Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$

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November 1994
Revised: August 1996

ABSTRACT

We study Weyl-Heisenberg (=Gabor) expansions for either $L_2(\mathbb{R}^d)$ or a subspace of it. These are expansions in terms of the spanning set

$$X = (E^k M^l \varphi : k \in K,\ l \in L, \varphi \in \Phi),$$

where K and L are some discrete lattices in $\mathbb{R}^d$, $\Phi \subset L_2(\mathbb{R}^d)$ is finite, E is the translation operator, and M is the modulation operator. Such sets X are known as *WH systems*. The analysis of the "basis" properties of WH systems (e.g. being a *frame* or a *Riesz basis*) is our central topic, with the fiberization-decomposition techniques of shift-invariant systems, developed in a previous paper of us, being the main tool.

Of particular interest is the notion of the *adjoint* of a WH set, and the *duality principle* which characterizes a WH (tight) frame in term of the stability (orthonormality) of its adjoint. The actions of passing to the adjoint and passing to the dual system commute, hence the dual WH frame can be computed via the dual basis of the adjoint.

Estimates for the underlying frame/basis bounds are obtained by two different methods. The *Gramian analysis* applies to all WH systems, albeit provides estimates that might be quite crude. This approach is invoked to show how, under only mild conditions on X, a frame can be obtained by oversampling a Bessel set. Finally, finer estimates of the frame bounds, based on the Zak transform, are obtained for a large collection of WH systems.

AMS (MOS) Subject Classifications: Primary 42C15, Secondary 42C30

Key Words: Riesz bases, stable bases, Gabor expansions, Weyl-Heisenberg systems, frames, dual frames, Bessel sequences, Zak transform.

This work was partially sponsored by by the National Science Foundation under Grants DMS-9000053, DMS-9102857, DMS-9224748, and by the United States Army Research Office under Contracts DAAL03-G-90-0090, DAAH04-95-1-0089.

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1. Introduction

1.1. Frames, Riesz bases, and their dual systems

The present paper is the second in a series of three, all devoted to the study of *shift-invariant frames* and *shift-invariant stable (=Riesz) bases* for $H := L_2(\mathbb{R}^d)$, $d \geq 1$, or a subspace of it. In the first paper, [RS1], we studied such bases under the mere assumption that the basis set can be written as a collection of **shifts** (namely, integer translates) of a set of *generators* Φ. The present paper analyses Weyl-Heisenberg (=:WH, known also as Gaborian) frames and stable bases. Aside from specializing the general methods and results of [RS1] to this important case, we exploit here the special structure of the WH set, and in particular the duality between the shift operator and the modulation operator, the latter being absent in the context of general shift-invariant sets. In the third paper, [RS3], we present applications of the results of [RS1] to wavelet (or affine) frames. The flavour of the results there is quite different: wavelet sets are not shift-invariant, and the main effort of [RS3] is to show that, nevertheless, the basic analysis of [RS1] does apply to that case as well.

Let $X \subset L_2(\mathbb{R}^d)$. We consider X as a possible "basis" set for $L_2(\mathbb{R}^d)$, or for some closed subspace of it. The various notions of a "basis set" are conveniently defined with the aid of the so-called **synthesis operator** or **reconstruction operator** $T := T_X$ defined by

$$T_X : \ell_0(X) \to L_2(\mathbb{R}^d) : \; c \mapsto \sum_{x \in X} c(x)x.$$

Here, $\ell_0(X)$ is the collection of all finitely supported sequences in $\ell_2(X)$. If T is bounded, it is extended by continuity to all of $\ell_2(X)$. We use the notation T for this extension, as well.

Definition 1.1. X is a **basis** whenever T is 1-1 on its domain. X is **fundamental** if $\operatorname{ran} T$ is dense in $L_2(\mathbb{R}^d)$. If T is bounded, X is **a Bessel set**. If T is bounded and $\operatorname{ran} T$ is closed, X is a **frame**. Finally, a frame which is also a basis is known as a **Riesz** (or **stable**) basis.

Remark. Some of the articles that deal with frames for $L_2(\mathbb{R})$, reserve the notions of "frame" and "Riesz basis" only to the case that we refer to here as a "fundamental frame" and "fundamental Riesz basis".

Note that, if X is a Riesz basis, T has a bounded inverse which acts from $\operatorname{ran} T$ onto $\ell_2(X)$. We denote that inverse by T^{-1}. If X is not a Riesz basis, but is still a frame, a bounded inverse still exists but acts from $\operatorname{ran} T$ onto $(\ker T)^{\perp}$. This pseudo-inverse is denoted hereafter by $T_{|}^{-1}$, and is referred to as the **partial inverse** of T.

Another way to define the above "basis" notions, is with the aid of the **analysis operator** or the **decomposition operator**, which is the formal adjoint $T^* := T_X^*$ of T, and is defined by

$$T^* : L_2(\mathbb{R}^d) \to \ell_2(X) : f \mapsto (\langle f, x \rangle)_{x \in X}.$$

The equivalent definitions via the adjoint map are entirely analogous: the definitions of frames and Bessel sets remain unchanged (just replace T by T^*). The fundamentality of X amounts to the injectivity of T^*, and a Riesz basis is a frame whose corresponding T^* is surjective.

Since usually the injectivity of a map is easier to check than its surjectivity, fundamental frames are usually studied via T^*, while Riesz bases are usually analyzed via T. That tradition, however, will not be followed in the present paper, since our techniques are critically based on a simultaneous analysis of both the decomposition and reconstruction operators.

In order to decompose and reconstruct functions in $L_2(\mathbb{R}^d)$, one needs to complement the given frame X by another one, the so-called **dual frame** which is defined as

$$\mathrm{R}X,$$

with

$$\mathrm{R} := \mathrm{R}_X := T_|^{*\,-1} T_|^{\,-1}.$$

The map

$$f \mapsto T_X T_{\mathrm{R}X}^* f = \sum_{x \in X} \langle f, \mathrm{R}x \rangle\, x$$

is then an orthogonal projector onto $\operatorname{ran} T_X$, hence is the identity on $L_2(\mathbb{R}^d)$ in case the frame X is fundamental. In this regard, computing $T_{\mathrm{R}X}^* f$ is the **decomposition** of f, and computing $T_X T_{\mathrm{R}X}^* f$ (from the given sequence $T_{\mathrm{R}X}^* f$) is the **reconstruction** of f. The operator norms $\|T_X\|^2$ and $\|T_{X|}{}^{-1}\|^{-2}$ are known as the **frame bounds** of X. A frame whose two frame bounds coincide is a **tight frame**, and it is well-known that the dual frame of a tight frame is (up to a multiplicative constant) the original frame X (i.e., $\mathrm{R} = c\,\mathrm{Id}$ on $\operatorname{ran} T$).

Frames were introduced in [DS] in the context of non-harmonic Fourier series. Frames for $L_2(\mathbb{R})$ were quite extensively studied in the literature, with the focus being on two special choices: *wavelet* (or affine) *frames*, and *Weyl-Heisenberg* (or short-windowed Fourier transform) frames. We refer to the surveys [BW], [HW] and [D1], the books [C] and [D2], and to the references therein for discussions of frames, and in particular wavelet and Weyl-Heisenberg frames.

1.2. Weyl-Heisenberg systems defined

A Weyl-Heisenberg (WH) system is defined here with respect to a pair of lattices. Here, a **lattice** (or, more precisely, a d-dimensional lattice) K is the image $A_K\mathbb{Z}^d$ of a linear invertible map $A_K : \mathbb{R}^d \to \mathbb{R}^d$. The column vectors of A_K **generate** K. The **volume** of the lattice

$$|K| := |\det A_K|$$

measures the sparsity of it, and depends only on K. The **dual lattice** $\widetilde{K}$ of K is the lattice defined by

$$\widetilde{K} := \{l \in \mathbb{R}^d : \ l \cdot k \in 2\pi\mathbb{Z}, \ \forall k \in K\}.$$

Note that we always have

$$|K|\,|\widetilde{K}| = (2\pi)^d. \tag{1.2}$$

Given a d-dimensional lattice K, we let

$$\Omega_K$$

stand for any fundamental domain for K, i.e., a set for which $\cup_{k\in K}(k+\Omega_K)$ is an (essential) partition of $\mathbb{R}^d$. A standard choice for Ω_K is the parallelepiped $A_K[0,1]^d$.

We now turn to the definition of a WH system. First, for $t \in \mathbb{R}^d$, let E^t denote the translation operator and M^t the modulation operator, i.e.,

$$E^t : f \mapsto f(\cdot + t), \quad M^t : f \mapsto e_t f, \quad t \in \mathbb{R}^d,\ f \in L_2(\mathbb{R}^d),$$

with e_t the exponential function

$$e_t : w \mapsto e^{it\cdot w}.$$

Further, let $\Phi \subset L_2(\mathbb{R}^d)$ be finite (though our analysis applies to infinite Φ as well, it suffices for all practical purposes to assume that Φ is finite). Finally, let K, L be two lattices in $\mathbb{R}^d$. Then, we call the set

$$X = (K,L)_\Phi := \{E^k M^l \varphi :\ \varphi \in \Phi,\ k \in K,\ l \in L\}$$

a **Weyl-Heisenberg system** generated by Φ. The WH system is **normalized** whenever $\|\varphi\| = 1$, for all $\varphi \in \Phi$. Also, whenever Φ is a singleton, we refer to X as a **principal Weyl-Heisenberg system** (PWH system, for short). The number

$$\frac{(2\pi)^d}{|K||L|}$$

is the **density parameter** $\operatorname{den}(K,L)$ of (K,L), and is also referred to as the density

$$\operatorname{den} X$$

of a PWH system X. We call a PWH system X a **high-density** system if $\operatorname{den} X \geq 1$ and a **low density** system if $\operatorname{den} X < 1$.

1.3. Layout of the paper

Since the theory introduced in this paper is fairly elaborated, we had made a special effort to present it in several "layers of difficulty", allowing the interested reader to adjust his/her level of reading to the level of interest in this topic. Specifically, we present in §2 relatively concise account of the development of the paper, including an array of selected results, while detailing more elaborated findings in §3 and §4.

Firstly, we introduce, in §2.1, the notion of the *adjoint* X^* of a PWH system X. Our *duality principle of WH sets* (Theorems 2.2 and 2.3) deals with the basic relations between a system, its adjoint, and their dual systems. In general terms, the duality principle exhibits an intimate relation between T_{X^*} and T_X^* (hence also between T_X and $T_{X^*}^*$, since $X^{**} = X$), which "almost" says that $T_{X^*} = c\,U_1 T_X^* U_2$, U_1, U_2 unitary, and is almost as useful as such connection. We present in §2.1 few applications of the duality principle, and a variety of other applications is collected in §3.1. Some of these applications generalize known univariate observations, and others are new even for univariate systems. As an example for the former, a fairly trivial argument is invoked to show (Corollary 2.7) that low-density frames are never fundamental.

The duality principle is a straightforward consequence of the fiberization of the WH system. In §2.2 we describe the three Gramian matrices that underly our fiberization techniques and recall the main results of [RS1]. That already allows us to provide a simple and complete proof of the duality principle. Furthermore, it shows how the Wexler-Raz identity, [DLL], can be equivalently formulated (thereby proved) as a matrix identity. A more elaborated discussion of the duality principle, including direct relations between the operators of a system and the operators of its adjoint, are given in §3.2.

In §2.4 and §3.4 we present our "Gramian analysis": analysis of the system that is based on inspection of the relevant Gramian matrices. In §2.4, the notion of *oversampling* is discussed, i.e., the possibility of converting a Bessel system into a frame by increasing the density of either the shifts or the modulations. In §3.4 we use Gramian analysis in estimating the frame bounds of a given system.

For a large family of WH systems, the Gramian matrices can be observed to be convolution operators, or, more generally, matrix-valued convolution operators. In such cases, a more subtle analysis, based on the symbols of the underlying convolution operators, is feasible. These symbols turn out to be multivariate analogs of the Zak transform, hence we refer to that course of study as "Zak transform analysis". In §2.4, we present the relatively simple case of a *sup-adjoint* system (which amounts in one dimension to a system whose density in an integer number). In this case the Gramian matrices are realized as a *vector* of convolution operators. In §4 we discuss the more general case of *compressible systems* (which, in one dimension, are systems whose density is a rational number). For these systems, the Gramian matrices are realized as arrays (of finite order) of convolution operators.

It should be understood that the goal of this paper is the application of the fiberization techniques of [RS1] to WH systems. This means that we consistently strive to explain the various phenomena of the WH systems via the Gramian matrices, even when a direct course is available as well (such as in the case of Zak transform analysis).

1.4. Acknowledgments and general remarks

Our original interest in shift-invariant bases stems from the role of such systems in Approximation Theory (e.g., Box Splines). There, non-fundamental bases are the rule rather than the exception, and this explains our genuine interest in and emphasis on non-fundamental sets. On the other hand, *frames* have hardly been considered in Approximation Theory as an object of interest, and, in fact, our initial development of the frame material in [RS1-3] was done "from scratch". While this somewhat cavalier approach might have had its own advantages, it also, inevitably, resulted in the re-invention of known and even classical results (the Zak transform and the [DGM] painless construction of WH frames were among our early "innovations"). Communications we had in late 1992 with Chris Heil had helped us in drawing connections between our work and the rich frame literature.

