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Absolute Value Programming

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Abstract. We investigate equations, inequalities and mathematical programs involving absolute values of variables such as the equation $Ax + B|x| = b$, where A and B are arbitrary $m \times n$ real matrices. We show that this absolute value equation is NP-hard to solve, and that solving it with $B = I$ solves the general linear complementarity problem. We give sufficient optimality conditions and duality results for absolute value programs as well as theorems of the alternative for absolute value inequalities. We also propose concave minimization formulations for absolute value equations that are solved by a finite succession of linear programs. These algorithms terminate at a local minimum that solves the absolute value equation in almost all solvable random problems tried.

Keywords: absolute value (AV) equations, AV algorithm, AV theorems of alternative, AV duality

1. Introduction

We consider problems involving absolute values of variables such as:

$$Ax + B|x| = b, \tag{1}$$

where $A \in R^{m\times n}$, $B \in R^{m\times n}$ and $b \in R^m$. As will be shown, the general linear complementarity problem [2, 3] which subsumes many mathematical programming problems can be formulated as an absolute value (AV) equation such as (1). Even though these problems involving absolute values are NP-hard, as will be shown in Section 2, they share some very interesting properties with those of linear systems. For example, in Section 4 we formulate optimization problems with AV constraints and give optimality and duality results similar to those of linear programming, even though the problems are inherently nonconvex.

Other results that the AV equation (1) shares with linear inequalities are theorems of the alternative that we give in 5. In Section 3 of the paper we propose a finite successive linearization algorithm for solving AV equations that terminates at a necessary optimality condition. This algorithm has solved all solvable random test problems given to it for which mostly $m \geq 2n$ or $n \geq 2m$, up to size (m, n) of $(2000, 100)$ and $(100, 2000)$. When $m = n$ and B is invertible, which is the case for the linear complementarity problem formulation, we give a simpler concave minimization formulation that is also solvable by a finite succession of linear programs. Problems with $m = n$ and n between 50 and 1000 were solved by this approach. Section 6 concludes the paper and poses some open questions.

This work is motivated in part by the recent interesting paper of Rohn [12] where a theorem of the alternative is given for a special case of (1) with square matrices A

and B, and a linear complementarity equivalent to (1) is also given. However, both of these results are somewhat unusual in that both alternatives are given in the primal space R^n of the variable x instead of the primal and dual spaces as is usual [5], while the linear complementarity equivalent to (1) is the following nonstandard complementarity problem:

$$x_+ = (A+B)^{-1}(A-B)(-x)_+ + (A+B)^{-1}b. \tag{2}$$

Here x_+ denotes suppression of negative components as defined below. In contrast, our result in Section 2 gives an explicit AV equation (1) in terms of the classical linear complementarity problem (3) below, while our theorems of the alternative of Section 5 are in the primal and dual spaces R^n and R^m.

We now describe our notation. All vectors will be column vectors unless transposed to a row vector by a prime $'$. The scalar (inner) product of two vectors x and y in the n-dimensional real space R^n is then $x'y$ and orthogonality $x'y = 0$ will be denoted by $x \perp y$. For $x \in R^n$, the 1-norm will be denoted by $\|x\|_1$ and the 2-norm by $\|x\|$, while $|x|$ will denote the vector with absolute values of each component of x and x_+ will denote the vector resulting from setting all negative components of x to zero. The notation $A \in R^{m\times n}$ will signify a real $m \times n$ matrix, with ith row A_i and transpose A'. A vector of zeros in a real space of arbitrary dimension will be denoted by 0, and the vector of ones is e. The notation $\underset{x\in X}{\arg\min}\, f(x)$ denotes the solution set of $\min_{x\in X} f(x)$.

2. Problem Sources and Hardness

We will first show how to reduce any linear complementarity problem (LCP) to the AV equation (1):

$$\text{(LCP)} \qquad 0 \le z \;\perp\; Mz+q \ge 0, \tag{3}$$

where $M \in R^{n\times n}$ and $q \in R^n$. Then we show that solving (1) is NP-hard.

