Rigid Transformations

Objects "interior" relationships preserved

- distance
- angles (dot products)
- handedness

Translations preserve this ← are a type of R.T.

$$x' = x + t$$

Rotations

$\exists\, x \therefore f(x) = x$

center of Rotation

Euler:

SINGLE! SINGLE!

Every RT is a TRANS + a ROT

~~(assuming rot about arb pt = 2 TRANS + 1~~

Simplification:

Assume C.O.R. is the origin

No Loss of Generality

TRANSLATE $C \to O$

ROTATE

TRANSLATE $O \to C$

Rotations are Linear

~~unclear if linear are the only functions that meet the restrictions~~

See CARL'S COOL PROOF

ROTATIONS IN $\mathbb{R}^N$

1. TRANSFORMATION $f: \mathbb{R}^N \to \mathbb{R}^N$
2. LINEAR (~~might~~ follows from other things)
3. $f(0) = 0$ (since linear)
4. $\|A - B\| = \|f(A) - f(B)\|$
5. handedness preserved $\det(f) > 0$

$\Rightarrow$ angles are preserved / dot products

Linear operators are matrices
$\mathbb{R}^N \Rightarrow \mathbb{R}^N$ is a square matrix

since $Av = v'$ $\|v\| = \|v'\|$ (zero is a pt. in #4)
stick in axis basis vectors, see matrix must be normal

since must preserve dot products
stick in axes $\to$ see must be orthogonal

an ortho-normal matrix w/ positive det is a ROTATION (actually det=1)

CAN REPRESENT ROTATIONS AS ELEMENTS OF AN $N \times N$ matrix

NOT A GOOD PARAMETERIZATION

The set of all rotations in $\mathbb{R}^N$ is the set of all orthonormal $N \times N$ matrices is the Special Orthogonal Group

This set is a group in the algebraic sense

$$f \in G + g \in G \Rightarrow f \circ g \in G$$

$$\exists\, I \,:\, I \in G + \forall f \in G \quad 1 \circ f = f \text{ and } f \circ 1 = f$$

HOW DO WE "NAME" THE ELEMENTS OF THIS GROUP?

EG: How do we PARAMETERIZE IT.

MANY THINGS WE WANT FROM PARAMETERIZATION
CAN'T HAVE THEM ALL

- concise
- intuitive
- stable
- invertable
- $1 \Rightarrow 1$
- metric
- efficient
- composable / interpolatable (easily)

The problem:

$\mathbb{R}$ is straight

rotations are not

Some problems w/ using rotation matrices

- given an M, is it a rotation?
- if M is not a rotation, can you tell me what rotation it is close to
  - ≣ if we perform a computation on a rotation matrix, we probably won't end up w/ a rotation matrix

- given M_1 and M_2
- can you tell me how far apart they are?
- can you tell me some things in between?
  interpolate this: $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$ $\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$

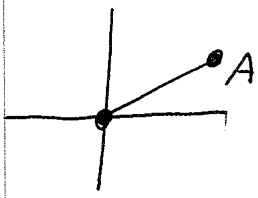
- if you change 1 entry, what happens
- what do the entries mean?

What can you do with rotation matrices:
  apply them to points
  compose them (multiply)
  be sure than any rotation can be expressed

NOTE: $AB \neq BA$ (in general)

Why is this so hard - and what to do about it?

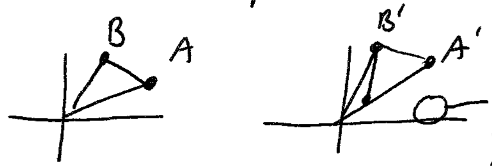
ROTATION IN 2D ← easy, but look at for practice



think of 1 pt (A) and where it goes f(A)

① the set of places that A can go is a circle

② EACH POINT ON THIS CIRCLE IS A ROTATION (The flip is ruled out by handed ness)



can't be B' even though dist OK

handedness means we know clockwise from counter clockwise

The set of rotations in $\mathbb{R}^2$ is the same as the set of points on a circle

NAMING POINTS ON A CIRCLE IS DIFFERENT than points on a number line

- things loop around

PARAMETERIZING POINTS ON THE CIRCLE

1. write the rotation matrix that rotates the unit X vector to the point

2. encode as the x, y position of the point
≈ quaternion

issues: - not all pairs on circle
but easy to find "nearby" point on circle
- hard to do math on them -
most operations move you off the circle
- hack: don't worry about circle, put yourself back on later
≡ only an issue if big differences
Interpolation as example

3. encode the distance from $\vec{0}$ (X vector) (in counter clockwise)
in 2D this works really well
Angles (see 2π is on trip around)

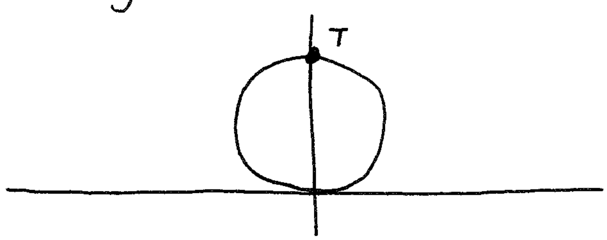
some issues do arrise -
- multiple names
- shortest direction
- roll over issues when do math
- non-metric (big number differences ≡ small angle differences)

CAN WE MAP THE CIRCLE TO THE REAL LINE ?

$1 \Leftrightarrow 1$

not everywhere - since we need to have that looping
if 2 pts are "next to each other" on circle, should be on line

Something that works "almost everywhere"



connect from T to point
point just to the right of T $\Rightarrow \infty$
left of T $\Rightarrow -\infty$

What about velocities?
- always tangent to the curve
- all possible velocities are in that line

Tangent Space -
1 less dimension than curve is in
linear space (flat) - can be encoded in $\mathbb{R}^N$

As A WAY TO DESCRIBE A ROTATION ?
Give velocity, assume that moves at that velocity
continually (but stays on the curve)
STARTING FROM ZERO

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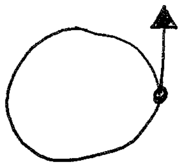
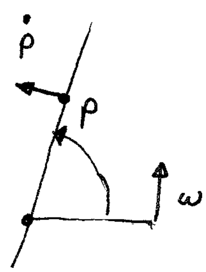
Angular Velocity $\hat{=} \omega$



$\leftarrow$ always in the "up" direction - tangent to the circle @ $x=1$

How to turn an angular velocity into a "regular" velocity (1D $\to$ 2D)

make it a vector - rotate it



LOCAL LINEARIZATION

$$\dot{p} = P\omega$$

$\uparrow$ the solution to this is an exponential

The operation that maps from the tangent space to a rotation (point on the sphere) is the Exponential Map.

If we were using complex numbers:
the angular velocity would be purely imaginary

Why is this good?
It's metric -
It's compact
The singularity is always "far away"

Now ONTO 3D

Suppose we have a point fixed distance from
origin = SPHERE

Second point can rotate around the vector
3rd Spherical Dimention
3D sphere - embedded in 4D space

Handedness rules out "half" of the set of S^3
(or makes half = to other half)
$\triangleq SO(3)$

3 degrees of freedom — but none map to a real line

How TO ENCODE ?
- ROTATION MATRIX
- EULER ANGLES
- AXIS / ANGLE FORM
- UNIT QUATERNIONS // Projected Quaternions
- EXPONENTIAL CO-ORDINATES