Primal-Dual Bilinear Programming Solution of the Absolute Value Equation

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Abstract

We propose a finitely terminating primal-dual bilinear programming algorithm for the solution of the NP-hard absolute value equation (AVE): $Ax - |x| = b$, where A is an $n \times n$ square matrix. The algorithm, which makes no assumptions on AVE other than solvability, consists of a finite number of linear programs terminating at a solution of the AVE or at a stationary point of the bilinear program. The proposed algorithm was tested on 500 consecutively generated random instances of the AVE with n =10, 50, 100, 500 and 1,000. The algorithm solved 88.6% of the test problems to an accuracy of $1e-6$.

Keywords: absolute value equation, bilinear programming, linear programming

1 INTRODUCTION

We consider the absolute value equation (AVE):

$$Ax - |x| = b, \tag{1.1}$$

where $A \in R^{n\times n}$ and $b \in R^n$ are given, and $|\cdot|$ denotes absolute value. A slightly more general form of the AVE, $Ax + B|x| = b$ was introduced in [14] and investigated in a more general context in [9]. The AVE (1.1) was investigated in detail theoretically in [11], and a bilinear program in the *primal* space of the problem was prescribed there for the special case when the singular values of A are not less than one. No computational results were given in either [11] or [9]. In contrast in [8], computational results were given for a linear-programming-based successive linearization algorithm utilizing a concave minimization model. As was shown in [11], the general NP-hard linear complementarity problem (LCP) [3, 4, 2], which subsumes many mathematical programming problems, can be formulated as an AVE (1.1). This implies that (1.1) is NP-hard in its general form. More recently a generalized Newton method was proposed for solving the AVE [10], while a uniqueness result for the AVE is presented in [15] and for a more general version of the AVE in [16], and finally existence and convexity results are given in [6].

Our point of departure here is to look at the AVE in its primal and dual spaces of the problem and formulate an algorithm that minimizes a bilinear function (that is the scalar product of two linear functions) in the combined primal-dual space which has a global minimum of zero that yields an exact solution of the AVE. In Section 2 we describe our bilinear formulation of the AVE and show that a zero minimum of the bilinear program yields a solution to the AVE. In Section 3 of the paper we state our algorithm for the bilinear program consisting of a succession of linear programs that terminate at a global solution of the the AVE or at a stationary point of the bilinear program. In Section 4 we give computational results that show the effectiveness of our approach by solving 88.6% of a sequence of 500

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randomly generated consecutive AVEs in R^{10} to $R^{1,000}$ to an accuracy of $1e-6$. Section 5 concludes the paper.

We describe our notation now. All vectors will be column vectors unless transposed to a row vector by a prime $'$. For a vector $x \in R^n$ the notation x_j will signify the j-th component. The scalar (inner) product of two vectors x and y in the n-dimensional real space R^n will be denoted by $x'y$. For $x \in R^n$, $\|x\|$ denotes the 2-norm: $(\sum_{i=1}^{n}(x_i)^2)^{\frac{1}{2}}$. The notation $A \in R^{m\times n}$ will signify a real $m \times n$ matrix. For such a matrix, A' will denote the transpose of A. A vector of ones in a real space of arbitrary dimension will be denoted by e. Thus for $e \in R^m$ and $y \in R^m$ the notation $e'y$ will denote the sum of the components of y. A vector of zeros in a real space of arbitrary dimension will be denoted by 0. The abbreviation "s.t." stands for "subject to".

2 Bilinear Formulation of the Absolute Value Equation

We begin with the linear program:

$$\min_{x,y} h'y \;\text{ s.t. }\; Ax - y = b,\; x + y \geq 0,\; -x + y \geq 0, \tag{2.2}$$

and its dual:

$$\max_{u,v,w} b'u \;\text{ s.t. }\; A'u + v - w = 0,\; -u + v + w = h,\; (v, w) \geq 0, \tag{2.3}$$

where h is some vector in R^n that will play a key role in a bilinear programming formulation. We now state the following simple lemma.