Our first presentation of the duality principle (in Oberwolfach, Summer 1993) had led to several very useful discussions with Hans Feichtinger and with some of his Vienna group people. In particular, numerical experiments conducted by Sigang Qiu had helped us in correcting the constant that appears in Theorem 2.3.

Selected results from §2 (such as Theorems 2.2 and 2.3) and §4 are announced in [RS2]. After receiving a copy of [RS2], Meir Zibulski from the Technion, Israel, had brought to our attention the articles [TO1,2] and [ZZ]. The former establishes the univariate equivalent of part (f) of Theorem 2.2, while the latter derives the univariate equivalents of part (a,b) of Theorem 4.5, as well as of some other results from §4.

About the time we were essentially done with the present endeavour, Ingrid Daubechies had brought to our attention Janssen's paper [J] and her joint paper with the Landaus [DLL]. Both papers deal with the same phenomenon that we describe in §2 here. Specifically, both contain statements equivalent to our Theorem 2.3, but under the additional assumptions that the underlying frame is univariate and fundamental. Both also contain results equivalent to the univariate case of parts (a,e) of our Theorem 2.2, including the connections between the frame bounds asserted in that theorem. It is probably correct to consider the three articles [J], [DLL], and ours as "simultaneous and independent", and it is worth mentioning that the techniques employed in these papers are quite different: [J] amplifies the approach of Tolimieri and Orr, [DLL] invokes what they call "the Wexler-Raz identity", while our development follows the fiberization techniques of [RS1]. We decided to abstain from expanding our paper in directions that may be suggested from the reading of [J] and [DLL], with the following single exception: in §3.3, we show how the Wexler-Raz identity can be observed by using our decomposition-fiberization techniques.

We would like to extend our thanks to all the people whose contributions are detailed above.

A final remark: our results are always derived in a multivariate setup, and deal with systems which are not necessarily fundamental in $L_2(\mathbb{R}^d)$. It is probably true that no significant simplification of arguments would have occurred, had we chosen to restrict attention to univariate systems. In contrast, the treatment of non-fundamental systems seems to be harder than their fundamental counterparts, at least from the standpoint of the tools we borrowed from [RS1] for either case.

2. Fiberization of WH systems and selected applications

2.1. The duality principle

We start our discussion here with the introduction of the (new) notion of the *adjoint* of the PWH system $(K, L)_\varphi$.

Given $\varphi \in L_2(\mathbb{R}^d)$, we associate each $X = (K, L)_\varphi$ with another PWH system, denoted by X^*, referred to hereafter as the **adjoint** system of X, and defined by

$$X^* := (\widetilde{L}, \widetilde{K})_\varphi.$$

We also refer to $(\widetilde{L}, \widetilde{K})$ as the adjoint $(K, L)^*$ of (K, L). Note that the density parameter of X^* is reciprocal to that of X:

$$\operatorname{den}(X)\operatorname{den}(X^*) = 1. \tag{2.1}$$

The system X is said to be **self-adjoint** if $X = X^*$ (i.e., if $K = \widetilde{L}$). Note that all self-adjoint systems are of density 1 (as follows at once from (2.1)), but not vice-versa, unless $d = 1$, as shown by the following example.

Example. Assume that X is a univariate PWH system. Then, $(K, L) = (p\mathbb{Z}, 2\pi q\mathbb{Z})$ for some parameters $p, q > 0$. Here, $\mathrm{den}(X) = (pq)^{-1}$. The adjoint system is $(\widetilde{L}, \widetilde{K}) = (\mathbb{Z}/q, 2\pi\mathbb{Z}/p)$, and its density is, indeed, pq. If $pq = 1$, the adjoint system coincides with the original system. Consequently, *a univariate PWH system is self-adjoint if and only if its density is* 1.

We prove in §2.2 the following result concerning the connection between a PWH system X and its adjoint X^*.

Theorem 2.2. *Let X be a normalized PWH system. Then:*

(a) X is a Bessel system if and only if X^* is one. In that case, $\|T_{X^*}\|^2 = \mathrm{den}(X^*)\|T_X\|^2$.

(b) Suppose that X is a Bessel system. Then X^* is a basis if and only if X is fundamental.

(c) X^* is a frame if and only if X is a frame. In that case, $\|{T_{X^*|}}^{-1}\|^{-2} = \mathrm{den}(X^*)\|{T_{X|}}^{-1}\|^{-2}$.

(d) X^* is a tight frame if and only if X is a tight frame.

(e) X^* is a Riesz basis if and only if X is a fundamental frame.

(f) X^* is an orthonormal basis if and only if X is a fundamental tight frame.

We note that the core claims in the above theorem are (a-c), with the rest being simple corollaries. Specifically, (d) follows directly from (a)+(c), (e) follows from (b)+(c), and (f) follows from (d)+(e). The statement in (b) is actually valid even when X is not Bessel, if one defines then appropriately the notion of a "basis". Finally, note that the roles of X and X^* in the theorem can certainly be interchanged (since $X^{**} = X$).

Remark. The paper [TO2] precedes us in observing the connection (in the univariate setup) between a set $(p\mathbb{Z}, 2\pi q\mathbb{Z})_\varphi$ and $(\mathbb{Z}/q, 2\pi\mathbb{Z}/p)_\varphi$. The univariate case of part (f) of Theorem 2.2 can already be found in that reference.

Theorem 2.2 is complemented by the following theorem concerning the relation between the dual systems of X and X^*. Roughly speaking, it shows that the actions of taking adjoint and taking dual commute. In phrasing this result, we already took advantage of the well-known (and simple) fact that the dual system of a PWH frame $(K, L)_\varphi$ is of the form $(K, L)_\psi$ (cf. Theorem 3.2 of [BW]), i.e., is also a PWH system. This means that the notion of "the adjoint $(\mathrm{R}_X X)^*$ of the dual system $\mathrm{R}_X X$ of X" is well-defined for every PWH system X.

Theorem 2.3. *Let X be a PWH frame generated by φ, X^* its adjoint frame. Then the dual of the adjoint is the adjoint of its dual; more precisely,*

$$\mathrm{R}_{X^*} X^* = (\mathrm{den}\, X)(\mathrm{R}_X X)^*.$$

In particular,

$$\mathrm{R}_{X^*}\varphi = (\mathrm{den}\, X)\mathrm{R}_X\varphi.$$

In words, the generator of the dual frame of X^*, is the same as the generator of the dual frame of X, up to the (important) multiplicative constant $\mathrm{den}\, X$.

Remark 2.4. Our argument in proving the above theorem (in fact, its first half) supports a broader claim: if $Y = (K, L)_\psi$ is *any* dual system of X (i.e., if $T_X T_Y^*$ is the identity on the closed span of X), then $(\mathrm{den}\, X)Y^*$ is a dual system for X^*. □

Though, for computing the dual frame of X, one merely needs to find its generator ψ, this task is rather difficult in general, since no bi-orthogonality relations exist between X and its dual frame. On the other hand, if X is a fundamental frame, its adjoint is a Riesz basis, hence the dual of this adjoint *is* characterized by the standard bi-orthogonality relations. Theorem 2.3 asserts that the generator of the dual frame of X^* is, up to the multiplicative constant $\operatorname{den} X$, the same as the generator of the dual frame of X, and thus suggests a simpler avenue for computing the generator of the dual frame of the fundamental X.

We refer to the relations between X and X^* that are expressed in Theorems 2.2 and 2.3 (as well as to (2.10), which is the technical ground for all these connections) as the **duality principle** of WH systems.

Remark. In [RS2], Theorem 2.3 is announced with a (-n incorrect) multiplicative constant $(\operatorname{den} X)^{1/2}$. The fact that our constant was flawed was revealed in numerical experiments conducted by Hans Feichtinger and Sigang Qiu (from Vienna), and we are grateful to them for pointing out this fact to us.

Three immediate, essentially well-known, corollaries are derived below as an illustration of the efficacy of the duality principle. Further corollaries and applications of the duality principle are collected in §3.1. These other ones include (a) review and extensions of the "painless construction of WH tight frames" from [DGM], (b) results on frames generated by functions that are non-negative or that their Fourier transform is such, (c) partial orthogonality relations that a tight WH frame of a special structure, dubbed "sup-adjoint" herein, must satisfy, and more.

Corollary 2.5. *A self-adjoint X is a fundamental frame if and only if it is a Riesz basis.*

This property follows directly from (e) of Theorem 2.2. The "only if" implication is well-known (cf. Theorem 3.7 of [BW] which contains a proof of that implication for cartesian lattices).

Another conclusion, whose univariate case can be found in p. 81 of [D2], is as follows.

Corollary 2.6. *Let X be a normalized fundamental PWH frame. Then the frame bounds A, B of X satisfy*

$$A \le \operatorname{den} X \le B.$$

In particular, the frame bound(s) of a normalized fundamental PWH tight frame is $\operatorname{den} X$.

Proof: By (e) of Theorem 2.2, the adjoint X^* of X is a Riesz basis. Denoting by A_0, B_0 the Riesz bounds of X^*, it is clear that $A_0 \le 1 \le B_0$ (since T_{X^*} maps a sequence of norm 1, viz. the δ-sequence, to the function φ, whose norm is 1 by assumption). An application of Theorem 2.2, parts (a,c), then yields the desired results on the frame bounds of X. □

The third illustrating corollary is the following:

Corollary 2.7. *Let X be a PWH system. Then, X can never be a fundamental frame for $L_2(\mathbb{R}^d)$ unless* $\operatorname{den} X \ge 1$.

The (strikingly simple) proof of that fact is based on the following

Lemma 2.8. *Let φ be the generator of a PWH fundamental frame X. Let ψ be the generator of the frame dual to X, i.e. $\psi = \mathrm{R}_X\varphi$. Then*

$$\langle \varphi, \psi \rangle = (\operatorname{den} X)^{-1}.$$

Proof: By (e) of Theorem 2.2, X^* is a Riesz basis, whose dual system is generated (Theorem 2.3) by $(\operatorname{den} X)\psi$. The bi-orthogonality relations between a Riesz basis and its dual basis then imply that $\langle \varphi, (\operatorname{den} X)\psi \rangle = 1$. $\square$

Proof of Corollary 2.7. Let $\mathrm{R}X$ be the frame dual to X. We recall that, for every $f \in L_2$, $T^*_{\mathrm{R}X} f$ is the (unique) minimal-norm sequence in the pre-image $T_X^{-1} f$. Choosing $f := \varphi$, and taking into account the fact that $T_X^{-1}\varphi$ contains one sequence of norm 1 (viz. the sequence that is 1 at φ and zero elsewhere), we conclude that $\|T^*_{\mathrm{R}X}\varphi\| \leq 1$. In particular, with ψ the generator of the dual frame, $|\langle \varphi, \psi \rangle| \leq 1$ (since this number is one of the values of the sequence $T^*_{\mathrm{R}X}\varphi$). Combining the above with Lemma 2.8, we conclude that

$$(\operatorname{den} X)^{-1} = \langle \varphi, \psi \rangle \leq 1.$$

Hence, indeed, X is a high-density system. $\square$

It is known that the univariate analog of Corollary 2.7 does not require X to be a frame, only a fundamental set. However, the simple argument for the univariate case (that appears in p. 978 of [D1]) applies only to the case when $\operatorname{den} X$ is rational (which is the univariate counterpart of what we call here *compressible PWH systems*, see §4, particularly Corollary 4.7). The original proof of this result for a general low-density invoked involved results from [R]. Since the completion of the present paper, several, relatively elementary proofs of this phenomenon were brought to our attention (cf. [RaS], [J], [DLL]). In particular, [RaS] establishes a general low-density result, without assuming X to be a frame and without assuming the modulation-shift lattice to be disconjugate (i.e., of the form $K \times L$).

The argument that was used to prove Corollary 2.7 easily implies the following further estimate.

Proposition 2.9. *Let Y be the frame dual to the PWH fundamental frame X. Let φ be the generator of the frame X, and ψ the generator of Y. Then,*

$$\sum_{y \in Y \setminus \psi} |\langle \varphi, y \rangle|^2 \leq 1 - (\operatorname{den} X)^{-2}.$$

$\square$

2.2. Fiberization and the Gramian matrices

Let $X = (K, L)_\Phi$ be a WH system (Φ, say, finite). Then, X is K-shift-invariant, in the sense that each of the shift operators

$$E^k : f \mapsto f(\cdot + k), \quad k \in K$$

maps X 1-1 onto itself. The reference [RS1] suggests, for a general K-shift-invariant X, the study of T_X and T_X^* via a decomposition process of these two operators into a collection of "fiber" operators that are simpler for analysis. For that purpose, we consider the set X as the collection of all K-shifts of the set

$$L_\Phi := \{e_l\varphi : \ \varphi \in \Phi, \ l \in L\}.$$

The basic object in the approach of [RS1] was the **pre-Gramian** matrix $J := J_X$. In the present case (i.e., of the WH system X), the pre-Gramian is an infinite matrix with L_2-valued entries, whose rows are indexed by $\widetilde{K}$, whose columns are indexed by L_Φ, and whose $(k,(\varphi,l)) \in (\widetilde{K} \times L_\Phi)$-entry is

$$|K|^{-1/2}\widehat{\varphi}(\cdot + k + l).$$

We let $J(w)$ be the evaluation of J at any $w \in \mathbb{R}^d$. Taking a standard matrix conjugation of J, we obtain the formal adjoint J^*. We use the collection $(J(w))_{w\in\mathbb{R}^d}$ for the representation of T_X, and the collection $(J^*(w))_{w\in\mathbb{R}^d}$ for the representation of T_X^*.

