (3) problems.
- Each problem must be done in a separate exam book.
- Please turn in THREE (3) exam books.
- Please write your code name, **NOT** your real name, on each exam book.

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1. Let X have probability density f_θ (with respect to some σ-finite measure), where $\theta \geq 0$ is an unknown parameter.

   (a) Let $\theta_1 > 0$ be a fixed value, $L = f_{\theta_1}(X)/f_0(X)$, and G be the distribution function of L under $\theta = 0$. Define $p = 1 - G(L)$ (the so-called p-value). Show that $p \leq 1/L$.

   (b) Assume in part (a) that G is continuous. Show directly that if $\theta = 0$, then p has a uniform distribution on (0,1).

   (c) Assume that the family $\{f_\theta : \theta \geq 0\}$ has monotone likelihood ratio in a real-valued statistic Y. Let H be the distribution function of Y when $\theta = 0$ and $p = 1 - H(Y)$. Show that, if H is continuous, then the uniformly most powerful (UMP) test of size $\alpha \in (0,1)$ for testing

   $$H_0 : \theta = 0 \quad \text{versus} \quad H_1 : \theta > 0$$

   rejects H_0 if and only if $p \leq \alpha$.

   (d) Suppose that we drop the condition that H is continuous in part (c). Derive a possibly randomized UMP test of size α.

2. Throughout this problem, the pair (X, Y) has a joint density $f(x, y)$ with respect to Lebesgue measure on R^2 and a joint distribution function $F(x, y)$. Let F_1 and F_2 be the marginal distribution functions of X and Y, respectively, and let Φ denote the standard normal distribution function. In statistical finance, the transformations $X \to Z = \Phi^{-1}(F_1(X))$, $Y \to W = \Phi^{-1}(F_2(Y))$ are used.

   (a) Show that the marginal distributions of Z and W are standard normal.

   (b) Give an example of a $F(.,.)$ where (Z, W) is not bivariate normal. Give an expression for the distribution function or density of (Z, W) for your example.

   (c) The *bivariate normal copula model* is defined by the assumption that the joint distribution of (Z, W) is bivariate normal with zero mean, unit variance, and correlation coefficient ρ. That is,

   $$F \in \mathcal{F} = \{F : (\Phi^{-1}(F_1(X)), \Phi^{-1}(F_2(Y))) \sim N(0, 0, 1, 1, \rho)\}.$$

   Let $\{(X_i, Y_i), i = 1, 2, \dots n\}$ be independent and identically distributed with distribution function F, and set $Z_i = \Phi^{-1}(F_1(X_i))$, $W_i = \Phi^{-1}(F_2(Y_i))$, $i = 1, 2, \dots, n$. If we (temporarily) assume that F_1 and F_2 are known, and if $F \in \mathcal{F}$, then $E(ZW) = \rho$ and a method of moments "estimate" of ρ is $\hat{\rho}_{\texttt{MOM}} = n^{-1} \sum_{i=1}^{n} Z_i W_i$. Find the asymptotic distribution of $\sqrt{n}(\hat{\rho}_{\texttt{MOM}} - \rho)$ when $F \in \mathcal{F}$. You may use the fact that $E(Z^4) = 3$.

   (d) With the same notation and assumptions as in part (c), that is, we assume F_1 and F_2 known, find the asymptotic distribution of $\sqrt{n}(\hat{\rho}_{\texttt{MLE}} - \rho)$, where $\hat{\rho}_{\texttt{MLE}}$ is the maximum likelihood "estimate" of ρ. You may use the fact that for $F \in \mathcal{F}$, (Z, W) has density

   $$\frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left\{ -\frac{z^2 - 2\rho z w + w^2}{2(1 - \rho^2)} \right\}.$$

   (e) i. Show that $\mathcal{F}$ is invariant under coordinate-wise strictly increasing transformations. That is, if $(X, Y) \sim F \in \mathcal{F}$ and $U = h_1(X)$, $V = h_2(Y)$ with h_1 and h_2 strictly increasing, then the distribution G of (U, V) is in $\mathcal{F}$.

   ii. Let $\hat{F}_1$ and $\hat{F}_2$ denote the empirical distributions of the X's and Y's respectively, and set $\tilde{F}_1 = n\hat{F}_1/(n + 1)$, $\tilde{F}_2 = n\hat{F}_2/(n + 1)$. Let $\hat{\rho}$ be the estimate of ρ obtained by replacing F_1 and F_2 by $\tilde{F}_1$ and $\tilde{F}_2$ in $\hat{\rho}_{\texttt{MOM}}$ as defined in part (c). Show that $\hat{\rho}$ is invariant under $X \to U$, $Y \to V$, where U and V are as defined in part (i) above.

3. Let $\{x_1, ..., x_m\}$ and $\{y_1, ..., y_n\}$ be two independent samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Denote $d = (x_1, ..., x_m, y_1, ..., y_n)$.