Proposition 1 **LCP as an AVE** *Given the LCP (3), without loss of generality, rescale M and q by multiplying by a positive constant if needed, so that no eigenvalue of the rescaled M is 1 and hence $(I-M)$ is nonsingular. The rescaled LCP can be solved by solving the following AV equation and computing z as indicated:*

$$\begin{aligned} (I+M)(I-M)^{-1}x - |x| &= -((I+M)(I-M)^{-1}+I)q, \\ z &= (I-M)^{-1}(x+q). \end{aligned} \tag{4}$$

Proof: From the simple fact that for any two real numbers a and b:

$$a+b = |a-b| \iff a \ge 0,\ b \ge 0,\ ab = 0, \tag{5}$$

it folows that the LCP (3) is equivalent to:

$$z + Mz + q = |z - Mz - q|. \tag{6}$$

Defining x from (4) as $x = (I - M)z - q$, equation (6) becomes:

$$(I + M)(I - M)^{-1}(x + q) + q = |x|, \tag{7}$$

which is the AV equation of (4). Hence, solving the AV equation (4) gives us a solution of the LCP (3) through the definition z of (4). ■

Conversely it can also be shown, under the somewhat strong assumption that B is invertible, and by rescaling so is $(I + B^{-1}A)$, that the AV equation (1) can be solved by solving an LCP (3). We will not give details of that reduction here because it is not essential to the present work but we merely state the LCP to be solved in terms of A, B and b of the AV equation (1):

$$0 \le z \ \perp \ (I - 2(I + B^{-1}A)^{-1})z + (I + B^{-1}A)^{-1}B^{-1}b \ge 0. \tag{8}$$

A solution x to the AV equation (1) can be computed from a solution z of (8) as follows:

$$x = (I + B^{-1}A)^{-1}(-2z + B^{-1}b). \tag{9}$$

There are other linear complementarity formulations of the AV equation (1) which we shall not go into here.

We now show that solving the AV equation (1) is NP-hard by reducing the NP-hard knapsack feasibility problem to an AV equation (1).

PROPOSITION 2 *Solving the AV equation (1) is NP-hard.*

Proof: The NP-hard knapsack feasibility problem consists of finding an n-dimensional binary variable z such that $a'z = d$ where a is an n-dimensional integer vector and d is an integer. It has been shown [1, 6] that this problem is equivalent to the LCP (3) with the following values of M and q:

$$M = \begin{bmatrix} -I & 0 & 0 \\ e' & -n & 0 \\ -e' & 0 & -n \end{bmatrix}, \ q = \begin{bmatrix} a \\ -b \\ b \end{bmatrix}. \tag{10}$$

Since eigenvalues of M are $\{-n, -n, -1, \ldots, -1\}$, Proposition 1 applies, without rescaling, and the NP-hard knapsack problem can be reduced to the AV equation (4) which is a special case of (1). Hence solving (1) is NP-hard. ■

Other sources for AV equations are interval linear equations [11] as well as constraints in AV programming problems where absolute values appear in the constraints and the objective function which we discuss in Section 4. Note that as a consequence of Proposition 2, AV programs are NP-hard as well.

We turn now to a method for solving AV equations.

3. Successive Linearization Algorithm via Concave Minimization

We propose here a method for solving AV equations based on a minimization of a concave function on a polyhedral set by solving a finite succession of linear programs. We shall give two distinct algorithms based on this method, one for $m \neq n$

and a simpler one for $m = n$. Before stating the concave minimization problem we prove a lemma which does not seem to have been given before and which extends the linear programming perturbation results of [8] to nondifferentiable concave perturbations.