Already at this initial stage, we are ready to present the most crucial observation concerning the duality principle. With X the PWH set $(K,L)_\varphi$, let us compute the w-value of the pre-Gramian $J_{X^*}(w)$ of the adjoint $X^* = (\widetilde{L},\widetilde{K})_\varphi$: the entries of the pre-Gramian of this adjoint are

$$|\widetilde{L}|^{-1/2}\widehat{\varphi}(w + l + k), \ (l,k) \in (L,\widetilde{K}), \quad w \in \mathbb{R}^d.$$

It follows, thus, that, for every $w \in \mathbb{R}^d$,

$$J_{X^*}(w) = \frac{|K|^{1/2}}{|\widetilde{L}|^{1/2}}\overline{J_X^*}(w) = (\operatorname{den} X)^{-1/2}\overline{J_X^*}(w). \tag{2.10}$$

Thus, roughly speaking (that is, ignoring the unitary transformation of taking complex conjugation, and ignoring the multiplicative constant $(\operatorname{den} X)^{-1/2}$), the pre-Gramian of X^* equals *pointwise* to the adjoint pre-Gramian of X; *this is the essence of the duality principle.*

At this point, the discussion can be advanced in two different complementary ways. The first, which we present here, fully invokes the fiberization results of [RS1]. Roughly speaking, these results imply that the basic norm bounds of T_X can be computed via a separate inspection of each fiber $J_X(w)$ of J_X. This, when combined with (2.10), will provide an immediate proof for almost all statements in Theorem 2.2. The other approach aims at direct connections between the two operators T_{X^*} and T_X^*, and is presented in §3.2.

Though the entire study of the decomposition idea can be performed by the pre-Gramian and its adjoint, we found it more convenient to express the various results in terms of Hermitian matrices. That is, the matrix

$$G(w) := G_X(w) := J^*(w)J(w) = |K|^{-1}(\sum_{k\in\widetilde{K}} \overline{\widehat{\varphi}}(w + k + l)\widehat{\varphi}'(w + k + l'))_{(\varphi,l),(\varphi',l')\in L_\Phi},$$

and

$$\widetilde{G}(w) := \widetilde{G}_X(w) := J(w)J^*(w) = |K|^{-1}(\sum_{l\in L,\varphi\in\Phi} \widehat{\varphi}(w + k + l)\overline{\widehat{\varphi}}(w + k' + l))_{k,k'\in\widetilde{K}}.$$

We refer to G as the **Gramian** of X and to $\widetilde{G}$ as the **dual Gramian** of X. The entries of both matrices can be easily shown to be locally integrable, hence, in particular, are well-defined a.e..

In what sense do the Gramian matrices represent the operators T_X^* and T_X? The detailed discussion of this point is provided in §3.2. An example of that representation is as follows: let $X = (K, L)_\varphi$ and $Y = (K, L)_\psi$ be two PWH systems. Let U_w be the map that assigns to $f \in L_2(\mathbb{R}^d)$ the $\widetilde{K}$-valued sequence $\widehat{f}(w + \widetilde{K})$. Then, for every $f \in L_2(\mathbb{R}^d)$ and a.e. $w \in \mathbb{R}^d$,

$$U_w(T_Y T_X^* f) = (J_Y J_X^*)(w)(U_w f). \tag{2.11}$$

In words, the value of the Fourier transform of $T_Y T_X^* f$ at w is the 0th entry of $(J_Y J_X^*)(w)\widehat{f}(w+\widetilde{K})$.
□

Discussion: observing the Wexler-Raz identity via fiberization. The Wexler-Raz identity from [WR] is a key ingredient in the approach taken in [DLL]. A complete proof of that identity via our fiberization techniques is given in §3.3. The essence of that analysis can already presented now: let X, Y, Z be three PWH systems generated by φ, ψ, g respectively and sharing the same shift lattice K and modulation lattice L. Then, the WR identity says that

$$T_\psi T_g^* \varphi = (\operatorname{den} Z) T_{\varphi^*} T_{g^*}^* \psi,$$

where we have denoted each lattice by its generator; i.e., T_ψ is the synthesis operator of Y, T_{φ^*} is the synthesis operator of X^*, etc.

In order to provide a taste of this identity, we transform it to the Fourier domain, and employ (2.11). Ignoring all normalization constants as well as possible complex conjugations, we obtain (since, up to multiplicative constants, $J_{\varphi^*} = J_\varphi^*$ and $J_{g^*}^* = J_g$)

$$J_\psi(w) J_g^*(w) U_w \varphi = J_\varphi^*(w) J_g(w) U_w \psi.$$

That last identity follows from the obvious relation

$$J_\psi(w) J_g^*(w) J_\varphi(w) = J_\varphi^*(w) J_g(w) J_\psi(w),$$

since $U_w f$ is the 0th column in the matrix $J_f(w)$. □

We turn now to a brief discussion of the core of the [RS1] observations. For that, we consider each (more precisely, almost each) $G(w)$ as a densely defined operator from $\ell_2(L_\Phi)$ into itself, and, similarly, each $\widetilde{G}(w)$ as a densely defined operator from $\ell_2(\widetilde{K})$ into itself. Whenever the relevant operator is well-defined and bounded, it is extended by continuity to the entire $\ell_2(L_\Phi)$ ($\ell_2(\widetilde{K})$). The inverse operators $G(w)^{-1}$, $\widetilde{G}(w)^{-1}$, and the pseudo-inverse operators $G(w)_|^{-1}$, $\widetilde{G}(w)_|^{-1}$ are defined in the same way described in the introduction.

The pertinent result here, which is stated below, follows from Corollary 3.2.2, Theorem 3.3.5 and Theorem 3.4.1 of [RS1].

Theorem 2.12. *Let X be a WH system as above, associated with a Gramian G and a dual Gramian $\widetilde{G}$. Consider the following functions (if $G(w)$ is the 0-operator, we define its partial inverse to be 0, too. Also, if the underlying operator is not well-defined or is unbounded, its norm equals ∞, by definition):*

$$\begin{array}{ccccccccccc}
\mathcal{G} & := & \mathcal{G}_X & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & w & \mapsto & \|G(w)\| & , \\
\mathcal{G}^* & := & \mathcal{G}^*_X & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & w & \mapsto & \|\widetilde{G}(w)\| & , \\
\mathcal{G}^- & := & \mathcal{G}^-_X & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & w & \mapsto & \|G(w)^{-1}\| & , \\
\mathcal{G}^{*-} & := & \mathcal{G}^{*-}_X & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & w & \mapsto & \|\widetilde{G}(w)^{-1}\| & , \\
\mathcal{G}^-_| & := & \mathcal{G}^-_{|X} & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & w & \mapsto & \|G(w)^{-1}_|\| & , \\
\mathcal{G}^{*-}_| & := & \mathcal{G}^{*-}_{|X} & : & \mathbb{R}^d & \to & \mathbb{R}_+ & : & w & \mapsto & \|\widetilde{G}(w)^{-1}_|\| & .
\end{array}$$

Then, the following is true.

(a) The following conditions are equivalent:
(i) X is a Bessel system.
(ii) $\mathcal{G} \in L_\infty$.
(iii) $\mathcal{G}^* \in L_\infty$.
Furthermore, $\|T\|^2 = \|T^*\|^2 = \|\mathcal{G}\|_{L_\infty} = \|\mathcal{G}^*\|_{L_\infty}$.

(b) Assume X is a Bessel system. Then the following conditions are equivalent.
(i) X is a frame.
(ii) $\mathcal{G}^-_| \in L_\infty$.
(iii) $\mathcal{G}^{*-}_| \in L_\infty$.
Furthermore, $\|T^{-1}_|\|^2 = \|{T^*}_|^{-1}\|^2 = \|\mathcal{G}^-_|\|_{L_\infty} = \|\mathcal{G}^{*-}_|\|_{L_\infty}$.

(c) Assume X is a Bessel system. Then the following conditions are equivalent:
(i) X is a Riesz basis.
(ii) $\mathcal{G}^- \in L_\infty$.
Furthermore, $\|T^{-1}\|^2 = \|\mathcal{G}^-\|_{L_\infty}$.

(d) Assume X is a Bessel system. Then the following conditions are equivalent:
(i) X is a fundamental frame.
(ii) $\mathcal{G}^{*-} \in L_\infty$.
Furthermore, $\|T^{*-1}\|^2 = \|\mathcal{G}^{*-}\|_{L_\infty}$.

(e) Assume X is a frame. Then the following conditions are equivalent:
(i) X is a tight frame.
(ii) $\mathcal{G}^* = \mathcal{G}^{*-}_|$ a.e.
(iii) $\mathcal{G} = \mathcal{G}^-_|$, a.e.
If X is fundamental, the last three conditions are also equivalent to:
(iv) $\mathcal{G}^* = \mathcal{G}^{*-}$ a.e.

(f) Assume X is a Riesz basis. Then the following conditions are equivalent:
(i) X is orthogonal.
(ii) $\mathcal{G} = \mathcal{G}^-$ a.e.

Since taking complex conjugation is certainly a unitary operation, we conclude from (2.10) that

$$\mathcal{G}^*_X = (\operatorname{den} X)\mathcal{G}_{X^*},$$

$$\mathcal{G}_X^{*\,-} = (\operatorname{den} X)\mathcal{G}_{X^*}^{-},$$

etc. Parts (a,c,e) of Theorem 2.2 then follow from these relations when combined with the various assertions of Theorem 2.12. As mentioned before, parts (d,f) of Theorem 2.2 follow from (a,c,e), and, finally, a direct proof of (b) is given in §3.2.

We now prove Theorem 2.3. The basic idea behind the proof is relatively simple, and supports a broader claim. To begin with, if $Y = (K, L)_\psi$ is the dual frame of $X = (K, L)_\varphi$, then the two operators $T_X T_Y^*$ and $T_Y^* T_X$ are orthogonal projectors, implying that the corresponding pre-Gramian products $J_X J_Y^*$ and $J_Y^* J_X$ are orthogonal projectors, too. At this point our decomposition techniques are employed: the analysis part of Lemma 4.1 of [RS1] asserts that almost each of the corresponding fiber operators is an orthogonal projector, (2.10) then converts that property to the fibers of the corresponding adjoint systems, and the synthesis part of that Lemma 4.1 is then invoked to conclude that (up to the right multiplicative constant) $T_{X^*}^* T_{Y^*}$ and $T_{Y^*} T_{X^*}^*$ are orthogonal projectors. However, that, in general, does not imply that Y^* is the dual frame of X^*, until one verifies that X^* and Y^* span the same subspace of $L_2(\mathbb{R}^d)$ (cf. Corollary 1.3.9 of [RS1]). The fact that X^* and Y^*, indeed, span the same space follows from the fact that the kernels of the synthesis operators of two dual systems (viz. X and Y) are identical (Proposition 1.3.7 of [RS1]).

However, the entire argument sketched above, including all missing details, had already been detailed in Corollary 4.4 of [RS1], which, as a matter of fact, was tailored there for the proof of Theorem 2.3. All we need here is to carefully compute the constants that arise when converting the shift-invariant setup of that corollary from the lattice $\mathbb{Z}^d$ to general lattices.

Proof of Theorem 2.3. With $\psi = \mathrm{R}_X\varphi$ the generator of the frame dual to X, set $\psi_0 := (\operatorname{den} X)\psi$, and $Y := (\widetilde{L}, \widetilde{K})_{\psi_0}$. Our objective is to show that Y is the dual frame of X^*.

The basis of the duality principle is the identification of J_{X^*} as $(\operatorname{den} X)^{-1/2}\overline{J_X^*}$. With $\mathrm{R}X$ the dual frame of X, our definition of ψ_0 and Y imply also that

$$J_Y = (\operatorname{den} X)^{1/2}\overline{J_{\mathrm{R}X}^*}.$$

This establishes the relation we need for the application of Corollary 4.4 of [RS1], with the following additional remark: the corollary assumes the underlying systems to be $\mathbb{Z}^d$-shift-invariant, hence requires relations such as $J_X J_{\mathrm{R}X}^* = J_{X^*}^* J_Y$. Though our systems are K- or $\widetilde{L}$-shift invariant, the same relation should still be required (i.e., without any normalization factors; this is due to the fact that we use throughout normalized pre-Gramians). We had normalized ψ_0 exactly for that purpose, and consequently obtain the relations "just right" (save an extra harmless complex-conjugation).