LEMMA 2.1. *Let (x, y) be a solution of the primal problem (2.2) and (u, v, w) be a solution of the corresponding dual problem (2.3). Then:*

$$v + w > 0 \implies Ax - |x| = b \tag{2.4}$$

Proof From the complementarity condition we have that:

$$v'(x + y) + w'(-x + y) = 0. \tag{2.5}$$

Hence, if $v + w > 0$ it follows that either $(x + y)_i = 0$, or $(-x + y)_i = 0$, for $i = 1, \ldots, n$. Hence $y = |x|$ and from the constraint $Ax - y = b$ it follows that $Ax - |x| = b$. $\square$

Based on this lemma it follows that for a primal-dual optimal solution (x, y, u, v, w), if $v + w \geq \epsilon e$ for a positive ϵ, then $Ax - |x| = b$. Furthermore, from the dual constraints we have that $h = -u + v + w$ and hence the difference between the primal and dual objective functions evaluated at a primal-dual feasible point becomes:

$$h'y - b'u = (-u + v + w)'y - b'u \geq 0, \tag{2.6}$$

where the inequality of (2.6) follows from the fact that at a primal-dual feasible point, the primal objective function exceeds or equals the dual objective function. At a primal-dual optimal point this difference is zero. Hence combining these statements with Lemma 2.1 and the extra imposed condition that $v + w \geq \epsilon e$, we have the following proposition.

PROPOSITION 2.2. **Equivalence of AVE and Zero Minimum of the Bilinear Program** *At a zero minimum of the following bilinear program:*

$$
\begin{array}{lrcl}
\min\limits_{x,y,u,v,w} & y'(-u+v+w)-b'u & & \\
\text{s.t.} & Ax-y & = & b \\
& x+y & \geq & 0 \\
& -x+y & \geq & 0 \\
& A'u+v-w & = & 0 \\
& v+w & \geq & \epsilon e \\
& (v,w) & \geq & 0
\end{array}
\tag{2.7}
$$

we have that $y=|x|$ and $Ax-|x|=b$ for any solution point (x,y,u,v,w).

We establish now the existence of a zero-minimum solution to the bilinear program (2.7) under the assumption that AVE (1.1) is solvable.

PROPOSITION 2.3. **Existence of a Zero-Minimum Solution to the Bilinear Program** *Under the assumption that the absolute value equation (1.1) is solvable, the bilinear program (2.7) has a zero minimum solution (x,y,u,v,w) such that x solves the absolute value equation (1.1).*

Proof Since AVE (1.1) has a solution, say x, then the feasible region of the bilinear program (2.7) is nonempty because the point $(x, y=|x|, u=0, v=w=\epsilon e/2)$ satisfies the constraints of (2.7). Hence the quadratic bilinear objective function of (2.7) which by Proposition 2.2 is bounded below by zero must by [5] have a solution. Since by Proposition 2.2 a zero-minimum solution solves AVE, and AVE is solvable by assumption, such a zero-minimum solution exists that solves AVE. $\square$

We now present a computational algorithm for solving the bilinear program (2.7) that consists of solving a finite number of linear programs.

3 Bilinear Programming Algorithm for the Absolute Value Equation

We begin by stating our bilinear algorithm as follows.

ALGORITHM 3.1. *Choose parameter value ϵ for the constraint of (2.7) (typically $\epsilon = 1\text{e}-2$), tolerance (typically tol$=1\text{e}-6$), and maximum number of iterations itmax (typically itmax$=40$).*

(I) Initialize the algorithm by determining an initial (x^0, y^0) by solving the following linear program:

$$
\begin{array}{lrcl}
\min\limits_{x,y} & e'y & & \\
\text{s.t.} & Ax-y & = & b \\
& x+y & \geq & 0 \\
& -x+y & \geq & 0
\end{array}
\tag{3.8}
$$

Set iteration number $i=0$.

(II) **While $\|Ax^i-|x^i|-b\| > tol$, the bilinear objective function of (2.7) is decreasing, and $i \leq itmax$ perform the following three steps.**

(III) Solve the following linear program for $(u^{i+1}, v^{i+1}, w^{i+1})$:

$$
\begin{array}{lrcl}
\min\limits_{u,v,w} & {y^{i}}'(-u+v+w)-b'u & & \\
\text{s.t.} & A'u+v-w & = & 0 \\
& v+w & \geq & \epsilon e \\
& (v,w) & \geq & 0
\end{array}
\tag{3.9}
$$

(IV) Solve the following linear program for (x^{i+1}, y^{i+1}):

$$\begin{array}{lrcl} \min\limits_{x,y} & (-u^{i+1}+v^{i+1}+w^{i+1})'y & & \\ \text{s.t.} & Ax-y & = & b \\ & x+y & \geq & 0 \\ & -x+y & \geq & 0 \end{array} \tag{3.10}$$

(V) $i = i+1$. Go to Step (II).

We establish now finite termination of our bilinear algorithm.