LEMMA 1 *Consider the concave minimization problem:*

$$\min_{z \in Z} d'z + \epsilon f(z),\ Z = \{z \mid Hz \leq h\} \neq \emptyset, \tag{11}$$

where $d \in R^\ell$, $f(z)$ is a concave function on R^ℓ, $H \in R^{k \times \ell}$ and $h \in R^k$. Suppose that $d'z + \epsilon f(z)$ is bounded below on Z for $\epsilon \in (0, \tilde{\epsilon}]$ for some $\tilde{\epsilon} > 0$. Suppose also that Z contains no lines going to infinity in both directions. Then there exists an $\bar{\epsilon} \leq \tilde{\epsilon}$ such that for all $\epsilon \in (0, \bar{\epsilon}]$ the concave minimization problem (11) is solvable by a vertex solution $\bar{z}$ that minimizes $f(z)$ over the nonempty solution set $\bar{Z}$ of the linear program $\min\limits_{z \in Z} d'z$.

Proof: By [10, Corollary 32.3.4] the concave minimization problem (11) has a vertex solution for each $\epsilon \in (0, \tilde{\epsilon}]$. Since Z has a finite number of vertices, we can construct a decreasing sequence $\{\epsilon_i \downarrow 0\}_{i=0}^{i=\infty}$ such that only the same vertex $\bar{z}$ of Z will occur as the solution of (11) for $\{\epsilon_i \downarrow 0\}_{i=0}^{i=\infty}$. Since for $i \neq j$:

$$\begin{aligned} d'\bar{z} + \epsilon_i f(\bar{z}) &\leq d'z + \epsilon_i f(z),\ \forall z \in Z, \\ d'\bar{z} + \epsilon_j f(\bar{z}) &\leq d'z + \epsilon_j f(z),\ \forall z \in Z, \end{aligned}$$

it follows that for $\lambda \in [0, 1]$:

$$d'\bar{z} + ((1-\lambda)\epsilon_i + \lambda\epsilon_j) f(\bar{z}) \leq d'z + ((1-\lambda)\epsilon_i + \lambda\epsilon_j) f(z),\ \forall z \in Z.$$

Hence:

$$d'\bar{z} + \epsilon f(\bar{z}) \leq d'z + \epsilon f(z),\ \forall z \in Z,\ \forall \epsilon \in (0, \bar{\epsilon} = \epsilon_0].$$

Letting $\epsilon \downarrow 0$ gives:

$$d'\bar{z} \leq d'z,\ \forall z \in Z.$$

Hence the solution set $\bar{Z}$ of $\min\limits_{z \in Z} d'z$ is nonempty and for $\epsilon \in (0, \bar{\epsilon} = \epsilon_0]$:

$$\epsilon f(\bar{z}) \leq (d'z - d'\bar{z}) + \epsilon f(z) = \epsilon f(z),\ \forall z \in \bar{Z} \subset Z.$$

■

We formulate now the concave minimization problem that solves the AV equation (1).

PROPOSITION 3 **AV Equation as Concave Minimization** *Consider the concave minimization problem:*

$$\begin{aligned} \min_{(x,t,s) \in R^{n+n+m}} \quad & \epsilon(-e'|x| + e't) + e's \\ \text{such that} \quad & -s \leq Ax + Bt - b \leq s, \\ & -t \leq x \leq t. \end{aligned} \tag{12}$$

Without loss of generality, the feasible region is assumed not to have straight lines going to ∞ in both directions. For each $\epsilon > 0$, the concave minimization problem (12) has a vertex solution. For any $\epsilon \in (0, \bar{\epsilon}]$ for some $\bar{\epsilon} > 0$, any solution $(\bar{x}, \bar{t}, \bar{s})$ of (12) satisfies:

$$\begin{aligned} (\bar{x}, \bar{t}) \in \overline{XT} &= arg \min_{(x,t)\in R^{n+n}} \|Ax + Bt - b\|_1, \\ \|\bar{t}\|_1 - \|\bar{x}\|_1 &= \min_{(x,t)\in \overline{XT}} \|t\|_1 - \|x\|_1. \end{aligned} \tag{13}$$