With these clarification in hand, we invoke the above-mentioned Corollary 4.4 to obtain that, indeed, Y is the dual frame of X^*. □

Remark. Here is, for a fundamental X, the argument that proves Remark 2.4: Y is a dual system for the fundamental X if and only if $T_X T_Y^* = \mathrm{id}$ if and only if $J_X(w)J_Y^*(w) = \mathrm{id}$ for a.e. w. A similar chain for the adjoint systems, together with (2.10) then completes the proof. □

2.3. Gramian analysis: oversampling

By *Gramian analysis* we mean any method that studies the WH system $X := (K, L)_\Phi$ via its Gramian and dual Gramian fibers without transforming these fibers any more.

For example, one may attempt to estimate the frame bounds of a fundamental X by computing the norm and inverse norm of each fiber $\widetilde{G}(w)$ of the dual Gramian. That, however, may appear to be impractical: after all $\widetilde{G}(w)$ is still a matrix of infinite order, hence represents an operator that, albeit much simpler than T, is not that transparent for analysis.

However, if one aims at *estimating* (rather then computing exactly) the frame bounds of X, then the Gramian matrices provide a convenient venue for it. For example, one may compute the ℓ_1-norm of a typical row of $\widetilde{G}(w)$, and then takes the maximum over all possible rows and all fibers $\widetilde{G}(w)$. The result, in view of Theorem 2.12, bounds the upper frame bound from above. In fact, it is easy to perform this computation to obtain that, with $X = (K, L)_\varphi$,

$$|K| \, \|T_X^*\|^2 \leq \| \sum_{k \in \widetilde{K}} | \sum_{l \in L} E^{k+l} \widehat{\varphi} \, \overline{E^l \widehat{\varphi}} | \|_{L_\infty(\mathbb{R}^d)}.$$

The detailed derivation of the various estimates on the frame bounds is found in §3.4 (in particular, Theorem 3.21 and Theorem 3.27). Since results analogous to some of those statements can be found in [D1] and in [TO2], it is worth mentioning here a difference between the results in the aforementioned references and the estimates provided by Gramian analysis. That difference may be easily understood by the following illustration: suppose that $(f_i)_{i \in I}$ is a collection of functions defined on some common domain A. Then, in general, the condition

$$B_1 := \sum_{i \in I} \sup |f_i| < \infty,$$

is stronger than the condition

$$B_2 := \sup \sum_{i \in I} |f_i| < \infty.$$

Furthermore, one usually has $B_2 < B_1$. The results and estimates of [D1] and [TO2] are based on the finiteness and "smallness" of quantities defined as B_1 above, while our results below are based on quantities defined similarly to the number B_2 above, and thus are finer than their univariate counterparts in the literature.

An interesting application of Gramian analysis, that is fully detailed here, is connected to the idea of *oversampling*, when one increases the density of X in order to stabilize the system. This idea appears, for example, in [FG] and [D1] (see also [D2]), and its particular implementation here is as follows: let $X = (K, L)_\varphi$ be a WH-system. Suppose that we fix L and φ, but vary the volume of K. As K becomes denser, X is more likely to become a frame. This prediction is nicely reflected in the dual Gramian: as K becomes denser, $\widetilde{K}$ become sparser; thus, while the diagonal entry of any row of $|K|\widetilde{G}(w)$, is independent of the choice of K, the 1-norm of the off-diagonal entries of this row is sensitive to the sparsity of $\widetilde{K}$. Specifically, if $\widehat{\varphi}$ decays reasonably fast (i.e., if φ is smooth enough), we expect this ℓ_1-norm to decay to zero, (uniformly so over all rows and fibers). Analogously, if φ is nicely localized, we may switch from $(K, L)_\varphi$ to $(L, K)_{\widehat{\varphi}}$ (hence will oversample the modulations rather than the shifts). So, roughly speaking, under some basic conditions on the smoothness (decay) of φ, we should be able to obtain a frame from the Bessel system X by increasing the density of K (L, respectively). A special result in this direction follows (we present only the PWH-version of this result; the extension to general WH systems is notational). In this result, as well as in other results of this section, we use the notation

$$\widehat{\beta}(s) := \widehat{\beta}_L(s) := \|\sum_{l\in L} E^l(\widehat{\varphi}\, E^s\overline{\widehat{\varphi}})\|_{L_\infty(\mathbb{R}^d)}.$$

Theorem 2.13. *Let $X := (K, L)_\varphi$ be a Bessel system. Assume that*
(a) The sequence $\{\widehat{\beta}(s)\}_{s\in\widetilde{K}}$ is summable (i.e., lies in $\ell_1(\widetilde{K})$).
(b) The shifts $\{\varphi(\cdot - l)\}_{l\in\widetilde{L}}$ form a Riesz basis (respectively, a frame) with Riesz (frame) bounds A and B.
Then, for all sufficiently large integer n, the PWH system $X_n := (K/n, L)_\varphi$ is a fundamental frame (frame, respectively). Furthermore,

$$\lim_{n\to\infty}\frac{\|T_{X_n}\|^2}{n} = \operatorname{den} X\, B, \qquad \lim_{n\to\infty}\frac{\|T_{X_n|}{}^{-1}\|^{-2}}{n} = \operatorname{den} X\, A.$$

Proof: We prove only the case when $F := \{\varphi(\cdot - l)\}_{l\in\widetilde{L}}$ is a Riesz basis. For that, set

$$\mathcal{A}(s,\cdot) := \sum_{l\in L} E^l(\widehat{\varphi}\, E^s\overline{\widehat{\varphi}}).$$

It is well-known (cf. e.g. [RS1], [BW]), that F being a Riesz basis (or frame) is equivalent here to the following two conditions: (i): $\widehat{\beta}(0) < \infty$ (which is equivalent to F being a Bessel system), and (ii): $\widehat{\alpha}(0) := \operatorname{ess\,inf}_{\mathbb{R}^d}\mathcal{A}(0,\cdot) > 0$. (for frame: $\widehat{\alpha}(0) := \operatorname{ess\,inf}_{\sigma(F)}\mathcal{A}(0,\cdot) > 0$, where $\sigma(F)$ is the spectrum of F; cf. [RS1] for details). It is furthermore known that the frame bounds of F are $B = \|T_F\|^2 = \widehat{\beta}(0)/|\widetilde{L}|$ and $A = \|T_F{}^{-1}\|^{-2} = \widehat{\alpha}(0)/|\widetilde{L}|$.

Define $K_n := K/n$. Then, $\widetilde{K}_n = n\widetilde{K}$, and hence, since $\widehat{\beta}_{|_{\widetilde{K}}}$, is summable, $c_n := \|\widehat{\beta}\|_{\ell_1(\widetilde{K}_n\backslash 0)} \to 0$ as $n\to\infty$. By Theorem 3.21,

$$\frac{\|T_{X_n}\|^2}{n} \le \frac{(\widehat{\beta}(0) + c_n)}{n|K_n|} = |\widetilde{L}||K|^{-1}(\|T_F\|^2 + |\widetilde{L}|^{-1}c_n) \to (\operatorname{den} X)\|T_F\|^2 = (\operatorname{den} X)B.$$

Further, the same theorem provides the simpler estimate

$$\frac{\|T_{X_n}\|^2}{n} \ge \frac{\widehat{\beta}(0)}{n|K_n|} = (\operatorname{den} X)\, B,$$

and the first claim is thus established.

We now prove the other claim. Choosing n large enough, we can guarantee that $(\widehat{\alpha}(0) - c_n) > 0$. Theorem 3.27 then implies that X_n is a fundamental frame and that

$$\frac{\|T_{X_n}{}^{-1}\|^{-2}}{n} \ge \frac{(\widehat{\alpha}(0) - c_n)}{n|K_n|} \to |\widetilde{L}||K|^{-1}A = (\operatorname{den} X)A.$$

Again, the converse inequality is straightforward. By the definition of $\widehat{\alpha}(0)$ we can find, for any $\varepsilon > 0$, a set of positive measure Ω_ε such that $\mathcal{A}(0,w) < \widehat{\alpha}(0) + \varepsilon$ for every $w \in \Omega_\varepsilon$. Since $|K_n|^{-1}\mathcal{A}(0,w)$ is a diagonal element of $\widetilde{G}_{X_n}(w)$, (regardless of the value of n), we conclude that $\mathcal{G}^{*-}_{X_n} \geq |K_n|(\widehat{\alpha}(0)+\varepsilon)^{-1}$ on Ω_ε. By letting $\varepsilon \to 0$, we obtain

$$n\|T_{X_n|}{}^{-1}\|^2 = n\|\mathcal{G}^{*-}_{X_n}\|_{L_\infty} \geq |K|/\widehat{\alpha}(0) = ((\operatorname{den} X)A)^{-1}.$$

□

The theorem suggests the construction of "snug frames" (Daubechies' terminology, meaning that the frame is "almost" tight) by "going to the limit" with a smooth orthonormal system: one starts with an orthonormal system $F := \{\varphi(\cdot - j)\}_{j\in\mathbb{Z}^d}$. If $\widehat{\varphi}$ decays fast enough, one expects the function $\widehat{\beta}_{\mathbb{Z}^d}$ to be summable. This means that for almost every lattice K, $\{\widehat{\beta}_{\mathbb{Z}^d}(k)\}_{k\in\widetilde{K}}$ is summable. By taking $X := (K/n, \mathbb{Z}^d)_\varphi$ for sufficiently large n, one obtains an "almost tight" frame for $L_2(\mathbb{R}^d)$. If F is "merely" tight frame, the same procedure applies, only that the fundamentality of X is lost.

As alluded to before, an alternative construction applies to functions which decay nicely, but, say, are not very smooth. In that case, one requires $\{\widehat{\varphi}(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d}$ to be orthonormal, defines the function β with respect to φ instead of $\widehat{\varphi}$, and proceeds by oversampling the modulations rather than the translations. Again, the construction in [D1] can be viewed as a particular instance of that strategy. Indeed, in case $d = 1$, the following result is intimately related to Theorems 2.5 and 2.6 of [D1].

Theorem 2.14. *Let $X := (K, L)_\varphi$ be a Bessel system. Define*

$$\beta(s) := \beta_K(s) := \|\sum_{k\in K} E^k(\varphi\,\overline{E^s\varphi})\|_{L_\infty(\mathbb{R}^d)}.$$

Assume that
(a) The sequence $\{\beta(s)\}_{s\in\widetilde{L}}$ is summable (i.e., lies in $\ell_1(\widetilde{L})$).
(b) The $\widetilde{K}$-modulations of φ form a Riesz basis (a frame), with Riesz (frame) bounds A and B.
Then, for all sufficiently large integer n, the PWH system $X_n := (K, L/n)_\varphi$ is a frame (which is fundamental in the Riesz case). Furthermore,

$$\lim_{n\to\infty}\frac{\|T_{X_n}\|^2}{n} = \operatorname{den} X\, B, \qquad \lim_{n\to\infty}\frac{\|T_{X_n|}{}^{-1}\|^{-2}}{n} = \operatorname{den} X\, A.$$

□

We also remark that some improvements are available in case $\widehat{\beta}_L$ (respectively, β_K) satisfies slightly better decay conditions than the mere assumption $\widehat{\beta}_L \in \ell_1(\widetilde{K})$ (respectively, $\beta_K \in \ell_1(\widetilde{L})$). For example, if $\widehat{\beta}_L$ (respectively, β_K) is majorized by a radially symmetric non-increasing summable function $\widehat{\beta}'$ (respectively, β'), then it is easy to see that for any lattice K, (respectively, L) and for sufficiently large *positive* (not necessarily integer) n, the system $(K/n, L)_\varphi$ (respectively, $(K, L/n)_\varphi$) is a frame.

2.4. Zak transform analysis

The Gramian analysis is crude in the sense that, generally speaking, it fails to take into account possible cancellations in the application of the operators $G(w)$ and $\widetilde{G}(w)$. However, there are cases when the Gramian $G(w)$ and the dual Gramian $\widetilde{G}(w)$ are intimately connected with a finite collection of convolution operators, and, in such cases, the Gramian matrix norms can be more accurately computed via the symbols of the underlying convolution operators. In this section we overview this approach, and provide the complete details in §4. The Zak transform enters the discussion naturally, as the symbol of the relevant convolution operator. As already mentioned in the introduction, the reference [ZZ] contains, in the univariate setup, results which are equivalent to some of the results obtained here in this direction.

In the context of the duality principle, we discussed various types of WH systems that admit special properties, for example, a self-adjoint system. Zak transform analysis leads to a further, more systematic, classification of WH systems as follows:

(a) A PWH self-adjoint system, $X = X^*$. In this case, the Gramian/dual Gramian fiber operator is a convolution operator. A single function, the symbol of that convolution operator, can then be used to analyse completely the "basis" properties of X.

(b) A WH system $X = (K, L)_\Phi$ that is self-adjoint in the sense that $\widetilde{K} = L$. The Gramian fibers are now shown to be a vector of convolution operators: a row vector in the dual Gramian case, and a column vector in the Gramian case. "Frame properties" of X can still be analysed here with the aid of a single function (viz., the sum of squares of symbols).

