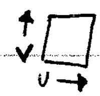
What is a surface?

A 2D Manifold

Every point is homeomorphic to $\mathbb{R}^2$

resembles (wikipedia)

$\mathbb{R}^2$ is parameterized 

① LOCALLY PARAMETERIZABLE — Analogy to Curves

② Differentially looks like a plane

Bigger region stops looking like a plane

"curviness"

~~Rough!~~

Turns out to be equivalent questions:

How curvy?

How "good" is the mapping?

some surfaces are curvy but mappable

twisty paper ← developable

A local parameterization always exists

we might not want to find it

we might not want to find a "good" one

we might not be able to find a "good" non-local one

Surfaces

What properties might we measure?

Normals / Tangents
Change in Normals / Tangents

Lengths
Angles
Areas

$\Big\}$ measures of distortion - how do these change
① when flattened
② when transformed

Curvatures - how normals / tangents change related to other quantities

Change in curvature

Connect to parametrization distortion

Differential Geometry of Surfaces

Chicken and egg:
params → diff geo
diff geo → params

Consider P on surface
consider all curves on surface through P
tangents all in a plane (except singular points)
→ tangent plane to surface

Normal = perpindicular to tangent plane

If normal is unit vector, it change must be in tangent plane
a 2D vector

Normal Sections = planes through normal
↑
spin around 360°
-really 180°

radial curvatures

Consider curves on normal sections
curvature as a function of "angle"
maximum
minimum
principle curvatures K_1 K_2

"can be shown that the directions are ⊥"

mean curvatre = $K_1 + K_2$
gaussian curvatre = $K_1 \cdot K_2$

Bowl shaped = GC > 0 = elliptic point
Saddle shaped = GC < 0 = hyperbolic point ← must have some direction curvature = 0
ridge = gc = 0 = flat in one direction
(parabolic ?)

Some Parameterization....

Surface $\triangleq$ locally mappable (homeomorphic) to $\mathbb{R}^2$

at any point has a tangent plane, 2 vectors span it

$X = f(u,v)$ $\quad \frac{\partial f}{\partial u} = X_1 \quad \frac{\partial f}{\partial v} = X_2$ $\quad$ (or X_u X_v)

↳ we're picking 2 curves

Normal $= X_1 \times X_2$

any tangent vector $= \alpha_1 X_1 + \alpha_2 X_2$

NOTE: it doesn't matter much how we get the tangent vectors

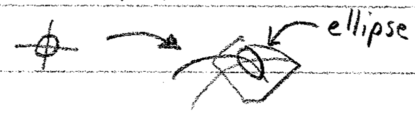
Intrinsic Property doesn't matter what parameterization is

wrong term

How does a little circle in U,V space end up in xyz space?

DISTORTED!

1ST ORDER MODEL (map to tangent plane)



FIRST FUNDAMENTAL FORM

if we have a vector in UV space, how long $\overset{\text{squared}}{}$ is it in tangent space?

Why? tells us how sizes change
unit area in param space →
Integrate to get areal properties

$$X = du\, X_1 + dv\, X_2$$

$$X \cdot X = du^2\, X_1 \cdot X_1 + 2\, du\, dv\, X_1 \cdot X_2 + dv^2\, X_2 \cdot X_2$$

$$u \underbrace{\begin{bmatrix} X_1 \cdot X_1 & X_2 \cdot X_1 \\ X_2 \cdot X_1 & X_2 \cdot X_2 \end{bmatrix}}_{\text{metric tensor } G} u$$

$$G = J^T J$$

Useful for making measurments
what happens to angles
sum up areas

How to compute the curvatures?

we don't know the principle curvatures

u an v may not be the right direction!

How does normal vector change in a direction
(must be in tangent plane! since unit vector)

How do the tangents change? (in the directions its easy to try)

$f_{uu} = \frac{\partial^2 f}{\partial u^2}$ f_{uv} f_{vv}

Only interested in the normal direction

$$\begin{bmatrix} n \cdot f_{uu} & n \cdot f_{uv} \\ n \cdot f_{uv} & n \cdot f_{vv} \end{bmatrix} = B$$ second fundamental form

Curvature is:

in the "normal" direction $\dfrac{u^T B u}{u^T G u}$

Curvature Tensor $S = G^{-1} B$

eigenvalues are principle curvatures

eigenvectors are principle directions

determinant is gauss curvature

$\frac{1}{2}$ trace is mean curvature

transforms vectors $\Rightarrow$ curvatures

This doesn't quite get to the punch line:
- the eigenvectors of S are the principle directions
- the eigenvalues of S are the principle curvatures
- the Gaussian curvature is the determinant of S
- the mean curvature is the trace of S