In particular, if the AV equation (1) is solvable then:

$$\begin{aligned} |\bar{x}| &= \bar{t}, \\ A\bar{x} + B|\bar{x}| &= b. \end{aligned} \tag{14}$$

Proof: Note that the feasible region of (12) is nonempty and its concave objective function is bounded below by zero, since $-e'|x| + e't \geq 0$ follows from the second constraint $|x| \leq t$. Hence by [10, Corollary 32.3.4] the concave minimization problem (12) has a vertex solution provided it has no lines going to infinity in both directions. The latter fact can be easily ensured by making the standard transformation $x = x^1 - x^2$, $(x^1, x^2) \geq 0$, since $(s, t) \geq 0$ follow from the problem constraints. For simplicity we shall not make this transformation and assume that (12) has a vertex solution. (That this transformation is not necessary is borne out by our numerical tests.) The results of (13) follow from Lemma 1, while those of (14) hold because the minimum in (13) is zero and $|\bar{x}| = \bar{t}$ renders the perturbation $(-e'|x| + e't)$ in the minimization problem (12) a minimum. ■

We state now a successive linearization algorithm for solving (12) that terminates at a point satisfying a necessary optimality condition. Before doing that we define a supergradient of the concave function $-e'|x|$ as:

$$\partial(-e'|x|)_i = -sign(x_i) = \left\{ \begin{array}{rl} 1 & if\ x_i < 0 \\ 0 & if\ x_i = 0 \\ -1 & if\ x_i > 0 \end{array} \right\},\ i = 1, \ldots, n. \tag{15}$$

Algorithm 1 Successive Linearization Algorithm (SLA1) for (1) *Let $z = \begin{bmatrix} x & t & s \end{bmatrix}'$. Denote the feasible region of (12) by Z and its objective function by $f(z)$. Start with a random $z^0 \in R^{n+n+m}$. From z^i determine z^{i+1} as follows:*

$$z^{i+1} \in\ arg\ vertex \min_{z\in Z} \partial f(z^i)'(z - z^i). \tag{16}$$

Stop if $z^i \in Z$ and $\partial f(z^i)'(z^{i+1} - z^i) = 0$.

We now state a finite termination result for the SLA1 algorithm.

Proposition 4 **SLA1 Finite Termination** *The SLA1 Algorithm 1 generates a finite sequence of feasible iterates $\{z^1, z^2, \ldots, z^{\bar{i}}\}$ of strictly decreasing objective* *function values:* $f(z^1) > f(z^2) > \ldots > f(z^{\bar{i}})$*, such that* $z^{\bar{i}}$ *satisfies the minimum principle necessary optimality condition:*

$$\partial f(z^{\bar{i}})'(z - z^{\bar{i}}) \geq 0, \ \forall z \in Z. \tag{17}$$

Proof: See [7, Theorem 3]. ■

Preliminary numerical tests on randomly generated solvable AV equations were encouraging and are depicted in Table 1 for 100 runs of Algorithm 1 for cases with $m \neq n$. All runs were carried out on a Pentium 4 3Ghz machine with 1GB RAM. The CPLEX linear programming package [4] was used within a MATLAB [9] code.

Since Algorithm 1 could not effectively solve the case $m = n$, we developed a simplified concave minimization algorithm for that case when B is invertible. This case is still NP-hard and covers the linear complementarity problem as shown in Proposition 1. The AV equation (1) degenerates to the following when $m = n$ and B^{-1} exists:

$$|x| = Gx + g, \tag{18}$$

where $G = -B^{-1}A$ and $g = B^{-1}b$. We note that since:

$$|x| \leq Gx + g \iff -Gx - g \leq x \leq Gx + g, \tag{19}$$

we immediately have that the following concave minimization problem will solve (18) if it has a solution:

$$\begin{array}{rl} \displaystyle\min_{x \in R^n} & e'Gx - e'|x| \\ \text{such that } (-I - G)x & \leq g, \\ (I - G)x & \leq g. \end{array} \tag{20}$$