(c) A **sup-adjoint** PWH system X. By definition, this means that $X^* \subset X$ (i.e., $\widetilde{K} \subset L$; in one dimension, this is the case of integral density). This system can be observed to be a special case of the systems discussed in (b). In fact, the same applies to sup-adjoint system generated by *several* functions: in all these cases the dual Gramian can be compressed into a single function, a function which, in a univariate setup, was already put into good use in [D1] (being the way Daubechies derives her "exact bounds", cf. p. 982 of [D1]). Due to the duality principle, the analysis of PWH sup-adjoint systems extends to their adjoint systems, termed here as **sub-adjoint**.

(d) **Compressible WH systems**. This is the most general case our Zak transform analysis applies to. It is defined as the case when the lattice $\widetilde{K} \cap L$ has a finite index in L; in one dimension this is equivalent to rational density. In this case, the Gramian and dual Gramian fibers are shown to be matrix-valued convolution operators. The order of the relevant matrix is the index of $\widetilde{K} \cap L$ in L (respectively, in $\widetilde{K}$) for the Gramian operator (respectively, for the dual Gramian operator).

In the rest of this subsection, we outline the analysis of *sup-adjoint systems.* The extension of these observations to compressible WH systems is discussed in §4. Note that the sup-adjoint assumption $\widetilde{L} \subset K$ implies, in the PWH case, that $\widetilde{K} \subset L$, implying thereby that $X^* = (\widetilde{L}, \widetilde{K})_\varphi \subset (K, L)_\varphi$. Thus, indeed, the condition $X^* \subset X$ is equivalent to $\widetilde{L} \subset K$. Note that $\operatorname{den} X$ here is the index of $\widetilde{L}$ in K, hence is an integral number. In fact, in one dimension sup-adjoint systems are *characterized* by their integral density.

Example. Assume $K = \mathbb{Z}^d$. Then the condition $\widetilde{L} \subset \mathbb{Z}^d$ is equivalent to the condition that L is a superset of $2\pi\mathbb{Z}^d$. Therefore, $(\mathbb{Z}^d, L)_\varphi$ is sup-adjoint if and only if L is superlattice of $2\pi\mathbb{Z}^d$. Thus, all PWH systems of the form $(\mathbb{Z}^d, 2\pi\mathbb{Z}^d/n)_\varphi$, n integer, are sup-adjoint. □

Given a sup-adjoint $X = (K, L)_\Phi$, we have that $\widetilde{K}$ is a subgroup of L. We set

$$\Gamma$$

for (any representer set of) the finite quotient group $L/\widetilde{K}$. Introducing the set

$$\Gamma_\Phi := \{M^\gamma \varphi : \ \varphi \in \Phi, \, \gamma \in \Gamma\},$$

we see that

$$(K, L)_\Phi = (K, \widetilde{K})_{\Gamma_\Phi}. \tag{2.15}$$

This means that every sup-adjoint system X can be viewed as a WH system $(K, \widetilde{K})_\Psi$. Hence we assume without loss that $X = (K, \widetilde{K})_\Phi$.

Let us inspect the pre-Gramian J_X^* of such X. Here, the rows of J_X^* are indexed by $\Phi \times \widetilde{K}$. We organize J_X^* in row-blocks $(\varphi, \widetilde{K})$, $\varphi \in \Phi$, and denote the corresponding block by J_φ^*. Then, we observe that, for any fixed $w \in \mathbb{R}^d$, the rows of $J_\varphi^*(w)$ comprise the set of all $\widetilde{K}$-shifts of sequence

$$a_{\varphi,w} : \widetilde{K} \to \mathbb{C} : \widetilde{k} \mapsto |K|^{-1/2} \overline{\widehat{\varphi}(w + \widetilde{k})}.$$

Consequently (with possible some minor re-indexing), the map

$$J_\varphi^*(w) : \ell_2(\widetilde{K}) \to \ell_2(\widetilde{K}) : c \mapsto J_\varphi^*(w) c$$

is the convolution

$$c \mapsto a_{\varphi,w} * c.$$

This implies that the entire map $J_X^*(w)$ is a vector-valued convolution operator:

$$J_X^*(w) : \ell_2(\widetilde{K}) \to \ell_2(\Phi \times \widetilde{K}) : c \mapsto (a_{\varphi,w} * c)_{\varphi \in \Phi}. \tag{2.16}$$

It easily follows from that that the operator $J_X(w)$ can be identified with the convolution operator

$$J_X(w) : \ell_2(\Phi \times \widetilde{K}) \to \ell_2(\widetilde{K}) : (c_\varphi)_{\varphi \in \Phi} \mapsto \sum_{\varphi \in \Phi} a_{\varphi,w}^* * c_\varphi,$$

where $a_{\varphi,w}^*$ is the sequence $t \mapsto \overline{a_{\varphi,w}(-t)}$. Altogether, we obtain the following.

Proposition 2.17. *Let X be the self-adjoint WH system $(K, \widetilde{K})_\Phi$, and $\Phi \subset L_2(\mathbb{R}^d)$ be finite. Then, for each $w \in \mathbb{R}^d$ the dual Gramian operator $\widetilde{G}(w)$ is a convolution operator of the form*

$$\widetilde{G}(w) : \ell_2(\widetilde{K}) \to \ell_2(\widetilde{K}) : c \mapsto \sum_{\varphi \in \Phi} a_{\varphi,w} * a_{\varphi,w}^* * c.$$

Theorem 2.12 reduces the study of various properties of X to a corresponding study of the fiber operators $(\widetilde{G}(w))_w$. In the present case, $\widetilde{G}(w)$ is a convolution operator, and therefore all properties and quantities in question may be studied via the *symbol* of the convolution, i.e., its Fourier series. In the present case, we need to compute the symbol of

$$\sum_{\varphi\in\Phi} a_{\varphi,w} * a^*_{\varphi,w}. \tag{2.18}$$

The relevant symbol is a multivariate variant of the **Zak transform**, a notion that we now define. First, let $\mathcal{L}$ be some lattice in $\mathrm{I\!R}^d$. Let I be any linear isomorphism between $\mathcal{L}$ and its dual $\widetilde{\mathcal{L}}$ (e.g., choose a basis for $\mathcal{L}$, map it to its dual basis in $\widetilde{\mathcal{L}}$ and extend the map by linearity). Then

$$Z^{\mathcal{L}} f(w,t) := \sum_{k\in\mathcal{L}} f(w+k) e_{Ik}(t), \quad f \in L_2(\mathrm{I\!R}^d).$$

When considering $Z^{\mathcal{L}} f$ as defined on $\Omega_{\mathcal{L}} \times \Omega_{\mathcal{L}}$, $Z^{\mathcal{L}}$ is an isometry from a dense subspace of $L_2(\mathrm{I\!R}^d)$ into a dense subspace of $L_2(\Omega_{\mathcal{L}} \times \Omega_{\mathcal{L}})$ (with the latter being normalized appropriately), hence extends to an isometry between these two spaces. This fact is well-known in the univariate case (cf. p. 976 of [D1]) and extends to the multivariate case with no difficulty. The details of the map I will be insignificant, and we will suppress it and write

$$Z^{\mathcal{L}} f(w,t) = \sum_{k\in\mathcal{L}} f(w+k) e_{\widetilde{k}}(t).$$

As one now easily computes, the symbol of (2.18) is the function

$$|K|^{-1} \sum_{\varphi\in\Phi} |\sum_{k\in K} \widehat{\varphi}(w+\widetilde{k}) e_k|^2 = |K|^{-1} \sum_{\varphi\in\Phi} |Z^{\widetilde{K}} \widehat{\varphi}(w,\cdot)|^2.$$

Before summarizing, we convert the above expression from the self-adjoint case to the sup-adjoint case. In the latter situation, the set Φ is replaced by the set Γ_Φ, $\Gamma := L/\widetilde{K}$ (cf. (2.15)) and therefore the symbol becomes

$$|K|^{-1} \sum_{\varphi\in\Phi} \sum_{\gamma\in L/\widetilde{K}} |Z^{\widetilde{K}} \widehat{\varphi}(w+\gamma,\cdot)|^2 =: \widetilde{\mathcal{Z}}_X(w,\cdot).$$

Corollary 2.19. *Let X be a sup-adjoint system $(K,L)_\Phi$. Then, for a.e. $w \in \mathrm{I\!R}^d$, the dual Gramian fiber $\widetilde{G}(w)$ is a convolution operator with symbol*

$$\widetilde{\mathcal{Z}}_X(w,\cdot) := |K|^{-1} \sum_{\varphi\in\Phi} \sum_{\gamma\in L/\widetilde{K}} |Z^{\widetilde{K}} \widehat{\varphi}(w+\gamma,\cdot)|^2.$$

Standard properties of convolution operators can now be combined with with Theorem 2.12, to obtain the following result concerning sup-adjoint X. The univariate counterpart of (a) and (b) of this result is known and can be found e.g., in [D1].

Theorem 2.20. *Let $X = (K, L)_\Phi$, $\Phi \subset L_2$ finite, $\widetilde{K} \subset L$. Set $\Gamma := L/\widetilde{K}$, and denote*

$$\widetilde{\mathcal{Z}}_X := |K|^{-1} \sum_{\varphi\in\Phi} \sum_{\gamma\in\Gamma} |Z^{\widetilde{K}}(E^\gamma \widehat{\varphi})|^2.$$

Then the following is true:

(a) X is a Bessel system if and only if $\widetilde{\mathcal{Z}}_X \in L_\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Furthermore, $\|T_X\|^2 = \|T_X^*\|^2 = \|\widetilde{\mathcal{Z}}_X\|_{L_\infty}$.

(b) Assume X is Bessel. Then X is a fundamental frame if and only if $\widetilde{\mathcal{Z}}_X$ is bounded below (away from zero). Furthermore, $\|{T_X^*}^{-1}\|^{-2} = \|1/\widetilde{\mathcal{Z}}_X\|_{L_\infty}$.

(c) Assume that X is Bessel. Then X is a frame if and only if $\widetilde{\mathcal{Z}}_X$ is bounded below on its support. *Furthermore, $\|{T_{X|}}^{-1}\|^{-2} = \|{T_{X|}^*}^{-1}\|^{-2} = \|1/\widetilde{\mathcal{Z}}_X\|_{L_\infty(\sigma)}$, with σ the relevant support.*

The above analysis also implies that, for a sup-adjoint $X = (K, L)_\Phi$, the operator $T_X T_X^*$ is realized on the "Zak transform domain" as the multiplication

$$L_2 \ni f \mapsto \widetilde{\mathcal{Z}}_X Z^{\widetilde{K}} f. \tag{2.21}$$

To make sure, deriving the last theorem directly from that observation is quite straightforward, in stark contrast with the amount of work required for proving Theorem 2.12 (which we invoked in our approach here). However, having already Theorem 2.12 at our disposition, our argument contributes further to the understanding of the connection between the Gramian analysis and the Zak transform analysis.

Since, in our Gramian analysis, we did not state results concerning the *fundamentality* of X, we had no result of this kind to transport into the Zak transform analysis. Nevertheless, it is easy to observe from (2.21) that X is fundamental if and only if $\widetilde{\mathcal{Z}}_X$ vanishes almost nowhere; further, the L_2-functions whose Zak transform is supported on the complement of $\operatorname{supp} \widetilde{\mathcal{Z}}_X$ comprise the orthogonal complement of X. Also, since the frame operator $T_X T_X^*$ is realized on the Zak transform domain as multiplication by $\widetilde{\mathcal{Z}}_X$, the dual frame operator is, necessarily, division by the same function, implying that the Zak transform of the generating functions of the dual frame is given by

$$Z^{\widetilde{K}} \widehat{\varphi} / \widetilde{\mathcal{Z}}_X,$$

a well-known phenomenon for univariate fundamental frames. Here, $0/0 := 0$. More discussion along these lines is given near the end of this section, when we compare sub-adjoint systems to PSI spaces (in this regard, compare, also, (b) in the corollary below to Theorem 2.2.16 of [RS1]).

Corollary 2.22. *Let X be the sup-adjoint WH system $(K, L)_\Phi$, with dual Zak function $\widetilde{\mathcal{Z}}_X$, whose support is $\sigma(X)$. Then:*

(a) The closed span of X consists of all L_2-functions whose Zak transform $Z^{\widetilde{K}}$ is supported in $\sigma(X)$.

(b) Assuming X to be a frame, its dual frame is of the form $(K, L)_\Psi$, where, for each $\psi \in \Psi$, $Z^{\widetilde{K}}\psi$ is supported on $\sigma(X)$ and defined there by

$$Z^{\widetilde{K}}\psi := (Z^{\widetilde{K}}\varphi)/\widetilde{\mathcal{Z}}_X.$$

□

In case Φ is a singleton, the duality principle (i.e. Theorem 2.2) allows us to obtain several useful results with respect to **sub-adjoint** systems, i.e., systems of the form $(K, L)_\varphi$ where $L \subset \widetilde{K}$ (or, equivalently, PWH systems X that satisfy $X \subset X^*$). Here, X^* is sup-adjoint, and the function $\widetilde{\mathcal{Z}}_{X^*}$, written directly in terms of X (rather than in terms of X^*) is

$$|\widetilde{L}|^{-1} \sum_{\delta \in \Delta} |Z^L(E^\delta \widehat{\varphi})|^2,$$

with

$$\Delta := \widetilde{K}/L.$$

The duality principle can then be used to convert Theorem 2.20 to the language of sub-adjoint systems, with the relevant function being

$$\mathcal{Z}_X := (\operatorname{den} X)\widetilde{\mathcal{Z}}_{X^*} = |K|^{-1} \sum_{\delta \in \Delta} |(Z^L \widehat{\varphi})(\cdot + \delta, \cdot)|^2.$$

We omit this restated result.