PROPOSITION 3.2. **Finite Termination of the Bilinear Algorithm** *Under the assumption that the absolute value equation (1.1) is solvable and the maximum number of iterations itmax is sufficiently large, the Bilinear Algorithm 3.1 terminates in a finite number of iterations at a global zero-minimum point that solves the absolute value equation (1.1), or at iteration i with a solution $(x^{i+1}, y^{i+1}, u^{i+1}, v^{i+1}, w^{i+1})$ that satisfies the following minimum principle necessary optimality condition for the bilinear program (2.7):*

$$\begin{array}{c}(-u^{i+1}+v^{i+1}+w^{i+1})'(y-y^{i+1}) - (y^{i+1}+b)'(u-u^{i+1}) + {y^{i+1}}'(v-v^{i+1}) + {y^{i+1}}'(w-w^{i+1}) \geq 0, \\ \forall x \in X,\ (u,v,w) \in U,\end{array} \tag{3.11}$$

where

$$X = \{(x,y) \mid Ax-y=b,\ x+y \geq 0,\ -x+y \geq 0\}, \tag{3.12}$$

$$U = \{(u,v,w) \mid A'u+v-w=0,\ v+w \geq \epsilon e,\ (v,w) \geq 0\}. \tag{3.13}$$

Proof Note first that the sets X and U defined above are nonempty because as pointed out earlier that under the assumption that AVE has a solution x then $(x, |x|) \in X$ and $(0, \epsilon e/2, \epsilon e/2) \in U$. To keep the proof simple we shall assume that neither X nor U have straight lines going infinity in both directions. This assumption which allows us to utilize [13, Corollary 32.3.4], can be easily achieved by defining $x = x_I - x_{II},\ x_I \geq 0,\ x_{II} \geq 0$ and $u = u_I - u_{II},\ u_I \geq 0,\ u_{II} \geq 0$. Hence, the bilinear program (2.7) with an objective function bounded below by zero, which is equivalent to a concave function minimization [1, Proposition 2.2], has a vertex solution on the polyhedral set $X \times U$. If for some ith iteration the bilinear objective function does not decrease, then each of the linear programs of steps (III) and (IV) of the algorithm must have returned (x^{i+1}, y^{i+1}) and $(u^{i+1}, v^{i+1}, w^{i+1})$ such that:

$${y^{i+1}}'(-u+v+w) - b'u \geq {y^{i+1}}'(-u^i+v^i+w^i) - bu^i = {y^{i+1}}'(-u^{i+1}+v^{i+1}+w^{i+1}) - bu^{i+1},\ \forall (u,v,w) \in U, \tag{3.14}$$

and

$$(-u^{i+1}+v^{i+1}+w^{i+1})'y \geq (-u^{i+1}+v^{i+1}+w^{i+1})'y^i = (-u^{i+1}+v^{i+1}+w^{i+1})'y^{i+1},\ \forall (x,y) \in X. \tag{3.15}$$

Combining the inequalities of (3.14) and (3.15) gives the minimum principle necessary optimality condition (3.11). Since there are a finite number of vertices of the set $X \times U$, and since each vertex visited by Algorithm 3.1 gives a lesser value for the bilinear objective than the previous vertex, no vertex is repeated. Thus our algorithm must terminate at either a global zero minimum solution or a point satisfying the minimum principle necessary optimality condition. □

We turn now to our computational results.

4 Computational Results

We implemented our algorithm by solving 500 solvable random instances of the absolute value equation (1.1) consecutively generated. Elements of the matrix A were random numbers picked from a uniform distribution in the interval $[-5, 5]$. A random solution x with random components from [-.5,.5] was generated and the right hand side b was computed as $b = Ax - |x|$. All computation was performed on 4 Gigabyte machine running i386 rhe15 Linux. We utilized the CPLEX linear programming code [7] within MATLAB [12] to solve our linear programs.

Of the 500 test problems, 88.6% were solved exactly to a tolerance set to $tol = 1e{-}6$. The maximum number of iterations was set at 40. The computational results are summarized in Table 1.

Problem Size n	Number of AVEs out of 100 with 2-norm error $\leq$ tol=1e-6	Time in Seconds for Solving 100 Equations
10	90	1.805
50	87	5.725
100	88	20.605
500	88	1,996.6
1,000	90	19,008

Table 1: Computational Results for 500 Randomly Generated Consecutive AVEs

5 Conclusion and Outlook

We have proposed a bilinear programming formulation for solving the NP-hard absolute value equation. The bilinear program was solved by a finite succession of linear programs. In 88.6% of 500 instances, for each solvable random test problem, the proposed algorithm solved the problem to an accuracy of $1e-6$. Possible future work may consist of precise sufficient conditions under which the proposed formulation and solution method is guaranteed to solve this NP-hard problem exactly.

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