The successive linearization Algorithm 1 and its finite termination is applicable to (20) and leads to the following simple linear program at iteration i:

$$\begin{array}{rl} \displaystyle\min_{x \in R^n} & (e'G - sign(x^i)')(x - x^i) \\ \text{such that } (-I - G)x & \leq g, \\ (I - G)x & \leq g. \end{array} \tag{21}$$

We shall assume that $\{x \mid Gx + g \geq 0\} \neq \emptyset$, which is the case if (18) is solvable. We also note that the concave objective function of (20) is bounded below on the feasible region of (20) by $-e'g$ on account of (19). Hence the supporting plane of that function at x^i: $e'Gx^i - e'|x^i| + (e'G - sign(x^i)')(x - x^i)$ is also bounded below by $-e'g$ on the feasible region of (21). Consequently, the objective function of (21) is bounded below by $-e'g - e'Gx^i + e'|x^i|$ on its feasible region and thus (21) is solvable. We state now the algorithm for solving (18).

Algorithm 2 Successive Linearization Algorithm (SLA2) for (18) *Let* $\{x \mid Gx + g \geq 0\} \geq \emptyset$*. Start with a random* $x^0 \in R^n$*. From* x^i *determine* x^{i+1} *by the solvable linear program (21). Stop if* x^i *is feasible and and* $(e'G - sign(x^i)')(x^{i+1} - x^i) = 0$*.*

Table 1. **Results of 100 runs of Algorithm 1 for the AV equation (1). Each case is the average of ten random runs for that case. The error is $\|Ax + B|x| - b\|$ and $\|t - |x|\|$ refers to problem (12).**

m	n	No. Iter. $\bar{i}$	Time Sec.	Error	$\|t - |x|\|$
50	100	2.40	0.58	0	0
100	50	1.0	0.18	0	0
150	250	3.5	13.47	0	0
250	150	1.0	3.33	0	0
100	1500	1.7	99.82	0	0
1500	100	1.0	106.35	0	0
100	2000	1.5	233.82	0	0
2000	100	1.0	236.44	0	0
300	500	3.3	109.60	0	0
500	300	1.0	26.78	0	0

Table 2. **Results of 100 runs of Algorithm 2 for the AV equation (18). Each case is the average of ten random runs for that case. The error is $\|Gx - |x| + g\|$. NNZ(Error) is the average number of nonzero $\|Gx - |x| + g\|$ over ten runs.**

n	No. Iter. $\bar{i}$	Time Sec.	Error	$NNZ(Error)$
50	1.1	0.018	0	0
50	2.7	0.037	0.0861	0.1
100	2.1	0.14	0	0
100	2.8	0.19	0.0696	0.1
200	1.2	0.58	0	0
200	2.3	1.10	0.280	0.1
500	1.9	11.84	0	0
500	2.5	15.89	0.00711	0.1
1000	1.7	75.94	0	0
1000	2.0	89.92	0.00487	0.1

Table 2 gives results for Algorithm 2 for a hundred randomly generated solvable problems (18) with n between 50 and 1000. Of these 100 problems only 5 were not solved by this algorithm.

We turn now to duality and optimality results for AV optimization.