Discussion: Sub-adjoint systems as the WH analogue of PSI spaces. A PSI (=principal shift-invariant) space $S(\varphi)$ is, by definition, the L_2-closure of the shifts of the L_2-function φ. The study of $F := (E^\alpha \varphi)_{\alpha \in \mathbb{Z}^d}$ as a potential basis for $S(\varphi)$ is the simplest among all studies of shift-invariant bases (cf. [RS1], [BW] and the references therein). Up to some technicalities, a function $f \in S(\varphi)$ is identified by the restriction of its Fourier transform to $[0, 2\pi]^d$, hence the rest of its Fourier transform values are somewhat "redundant" information. Closely related to that is the fact that each of the "fiber spaces" (obtained by restricting the Fourier transform of $S(\varphi)$ to a lattice of the form $w + 2\pi\mathbb{Z}^d$) is either one- or zero-dimensional. The Gramian matrix here is of order 1, and its single entry is the function $[\widehat{\varphi}, \widehat{\varphi}] := \sum_{\alpha \in 2\pi\mathbb{Z}^d} |\widehat{\varphi}(\cdot + \alpha)|^2$, which simply computes the square of the norm of all fibers of φ. It follows that, on the frequency domain, the relevant operator $T_F^* T_F$ is realized as the multiplication operator

$$\ell_2(\mathbb{T}^d) \ni \tau \mapsto [\widehat{\varphi}, \widehat{\varphi}]\tau \in L_2(\mathbb{R}^d).$$

The support of $[\widehat{\varphi}, \widehat{\varphi}]$ is the **spectrum of** $S(\varphi)$ (i.e., it is the set of all one-dimensional fibers) and is useful for finding $\ker T_X$ and its orthogonal complement.

Analogously, in the case of a sub-adjoint WH $X := (K, L)_\varphi$, a function f in the closed span of X is essentially identified by the restriction of its Fourier transform to $\Omega_{\widetilde{K}} + L$. The fibers here are indexed by $\Omega_{\widetilde{K}} \times \Omega_L$, and each fiber space is an one- or zero-dimensional subspace of $\ell_2(\Delta)$ (spanned by the evaluation of the pre-Zak vector at the corresponding (w, t)). The operator $T_X^* T_X$ is realized, on "the Zak transform domain" as multiplication by $\mathcal{Z}_X$. We define the **WH spectrum of** X to be the support $\sigma(X) \subset \Omega_{\widetilde{K}} \times \Omega_L$ of $\mathcal{Z}_X$. The generator ψ of the dual frame of X can be found exactly as in the PSI case (cf. (2.2.13) in [RS1]), that is, it can be computed fiber by fiber:

$$Z^L \widehat{\psi} := Z^L \widehat{\varphi} / \mathcal{Z}_X,$$

with the division carried out only on the WH spectrum of X (more precisely, on the L-periodization of that spectrum; one should be somewhat careful here, since $Z^L\widehat{\varphi}$ has $\Omega_L \times \Omega_L$ as its fundamental domain, while $\mathcal{Z}_X$ has $\Omega_{\widetilde{K}} \times \Omega_{L}$ as its fundamental domain. Thus, these expressions should be extended to an L-periodic expression in the first argument before the division can take place). The analogy goes on and on. For example, a suitable unitary transformation which maps $\ell_2(X)$ onto $L_2(\Omega_{\widetilde{K}} \times \Omega_{L})$ maps $\ker T_X$ onto $L_2(\Omega_{\widetilde{K}} \backslash \sigma(X), \Omega_L)$ (compare with (2.2.8) and its previous display in [RS1]). □

Our final discussion in this section concerns with *oversampling* and *undersampling* of WH systems: Each sup-adjoint system can be realized as the result of oversampling a self-adjoint one, and each sub-adjoint system can be realized as the result of undersampling a self-adjoint system. Let $X = (K, \widetilde{K})_\varphi$ be the self-adjoint system, $Y = (K, L)_\varphi$ its sup-adjoint counterpart (that is $\widetilde{K} \subset L$), and Y^* the corresponding sub-adjoint. If X is a *fundamental* frame, then it is trivial to see that Y is such, too, and hence (by the duality principle, say) that Y^* is a Riesz basis. It is much less obvious to see that if X is a frame (not necessary fundamental), then both Y and Y^* are frames, too; it follows, however, from the analysis of the present subsection. In contrast, the *converse does not follow*: thus, while X can be viewed as an oversampled version of Y^*, the possible frame property of Y^* may be lost during that process. In summary, *the "oversampling" of a sub-adjoint frame does not preserve in general the frame property.*

3. Detailed Gramian analysis

3.1. More applications of the duality principle

We first discuss the construction of WH systems generated by compactly supported functions. We then discuss partial orthogonality relations that are satisfied by tight sup-adjoint systems, and conclude the section with various miscellaneous observations concerning the connection between a frame and its dual system.

The duality principle yields a new proof (and in several dimensions) for the "painless construction of tight WH frames" that was done in [DGM]. For that, suppose that we want to construct a fundamental frame for $L_2(\mathbb{R}^d)$ of the form $(K, L)_\varphi$, φ compactly supported. By the duality principle, this is equivalent to constructing a Riesz basis $(\widetilde{L}, \widetilde{K})_\varphi$. Since φ is compactly supported, we can choose $\widetilde{L}$ sufficiently sparse to guarantee that the sets

$$l + \operatorname{supp}\varphi, \quad l \in \widetilde{L}$$

are *disjoint*. In such case, the task of ensuring that $(\widetilde{L}, \widetilde{K})_\varphi$ is a Riesz basis is reduced to ensuring that $Y := (e_k\varphi)_{k\in\widetilde{K}}$ is a Riesz basis. Furthermore, if Y is only a frame, then X^* is a frame, too, with the same frame bounds, hence X is a frame, though not a fundamental one.

Corollary 3.1. *Let $X = (K, L)_\varphi$ be a normalized PWH system generated by a compactly supported function φ, and assume that the $\widetilde{L}$-shifts of φ have pairwise disjoint supports. Then $(K, L)_\varphi$ is a frame if and only if the set $Y := (e_k\varphi)_{k\in\widetilde{K}}$ is a frame. The frame bounds of X are* $\operatorname{den} X$ *times the frame bounds of Y. Furthermore, X is fundamental if and only if Y is a Riesz basis, and X is fundamental and tight if and only if Y is orthonormal.*

As is well-known (and not hard to prove), Y above is a Riesz basis if and only if the function

$$\widetilde{\varphi}_K := \Big(\sum_{k\in K} |\varphi(\cdot + k)|^2\Big)^{1/2}$$

is bounded above and below (i.e., away from 0). Also, (by Theorem 2.2.7 of [RS1]; see also §7 of [BW]) Y is a frame if and only if $\widetilde{\varphi}_K$ is bounded above and below *on its support* σY, and the frame bounds are

$$|K|\,\|\widetilde{\varphi}_K\|^2_{L_\infty(\sigma Y)}, \quad \text{and } |K|\,\|1/\widetilde{\varphi}_K\|^{-2}_{L_\infty(\sigma Y)}. \tag{3.2}$$

This yields, in view of Corollary 3.1, that the frame bounds of X are

$$|\widetilde{L}|\|\widetilde{\varphi}_K\|^2_{L_\infty(\sigma Y)}, \quad \text{and } |\widetilde{L}|\|1/\widetilde{\varphi}_K\|^{-2}_{L_\infty(\sigma Y)}. \tag{3.3}$$

The computation of the dual frame is straightforward here. Indeed, the orthogonality between the $\widetilde{L}$-shifts of φ implies that the function ψ that generates the dual frame of Y generates also the entire dual frame of X^*. The simplification is then due to the fact that computing the dual frame of Y is relatively easy (cf. [RS1]. The special case of a Riesz basis is well-known). The generator ψ of the dual frame of Y is the function

$$\psi := \frac{\varphi}{|K|\widetilde{\varphi}_K^2},$$

with the understanding that $0/0 := 0$. This function generates the dual frame of Y, hence the dual frame of X^*. In view of Theorem 2.3, we obtain that the frame dual to X is generated by

$$\psi := |\widetilde{L}|^{-1}\frac{\varphi}{\widetilde{\varphi}_K^2}.$$

We summarize these observations as follows:

Corollary 3.4. *Let $X = (K, L)_\varphi$ be a normalized PWH (fundamental) system, and assume that the $\widetilde{L}$-shifts of φ have pairwise disjoint supports. Then:*

(a) X is a frame if and only if the function $\widetilde{\varphi}_K$ is bounded above and away from 0 on its support σY (with $\sigma Y = \mathbb{R}^d$ in the fundamental case). Furthermore, the frame bounds of X are

$$|\widetilde{L}|\|\widetilde{\varphi}_K\|^2_{L_\infty(\sigma Y)}, \quad \text{and } |\widetilde{L}|\|1/\widetilde{\varphi}_K\|^{-2}_{L_\infty(\sigma Y)}.$$

(b) Assuming X to be a frame, the dual frame of X is the system $(K, L)_\psi$, where

$$\psi := |\widetilde{L}|^{-1}\frac{\varphi}{\widetilde{\varphi}_K^2}.$$

The analysis is applicable also in case $\widehat{\varphi}$ rather than φ is compactly supported. One simply applies the same arguments to the system $(L, K)_{\widehat{\varphi}}$ instead of $(K, L)_\varphi$. Thus, for this case, the $\widetilde{K}$-shifts of $\widehat{\varphi}$ are required to have disjoint supports.

We had mentioned before the basic fact that a frame and its dual system fail to satisfy the bi-orthogonality relations that are satisfied in the Riesz basis case. However, for sup-adjoint PWH systems X, the bi-orthogonality relations are preserved, though only to a limited extent:

Theorem 3.5. *Every fundamental sup-adjoint (tight) frame $X := (K, L)_\varphi$ is a union of the form*

$$(K, L)_\varphi = \cup_{g \in G} (K, L)^*_g$$

of $(\operatorname{den} X)^2$ *Riesz (orthogonal) bases. Here,* $G \subset (K, L)_\varphi$.

Proof: Let $(X_i)_{i \in I}$ be the cosets of X^* in X, and, with Y the dual frame of X, let $(Y_i)_i$ be the corresponding cosets of Y^* in Y. By Theorem 2.3, $(\operatorname{den} X)Y^*$ is the dual frame of X^*. Assuming X to be fundamental, we know (from (e) of Theorem 2.2) that the pair $(X^*, (\operatorname{den} X)Y^*)$ is bi-orthogonal, and from that it easily follows that each pair $(X_i, (\operatorname{den} X)Y_i)$ also consists of bi-orthogonal Riesz bases. Since each X_i is of the form $g + X^*$, $g \in X$, the result for a fundamental frame follows, while the result for a fundamental tight frame follows from the orthonormality of X^* (Theorem 2.2, part (f)). □

Example. Suppose that $X = (\mathbb{Z}^d, 2\pi\mathbb{Z}^d/n)_\varphi$, with n positive integer. Then $X^* = (n\mathbb{Z}^d, 2\pi\mathbb{Z}^d)_\varphi$, hence X is indeed sup-adjoint. The cosets of X^* in X are

$$(k + n\mathbb{Z}^d, l/n + 2\pi\mathbb{Z}^d)_\varphi, \quad k, l = 1, \ldots, n.$$

A necessary and sufficient condition for X to be a fundamental tight frame is the orthogonality of the subset $(n\mathbb{Z}^d, 2\pi\mathbb{Z}^d)_\varphi$. In particular, one cannot get a sup-adjoint fundamental tight frame whose elements are not partially orthogonal in the above sense!

While fundamental frames and fundamental tight frames can be constructed with some ease if the generator φ is known to be either compactly supported or band-limited, such frames can also be generated by functions which are neither compactly supported, nor band-limited. However, if the generator φ or its Fourier transform $\widehat{\varphi}$ is *positive* (as is the case with the Gaussian kernel) *one cannot use φ to construct a fundamental tight frame.* In fact, a slightly more general result is true:

Theorem 3.6. *Let φ be a generator of a fundamental frame X. Let ψ be the generator of the dual frame. Then:*

(a) If $\varphi > 0$ a.e., then ψ assumes positive and negative values in a non-trivial way (i.e., on sets of positive measures).

(b) If $\widehat{\varphi} > 0$ a.e., then $\widehat{\psi}$ assumes positive and negative values in a non-trivial way.

In particular, if $\varphi \in L_2$, and either $\varphi > 0$ a.e. or $\widehat{\varphi} > 0$ a.e, then, there exists no PWH fundamental tight frame generated by φ.