Denote $\bar{y}$ to be the sample mean of the y_i's, and $\bar{x}$ to be the sample mean of the x_i's, $s_1^2 = \sum_{i=1}^m (x_i - \bar{x})^2/m$, and $s_2^2 = \sum_{i=1}^n (y_i - \bar{y})^2/n$. Let Ψ_v be the cumulative distribution function of a t-distribution with v degrees of freedom. Consider the one-sided test problem $H_0 : \mu_1 - \mu_2 \leq 0$ versus $H_1 : \mu_1 - \mu_2 > 0$.

(a) Prove that under the non-informative prior $\pi(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \propto \sigma_1^{-2}\sigma_2^{-2}$, the posterior probability that $H_0 : \mu_1 - \mu_2 \leq 0$ is true has the form:

$$p(d) \equiv E_B \left\{ \Psi_{m+n-2} \left((\bar{y} - \bar{x}) \sqrt{\frac{m+n-2}{B^{-1}s_1^2 + (1-B)^{-1}s_2^2}} \right) \right\},$$

where $B \sim beta((m-1)/2, (n-1)/2)$.

(b) Define

$$f(b) = \Psi_v \left(\frac{z}{\sqrt{b^{-1}t_1 + (1-b)^{-1}t_2}} \right), \quad \text{for } b \in (0, 1),$$

where $z \leq 0$, $t_1 > 0$, $t_2 > 0$, and $v > 0$ are fixed constants. It is known that $f(b)$ is a convex function of b and that $f(b)$ is strictly convex if $z < 0$. Let Ψ_v^{-1} be the inverse function of Ψ_v and let Φ be the standard normal distribution function. Define

$$\begin{aligned} g(a) &\equiv P\left\{ \Psi_{m+n-2} \left(\frac{Z}{\sqrt{(m-1)^{-1}aC_{m-1} + (n-1)^{-1}(1-a)C_{n-1}}} \right) \leq r \right\} \\ &= E_{C_{m-1}, C_{n-1}} \left[\Phi \left\{ \sqrt{\frac{aC_{m-1}}{m-1} + \frac{(1-a)C_{n-1}}{n-1}} \left(\Psi_{m+n-2}^{-1}(r) \right) \right\} \right] \end{aligned}$$

where Z, C_{m-1} and C_{n-1} are independent random variables such that $Z \sim N(0,1)$, $C_{m-1} \sim \chi_{m-1}^2$, and $C_{n-1} \sim \chi_{n-1}^2$. Prove that $g(a)$ is a convex function of a.

(c) Prove that the posterior probability $p(d)$ in part (a) has the following repeated sampling distribution property with respect to the distribution of d:

$$P_d(p(d) \leq r) \leq \Psi_{\min\{m-1, n-1\}}(\Psi_{m+n-2}^{-1}(r)).$$

4. Let $x_1, x_2, \ldots, x_n$ be n fixed real numbers such that $0 < \sum_{i=1}^n (x_i - \bar{x})^2 < \infty$ where $\bar{x} = n^{-1} \sum_{i=1}^n x_i$. We observed data $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, where

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad (\dagger)$$

and the ε_i are independent random variables with $E(\varepsilon_i) = 0$ and $E(\varepsilon_i^2) = \sigma^2$, $i = 1, 2, \ldots, n$. Let $\hat{\beta}_0$ and $\hat{\beta}_1$ denote the least-squares estimates of β_0 and β_1, respectively. Given a fixed number z, we wish to estimate $\mu = E(y)$ at z. Thus, $\mu = \beta_0 + \beta_1 z$. Let $\hat{\mu}$ denote the least-squares estimate of μ.

(a) Suppose that instead of fitting the true model ($\dagger$), we fit the incorrect model

$$y_i = \eta + \varepsilon_i, \quad i = 1, 2, \ldots, n$$

where η is a constant. Let $\hat{\mu}_1$ denote the least-squares estimate of η. Find a necessary and sufficient condition on $\beta_0, \beta_1, \sigma^2$, and $x_1, \ldots, x_n$ for $E(\hat{\mu}_1 - \mu)^2 < E(\hat{\mu} - \mu)^2$.

(b) Next suppose that we fit the incorrect (no-intercept) model

$$y_i = \gamma x_i + \varepsilon_i, \quad i = 1, 2, \ldots, n.$$

Let $\hat{\gamma}$ denote the least-squares estimate of γ and define $\hat{\mu}_2 = \hat{\gamma} z$. Find a necessary and sufficient condition for $E(\hat{\mu}_2 - \mu)^2 < E(\hat{\mu} - \mu)^2$.

Notes:

(i) All expectations are taken with $x_1, \ldots, x_n$ and z fixed.

(ii) You may use the formulas: $\hat{\beta}_1 = s_x^{-1} \sum_i (x_i - \bar{x})(y_i - \bar{y})$, $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$,

$$\begin{aligned} \text{Var}(\hat{\beta}_0) &= \sigma^2 (n^{-1} + s_x^{-1} \bar{x}^2) \\ \text{Var}(\hat{\beta}_1) &= \sigma^2 s_x^{-1} \\ \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) &= -\sigma^2 \bar{x} s_x^{-1} \end{aligned}$$

where $\bar{y} = n^{-1} \sum_i y_i$ and $s_x = \sum_i (x_i - \bar{x})^2$.