4. Duality and Optimality for Absolute Value Programs

We derive in this section a weak duality theorem and sufficient optimality conditions for AV programs. These results are somewhat curious in the sense that they hold for nonconvex problems. However, we note that we are missing strong duality and necessary optimality conditions for these AV programs. We begin by defining our primal and dual AV programs as follows.

$$\underline{\text{Primal AVP}} \qquad \min_{x\in X} c'x + d'|x|, \; X = \{x \in R^n \;\mid\; Ax + B|x| = b, \; Hx + K|x| \geq p\}. \tag{22}$$

$$\underline{\text{Dual AVP}} \qquad \max_{(u,v)\in U} b'u + p'v, \; U = \{(u,v) \in R^{m+k} \;\mid\; |A'u + H'v - c| + B'u + K'v \leq d, \; v \geq 0\}. \tag{23}$$

Proposition 5 **Weak Duality Theorem**

$$x \in X, \; (u,v) \in U \Longrightarrow c'x + d'|x| \geq b'u + p'v \tag{24}$$

Proof:

$$\begin{array}{rcl} d'|x| & \geq & |x'| \cdot |A'u + H'v - c| + |x'|(B'u + K'v) \\ & \geq & u'Ax + v'Hx - c'x + u'B|x| + v'K|x| \\ & \geq & b'u + p'v - c'x. \end{array}$$

. ■

Our next result gives sufficient conditions for primal and dual optimality.

Proposition 6 **Sufficient Optimality Conditions** *Let $\bar{x}$ be feasible for the primal AVP (22) and $(\bar{u}, \bar{v})$ be feasible for the dual AVP (23) with equal primal and dual objective functions, that is:*

$$c'\bar{x} + d'|\bar{x}| = b'\bar{u} + p'\bar{v}. \tag{25}$$

Then $\bar{x}$ is primal optimal and $(\bar{u}, \bar{v})$ is dual optimal.

Proof: Let $x \in X$. Then:

$$\begin{array}{rl} c'x + d'|x| & -c'\bar{x} - d'|\bar{x}| \geq \\ & c'x + |x'| \cdot |A'\bar{u} + H'\bar{v} - c| + |x'|(B'\bar{u} + K'\bar{v}) - c'\bar{x} - d'|\bar{x}| \geq \\ & c'x + \bar{u}'Ax + \bar{v}'Hx - c'x + \bar{u}'B|x| + \bar{v}'K|x| - c'\bar{x} - d'|\bar{x}| \geq \\ & b'\bar{u} + p'\bar{v} - c'\bar{x} - d'|\bar{x}| = 0 \end{array}$$

Hence $\bar{x}$ is primal optimal.

Now let $(u, v) \in U$. Then:

$$\begin{array}{ll} b'\bar{u} + p'\bar{v} & -b'u - p'v \geq \\ & c'\bar{x} + d'|\bar{x}| - u'A\bar{x} - u'B|\bar{x}| - v'H\bar{x} - v'K|\bar{x}| \geq \\ & c'\bar{x} + |\bar{x}'| \cdot |A'u + H'v - c| + |\bar{x}'|(B'u + K'v) - u'A\bar{x} - u'B|\bar{x}| - v'H\bar{x} - v'K|\bar{x}| \geq \\ & c'\bar{x} + u'A\bar{x} + v'H\bar{x} - c'\bar{x} + u'B|\bar{x}| + v'K|\bar{x}| - u'A\bar{x} - u'B|\bar{x}| - v'H\bar{x} - v'K|\bar{x}| = 0. \end{array}$$

Hence $(\bar{u}, \bar{v})$ is dual optimal. ■

We turn now to our final results of theorems of the alternative.

5. Theorems of the Alternative for Absolute Value Equations and Inequalities

We shall give two propositions of the alternative here. The first one applies under no assumptions and the second one under one assumption.

PROPOSITION 7 **First AV Theorem of the Alternative** *Exactly one of the two following alternatives must hold:*

I. $Ax + Bt = b,\ Hx + Kt \geq p,\ t \geq |x|$ has solution $(x, t) \in R^{n+n}$.

II. $|A'u + H'v| + B'u + K'v \leq 0,\ b'u + p'v > 0\ v \geq 0$ has solution $(u, v) \in R^{m+k}$.

Proof: Let $\bar{I}$ and $\overline{II}$ denote negations of I and II respectively.