Proof: If $X = (K, L)_\varphi$ is a fundamental frame, then, by Theorem 2.2, X^* is a Riesz basis, and, by Theorem 2.3, ψ is perpendicular to $X^* \backslash \varphi$. Since this latter set contains functions that are positive a.e. (of the form $E^l\varphi$, $l \in \mathbb{R}^d$), it follows that ψ cannot be essentially of one sign. This proves (a), while (b) follows by switching to the fundamental frame $(L, K)_{\widehat{\varphi}}$ and invoking (a). □

As Corollary 3.1 already indicates, the last result cannot be extended to functions φ which are only positive *on their support*. However, the argument used above leads to the following partial converse of Corollary 3.1.

Corollary 3.7. *Let $X = (K, L)_\varphi$ be a fundamental tight frame, and assume that $\varphi > 0$ (respectively, $\widehat{\varphi} > 0$) a.e. on its support. Then the sets $(l + \operatorname{supp}\varphi)_{l\in\widetilde{L}}$ (respectively, $(k + \operatorname{supp}\widehat{\varphi})_{k\in\widetilde{K}}$) are essentially disjoint.*

Proof: Since X is fundamental and tight, its adjoint X^* is orthogonal. Thus, if $\operatorname{supp}\varphi \cap (l + \operatorname{supp}\varphi)$ has a positive measure, the positivity of φ implies that $\langle \varphi, E^{-l}\varphi\rangle > 0$, and hence $l \notin \widetilde{L}$. The argument for $\widehat{\varphi}$ is essentially the same. □

Example. If $\operatorname{supp}\varphi = [0, a]$ and $\varphi > 0$ on $(0, a)$, then Corollary 3.1 combined with Corollary 3.7 show that $(p\mathbb{Z}, q\mathbb{Z})_\varphi$ is a fundamental tight frame if and only if the following two conditions hold:
(a) $aq \le 2\pi$.
(b) $\sum_{j\in\mathbb{Z}} |\varphi(\cdot + jp)|^2$ is constant.

Finally, we have already commented on the value of Theorem 2.3 for the computation of the generator of the dual frame of a fundamental frame. We add below some further details in that direction.

Theorem 2.3, together with the duality principle, allows us to derive the following characterization of the generator ψ of the dual of a fundamental PWH frame.

Corollary 3.8. *Let $X = (K, L)_\varphi$ be a fundamental frame. Then the generator ψ of the dual frame of X is the only function in $L_2(\mathbb{R}^d)$ that satisfies the following two conditions:*
(a) $\psi \in \operatorname{ran} T_{X^*}$.
(b) $\sum_{k\in K} E^k\psi\overline{E^{l+k}\varphi} = |\widetilde{L}|^{-1}\delta_{l,0}, \quad l \in \widetilde{L}$.

Proof: Since X is a fundamental frame, its adjoint X^* is a Riesz basis. By Theorem 2.3, $(\operatorname{den} X)\psi$ generates the dual basis of $X^* = (\widetilde{L}, \widetilde{K})_\varphi$. Thus, ψ is characterized by the condition $\psi \in \operatorname{ran} T_{X^*}$ together with the bi-orthogonality conditions

$$\langle \psi, E^l M^k \varphi\rangle = (\operatorname{den} X)^{-1}\delta_{l,0}\delta_{k,0}, \quad (l, k) \in \widetilde{L} \times \widetilde{K}.$$

This last condition is equivalent to the stated condition (b) (after taking into account the fact that $(\operatorname{den} X)^{-1} = |\widetilde{L}|^{-1}|K|$.) □

Remark. Note that condition (a) trivially holds in the *tight* case. Thus one can easily conclude from the last result that a normalized fundamental tight frame $(K, L)_\varphi$ is characterized by the condition

$$\sum_{k\in K} E^k\varphi\overline{E^{k+l}\varphi} = |K|^{-1}\delta_{l,0}, \quad l \in \widetilde{L}. \tag{3.9}$$

The case discussed in Corollary 3.1 can be identified as a simple instance where condition (3.9) automatically holds for $l \neq 0$. □

The result reduces the computation of ψ to solving a bi-infinite linear system of equations. Under more restrictive conditions on the pair (K, L), we will show in section 4 that the computation of ψ can be achieved by solving a *finite* system of linear equations (with function-valued coefficients).

3.2. Unitary relations between the analysis operator of a system and the synthesis operator of its adjoint

We have not discussed so far the exact meaning of the presentation provided by $(J(w))_w$ to T (and, analogously, by $(J^*(w))_w$ to T^*). A complete discussion of that point (in a more general setup), including the issue of well-definedness of these operators can be found in §1 of [RS1]. In §2, we had circumvented entirely that point, and invoked instead the ready-to-use Theorem 2.12. However, for the sake of the present development, we need at least a basic grasp of that representation. All WH systems considered in the present subsection are assumed to be Bessel.

Given an indexed set K, and an open set $\Omega \subset \mathbb{R}^d$, let

$$L_2(\Omega, K)$$

be the Hilbert space of all functions from Ω to $\ell_2(K)$ which are measurable and square-integrable (in the sense of [H]). Given a WH system $X = (K, L)_\varphi$, we may identify the space $\ell_2(X)$ with $L_2(\Omega_{\widetilde{K}}, L)$ with the aid of the unitary map

$$\ell_2(X) \ni c \mapsto U_1 c,$$

where

$$U_1 c(w, l) := \widehat{c_{|(K,l)}}(w). \tag{3.10}$$

Here, $\widehat{c_{|(K,l)}}$ is the Fourier series of the restriction of c to (K, l), and $\Omega_{\widetilde{K}}$ is any fundamental domain for $\widetilde{K}$ (i.e., $\Omega_{\widetilde{K}} + \widetilde{K}$ is an essential partition of $\mathbb{R}^d$). We also identify $L_2(\mathbb{R}^d)$ with $L_2(\Omega_{\widetilde{K}}, \widetilde{K})$ via the map

$$L_2(\mathbb{R}^d) \ni f \mapsto U_2 f, \tag{3.11}$$

with $U_2 f(w, \widetilde{k}) := \widehat{f}(w + \widetilde{k})$. Interpreting J_X as the operator

$$J_X : L_2(\Omega_{\widetilde{K}}, L) \to L_2(\Omega_{\widetilde{K}}, \widetilde{K}),$$

defined by $(J_X \tau)(w, \cdot) := J_X(w)\tau(w, \cdot)$, it follows, [RS1], that

$$U_2 T_X = J_X U_1, \tag{3.12}$$

hence that also

$$U_1 T_X^* = J_X^* U_2. \tag{3.13}$$

However, it will be erroneous to conclude, in view of the above and (2.10), that we had established an operator relation between, say, T_{X^*} and T_X^*. The reason is that, while, up to a constant, (2.10) indeed shows that both J_X^* and $\overline{J_{X^*}}$ are synthesized from the same fiber matrices, the two operators differ in their domain and target spaces: while J_X^* acts from $L_2(\Omega_{\widetilde{K}}, \widetilde{K})$ into $L_2(\Omega_{\widetilde{K}}, L)$, J_{X^*} acts from $L_2(\Omega_L, \widetilde{K})$ into $L_2(\Omega_L, L)$. Thus, finding exact connections between the relevant operators is more subtle than it may look like.

Before we proceed, we sidetrack to list one corollary of the above discussion of independent interest (though quite unrelated to the present course of development): in the form described above, it is clear that every space $L_2(\Omega, \widetilde{K}) \subset L_2(\Omega_{\widetilde{K}}, \widetilde{K})$ is an invariant subspace of the operator $J_Y J_X^*$, for any Bessel $X = (K, L)_\varphi$ and $Y = (K, L)_\psi$. Since the space $L_2(\Omega, \widetilde{K})$ represents all functions whose Fourier transform is supported on $\Omega + \widetilde{K}$, we arrive at the following conclusion.

Corollary 3.14. *Let $X = (K, L)_\varphi$ and $Y = (K, L)_\psi$ be any WH Bessel systems. Let σ be a measurable subset of $\mathbb{R}^d$. Then:*

(a) If σ is invariant under $\widetilde{K}$-shifts, then the space of all L_2-functions whose Fourier transform is supported on σ is an invariant subspace of $T_Y T_X^*$.

(b) If σ is invariant under $\widetilde{L}$-shifts, then the space of all L_2-functions supported on σ is an invariant subspace of $T_Y T_X^*$.

Proof: The first claim was proved in the paragraph preceding the corollary (the $\widetilde{K}$-invariance of σ implies the existence of $\Omega \subset \Omega_{\widetilde{K}}$ with $\sigma = \Omega + \widetilde{K}$). The second claim follows by an application of the first to $\widehat{X} := (L, K)_{\widehat{\varphi}}$. □

It will be technically more convenient to state our subsequent results in terms of the **modified adjoint**

$$X' := (\widetilde{L}, \widetilde{K})_{\varphi^*}$$

with $\varphi^* : t \mapsto \overline{\varphi(-t)}$. The reason is that $J_{X'} = \overline{J_{X^*}}$, and hence the connection between J_X^* and $J_{X'}$ is somewhat nicer than (2.10):

$$J_{X'}(w) = (\operatorname{den} X)^{-1/2} J_X^*(w). \tag{3.15}$$

This implies that the details of unitary connections that we are going to discuss below are slightly simplified if we replace X^* by X'.

We have just mentioned the main difficulty in establishing operator identities that connect the X and X^* (or X'). This difficulty disappears if we assume X to be *self-adjoint*. Indeed, in this case $L = \widetilde{K}$, hence the pre-Gramian identities (2.10), (3.15) can be interpreted as identities between operators. Thus, in the self-adjoint case, (2.10), (3.12) and (3.13) imply the following:

Theorem 3.16. *Let X be a self-adjoint WH system. Then there exists an isometry $U : L_2(\mathbb{R}^d) \to \ell_2(X)$ such that*

$$T_X^* = U T_{X'} U.$$

Consequently, the operators $T_X T_X^*$ and $T_{X'}^* T_{X'}$ are unitarily equivalent.

In fact, the previous discussion reveals a possible operator U. It is composed of the Fourier transform, followed by T_Y^*, when $Y = (K, L)_\chi$, and χ the characteristic function of $\Omega_{\widetilde{K}} = \Omega_L$.

In more general setups, it is easier to derive connections between T_X^* and $T_{X'}$ if one is willing to restrict the domain of these operators. Specifically, let Ω be any measurable subset of $\Omega_{\widetilde{K}} \cap \Omega_L$. Treating $L_2(\Omega, \mathcal{L})$ (with $\mathcal{L}$ either $\widetilde{K}$ or L) as a subspace of $L_2(\Omega_{\mathcal{L}}, \mathcal{L})$, and recalling the isometry between this latter space and $L_2(\mathbb{R}^d)$ (cf. (3.11)), one can find a subspace $H := H_{\Omega,\mathcal{L}}$ of $L_2(\mathbb{R}^d)$ (which is the space of all functions with Fourier transform supported on $\Omega + \mathcal{L}$) such that $U_{2,\mathcal{L}} H = L_2(\Omega, \mathcal{L})$, with $U_{2,\mathcal{L}}$ defined similarly to (3.11). In the same manner, following on the definition of U_1, (3.10), we may find subspaces $S_{\Omega,L}$ of $\ell_2(X')$ and $S_{\Omega,\widetilde{K}}$ of $\ell_2(X)$, together with associated unitary maps $U_{1,L}$ (respectively $U_{1,\widetilde{K}}$) that map $S_{\Omega,L}$ (respectively, $S_{\Omega,\widetilde{K}}$) onto $L_2(\Omega, \widetilde{K}) \subset L_2(\Omega_L, \widetilde{K})$ (respectively, $L_2(\Omega, L) \subset L_2(\Omega_{\widetilde{K}}, L)$). Finally, we denote by $I_{\widetilde{K}}$ the "identity map" that transforms $L_2(\Omega, \widetilde{K})$ from a subspace of $L_2(\Omega_{\widetilde{K}}, \widetilde{K})$ to a subspace of $L_2(\Omega_L, \widetilde{K})$. Thus we have the chain of isometries

$$H_{\Omega,\widetilde{K}} \underset{U_{2,\widetilde{K}}}{\longrightarrow} L_2(\Omega, \widetilde{K}) \underset{I_{\widetilde{K}}}{\longrightarrow} L_2(\Omega, \widetilde{K}) \underset{U_{1,L}^*}{\longrightarrow} S_{\Omega,L}.$$

Setting

$$V_{\widetilde{K}} := U_{1,L}^* I_{\widetilde{K}} U_{2,\widetilde{K}},$$

we conclude that $V_{\widetilde{K}}$ is an isometry between $H_{\Omega,\widetilde{K}}$ and $S_{\Omega,L}$. In a similar fashion, one can construct an isometry $V_L = U_{1,\widetilde{K}}^* I_L U_{2,L}$ between $H_{\Omega,L}$ and $S_{\Omega,\widetilde{K}}$.