$(I \Longrightarrow \overline{II})$ If both I and II hold then we have the contradiction:

$$\begin{array}{l} 0 \geq t'|A'u + H'v| + t'B'u + t'K'v \geq \\ |x'| \cdot |A'u + H'v| + t'B'u + t'K'v \geq \\ u'Ax + v'Hx + u'Bt + v'Kt \geq b'u + v'p > 0. \end{array}$$

$(I \Longleftarrow \overline{II})$ If II does not hold then:

$$0 = \max_{u,v} \{b'u + p'v \ \mid\ |A'u + H'v| + B'u + K'v \leq 0,\ v \geq 0\}.$$

This implies that:

$$0 = \max_{u,v,s} \{b'u + p'v \mid s + B'u + K'v \leq 0,\ |A'u + H'v| \leq s,\ v \geq 0\}.$$

This implication is true because when (u, v, s) is feasible for the last problem then (u, v) is feasible for the previous problem. By linear programming duality after replacing $|A'u + H'v| \leq s$ by the equivalent constraint $-s \leq A'u + H'v \leq s$, we have that:

$$0 = \min_{t,y,z} \{0 \ \mid\ t - y - z = 0,\ Bt + A(y - z) = b,\ Kt + H(y - z) \geq p,\ (t, y, z) \geq 0\}.$$

Setting $x = y - z$ and noting that $t = y + z \geq \pm(y - z) = \pm x$ we have that feasibility of the last problem implies that the following system has a solution:

$$Ax + Bt = b,\ Hx + Kt \geq p,\ t \geq |x|,$$

which is exactly I. ■

We state and prove now our last proposition.

Proposition 8 **Second AV Theorem of the Alternative** *Exactly one of the two following alternatives must hold:*

I. $Ax + B|x| = b,\ Hx + K|x| \geq p$ has solution $x \in R^n$,

II. $|A'u + H'v| + B'u + K'v \leq 0,\ b'u + p'v > 0,\ v \geq 0$ has solution $(u, v) \in R^{m+k}$,

under the assumption that:

$$0 = \max_{x,t} \{e'|x| - e't \mid Ax + Bt = b,\ Hx + Kt \geq p,\ t \geq |x|\}. \tag{26}$$

Note The assumption (26) is needed only in establishing that $I \Longleftarrow \overline{II}$.
Proof:
($I \Longrightarrow \overline{II}$) If both I and II hold then we have the contradiction:

$$0 < b'u + p'v \leq u'Ax + u'B|x| + v'Hx + v'K|x| \leq$$
$$|x'|(|A'u + H'v| + B'u + K'v) \leq 0.$$

($I \Longleftarrow \overline{II}$) By Proposition 7 above we have that $\overline{II}$ implies I of Proposition 7, which in turn implies I of this proposition under the assumption (26). ■

We conclude with some remarks and some possible future extensions.

6. Conclusion and Outlook

We have investigated equations, inequalities and mathematical programs involving absolute values of variables. It is very interesting that these inherently nonconvex and difficult problems are amenable to duality results and theorems of the alternative that are typically associated with linear programs and linear inequalities. Furthermore, successive linearization algorithms appear to be effective in solving absolute value equations for which m and n are substantially different or when $m = n$. Even though we give sufficient optimality conditions for our nonconvex absolute value programs, we are missing necessary optimality conditions. This is somewhat in the spirit of the sufficient saddlepoint optimality condition of mathematical programming [5, Theorem 5.3.1] which holds without any convexity. However, necessity of a saddlepoint condition requires convexity as well as a constraint qualification [5, Theorem 5.4.7], neither of which we have here.

An interesting topic of future research might be one that deals with absolute value programs for which necessary optimality conditions can be obtained as well as strong duality results and globally convergent algorithms. Also interesting would be further classes of problems that can be cast as absolute value equations, inequalities or optimization problems that can be solved by the proposed algorithms here or their variants.

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