The key point in all this development is that, on $L_2(\Omega, \widetilde{K})$, (2.10) gives rise to the rigorous operator relation

$$(\operatorname{den} X)^{-1/2} J_X^* = I_L J_{X'} I_{\widetilde{K}}.$$

Therefore, we may combine the relations (3.12), (3.13) (applied to X and X') to conclude that *on* $H := H_{\Omega,\widetilde{K}}$

$$(\operatorname{den} X)^{-1/2} T_X^* = (\operatorname{den} X)^{-1/2} U_{1,\widetilde{K}}^* J_X^* U_{2,\widetilde{K}} = U_{1,\widetilde{K}}^* I_L J_{X'} I_{\widetilde{K}} U_{2,\widetilde{K}} = V_L U_{2,L}^* J_{X'} U_{1,L} V_{\widetilde{K}} = V_L T_{X'} V_{\widetilde{K}}.$$

We have, thus, proved the following:

Theorem 3.17. *Let $X = (K, L)_\varphi$ be a Bessel system. Let X' be its modified adjoint. Let Ω be any measurable subset of $\mathbb{R}^d$, such that $\{\Omega + l\}_{l \in \mathcal{L}}$ are pairwise disjoint, for $\mathcal{L} = L, \widetilde{K}$. Let H be the subset of all $L_2(\mathbb{R}^d)$-functions whose Fourier transform is supported in $\Omega + \widetilde{K}$. Then, there exist unitary maps*

$$V_{\widetilde{K}} : H \to \ell_2(X'),$$

and

$$V_L : T_{X'} V_{\widetilde{K}} H \to \ell_2(X),$$

both independent of φ, such that the identity

$$(\operatorname{den} X)^{-1/2} T_X^* = V_L T_{X'} V_{\widetilde{K}}$$

holds on H. Consequently, given any Bessel system $Y = (K, L)_\psi$, $(\operatorname{den} X)^{-1} T_Y T_X^*$ is unitarily equivalent on H to the restriction of $T_{Y'}^* T_{X'}$ to $\operatorname{ran} V_{\widetilde{K}} \subset \ell_2(X')$.

Remark. Note that T_X^* maps into $\ell_2(X)$ while T_Y is defined on $\ell_2(Y)$, hence $T_Y T_X^*$ is, formally, not well-defined. However, the two index sets X, Y are naturally identified with the set (K, L), and this is the way $T_Y T_X^*$ should be interpreted. $\square$

Theorem 3.17 falls short of implying directly Theorem 2.2. The minor reason is that it requires X to be Bessel (that can be overcome by a more careful analysis). The major reason is that we do not know in advance that the various norm bounds that we are after are realized on spaces H of the form that appear in the theorem. The fact that the relevant bounds are realized, indeed, on such spaces H is an immediate consequence of Theorem 2.12, which, however, leads to the direct proof of Theorem 2.2, presented in the previous subsection.

Nevertheless, Theorem 3.17 does admit some applications. For example, it provides the missing proof for assertion (b) of Theorem 2.2:

Proof of (b) of Theorem 2.2. If X is not fundamental, then $T_X^* f = 0$, for some $f \in L_2(\mathbb{R}^d)\backslash 0$. Let $X = (K, L)_\varphi$, and let Ω be any subset of $\mathbb{R}^d$ such that (a): $\widehat{f}$ does not vanish identically on $A := \Omega + \widetilde{K}$, (b): The $\widetilde{K}$-shifts, as well as the L-shifts, of Ω are pairwise disjoint. Without loss, we may assume $\widehat{f}$ to be supported on A (otherwise, we define $g \in L_2$ by $\widehat{g} := \widehat{f}\chi$, with χ the support function of A, and apply the representation of T_X^* in terms of J_X^* to conclude that $T_X^* g = 0$, too). Defining H in Theorem 3.17 with respect to the present Ω, the theorem provides us with the relation

$$0 = T_X^* f = V_L T_{X'} V_{\widetilde{K}} f.$$

Since $V_L, V_{\widetilde{K}}$ are partial isometries, this readily implies that $T_{X'}$ is not injective, hence so is T_{X^*}.

The converse is obtained by a similar argument. $\square$

A stronger version of Theorem 3.17 is available under slightly more restrictive conditions.

Corollary 3.18. *Let $X = (K, L)_\varphi$ be a Bessel system, X' its modified adjoint. Let $\Omega_{\widetilde{K}}$ and Ω_L be fundamental domains for the lattices $\widetilde{K}$, L, respectively.*

(a) If the L-shifts of $\Omega_{\widetilde{K}}$ are pairwise disjoint, then there exist unitary maps $V_1 := L_2(\mathbb{R}^d) \to \ell_2(X')$, and V_2, such that

$$(\operatorname{den} X)^{-1/2} T_X^* = V_2 T_{X'} V_1.$$

(b) If the $\widetilde{K}$-shifts of Ω_L are pairwise disjoint, then there exist unitary maps $V_1 : \ell_2(X') \to L_2(\mathbb{R}^d)$, and V_2 such that

$$(\operatorname{den} X)^{-1/2} V_2 T_X^* V_1 = T_{X'}.$$

In particular, (a) is satisfied by all univariate systems of low-density, and (b) is satisfied by all univariate systems of high-density.

Proof: For the proof of (a), we choose $\Omega := \Omega_{\widetilde{K}}$ in Theorem 3.17. We then recognize that $H_{\Omega,\widetilde{K}} = L_2(\mathbb{R}^d)$, and the asserted result then follows from that theorem.

For (b), we choose $\Omega := \Omega_L$ in Theorem 3.17. The construction details of $V_{\widetilde{K}}$ that precede the theorem then imply that $V_{\widetilde{K}}$ maps $H_{\Omega,\widetilde{K}}$ *onto* $\ell_2(X')$. We then take $V_1 := V_{\widetilde{K}}^*$ and $V_2 := V_L^*$.

For the univariate system $(2\pi p\mathbb{Z}, q\mathbb{Z})_\varphi$, we may choose $\Omega_{\widetilde{K}} := (0, 1/p)$, and $\Omega_L = (0, q)$. Therefore, one of the intervals is included in the other, and it is easy to relate the type of inclusion to the "right" type of density. $\square$

Discussion. The last corollary applies to multivariate systems, such as sup-adjoint, sub-adjoint (cf. §4), and many others, but certainly there are multivariate systems that satisfy neither condition. To make sure, only low-density systems can satisfy condition (a), and only high-density systems can satisfy condition (b). This is in analogy with the self-adjointness, a property which in general implies, and in the univariate is also implied by, the $\operatorname{den} X = 1$ property.

3.3. The Wexler-Raz identity

We sidetrack briefly here to discuss the Wexler-Raz identity [WR], whose proof in [DLL] was a key ingredient in the approach there. As we mention in the introduction, this section was added to the present article only after we became aware of the [DLL] work and its details.

We will show that the identity reads at the fiber level as

$$vAu^* = \overline{uA^*v^*},$$

with A a fiber of the pre-Gramian matrix, and u, v some ℓ_2-vectors (of the right "order"). Such identity, of course, holds, but only after one verifies that all sums above are well-defined, in the sense that each one of the rows and columns of A is in ℓ_2 (a condition which is self-evident for a pre-Gramian fiber A), that Au^* as well as vA are in ℓ_2, and that $v(Au^*) = (vA)u^*$. For the satisfaction of these latter requirements, we will assume that all our WH systems are Bessel, and will invoke Theorem 3.2.3 of [RS1] which rigorously justifies the matrix manipulations we employ.

Theorem 3.19 [WR], [DLL]. *Let $X = (K, L)_\varphi$, $Y = (K, L)_\psi$, $Z = (K, L)_g$ be three WH systems. Assume that X^*, Y, Z, (and Z^*) are Bessel. Then,*

$$T_Y T_Z^* \varphi = (\operatorname{den} Z)\, T_{X^*} T_{Z^*}^* \psi. \tag{3.20}$$

Proof: The Bessel assumption on the systems involved ensures us that both sides of (3.20) are well-defined $L_2(\mathbb{R}^d)$-functions. Let r (respectively, $\widetilde{r}$) be the Fourier transform of the left (resp. right) hand side of (3.20). We will show that $r = \widetilde{r}$, a.e.

The pre-Gramian setup detailed at the beginning of §3.2 shows that, for a.e. $w \in \mathbb{R}^d$,

$$r_{|_{w+\widetilde{K}}} = J_Y(w) J_Z^*(w) v,$$

with $v := \widehat{\varphi}_{|_{w+\widetilde{K}}}$. Here, we wish only to compute $r(w)$, hence are interested only at the ($\widetilde{k} = 0$)-row of $J_Y(w)$. That row consists of (cf. the third display of §3.1) $u := |K|^{-1/2} \widehat{\psi}_{|_{w+L}}$. Altogether, for a.e. w,

$$r(w) = |K|^{-1/2} u^t J_Z^*(w) v.$$

Analogous computation reveals that, for a.e. w,

$$\widetilde{r}(w) = (\operatorname{den} Z) |\widetilde{L}|^{-1/2} v^t J_{Z^*}^*(w) u = |K|^{-1/2} (\operatorname{den} Z)^{1/2} u^t \overline{J_{Z^*}}(w) v.$$

Invoking (2.10) (with X there replaced by Z^*), we conclude that, indeed, $r(w) = \widetilde{r}(w)$. $\square$

3.4. Gramian estimates

We elaborate here further on the the topic of Gramian estimates: the estimation of the frame bounds by an inspection of the Gramian matrices (cf. §2.3). The arguments that lead to such estimates are only sketched. In any event, these are straightforward arguments which were already discussed in more detail (and in a more general context) in §1.6 of [RS1]. For the simplicity of the presentation, we will consider only the case of a singleton Φ. The extension to a finite Φ is mainly notational.

Our first objective is the derivation of upper and lower bounds for $\|T\|$ that are verifiable by a mere inspection of the entries of G and $\widetilde{G}$. For that purpose, we introduce the following map, which will be referred to as the **$\mathcal{A}$-transform** ("$\mathcal{A}$" for $\mathcal{A}$mbiguity):

$$\mathcal{A} := \mathcal{A}_{\mathcal{L}} : L_2(\mathbb{R}^d) \to L_1(\mathbb{R}^d \times \Omega_{\mathcal{L}}) : f \mapsto \mathcal{A}f(s, \cdot) := \sum_{l \in \mathcal{L}} E^{s+l} f \, \overline{E^l f}.$$

Here, $\mathcal{L}$ is some fixed lattice, and $\Omega_{\mathcal{L}}$ is a fundamental domain for that lattice.

In terms of the $\mathcal{A}$-transform, the (l, l')-entry of the Gramian $G(w)$ of the WH system $(K, L)_\varphi$ is

$$|K|^{-1} \mathcal{A}_{\widetilde{K}} \widehat{\varphi}(l' - l, w + l).$$

The $(\widetilde{k}, \widetilde{k}')$-entry of the dual Gramian $\widetilde{G}(w)$ is

$$|K|^{-1} \mathcal{A}_L \widehat{\varphi}(\widetilde{k} - \widetilde{k}', w + \widetilde{k}').$$

These observations lead us to the following estimates:

Theorem 3.21. *Let X be the WH system $(K, L)_\varphi$, $\varphi \in L_2(\mathbb{R}^d)$. Then:*

(a) X is Bessel system if the function

$$f := \sum_{k \in \widetilde{K}} |\mathcal{A}_L \widehat{\varphi}(k, \cdot)|$$

is essentially bounded and only if the function

$$g := (\sum_{k \in \widetilde{K}} |\mathcal{A}_L \widehat{\varphi}(k, \cdot)|^2)^{1/2}$$

is essentially bounded, and we have

$$\|g\|_{L_\infty(\mathbb{R}^d)} \le |K| \|T_X\|^2 \le \|f\|_{L_\infty(\mathbb{R}^d)}.$$

(b) X is Bessel system if the function

$$f_1 := \sum_{l \in L} |\mathcal{A}_{\widetilde{K}} \widehat{\varphi}(l, \cdot)|$$

is essentially bounded and only if the function

$$g_1 := (\sum_{l \in L} |\mathcal{A}_{\widetilde{K}} \widehat{\varphi}(l, \cdot)|^2)^{1/2}$$

is essentially bounded, and we have

$$\|g_1\|_{L_\infty(\mathbb{R}^d)} \leq |K| \|T_X\|^2 \leq \|f_1\|_{L_\infty(\mathbb{R}^d)}.$$

Proof: In view of Theorem 2.12, computing $\|T_X\|$ (and thereby verifying the Bessel property of X) is equivalent to computing the ∞-norm of either $\mathcal{G}$ of $\mathcal{G}^*$. Since $G(w)$ is Hermitian, we may bound its 2-norm $\mathcal{G}(w)$ from above by its 1-norm. With f_1 as in (b), $\|f_1\|_{L_\infty}$ can be easily proved to coincide with the essential supremum (over w) of the 1-norms of $G(w)$. A similar argument, with $\widetilde{G}$ replacing G, leads to the estimate that involves f.

Again, since $G(w)$ is non-negative, we can also bound $\mathcal{G}(w)$ from *below* by the ℓ_2-norm of each row of it. This leads to the estimate that involves g_1. Replacing G by $\widetilde{G}$ and repeating the same idea, leads to the estimate that involves g. □

Discussion. It seems instructive to pause and compare the different estimates provided by (a) and (b) in the above theorem. For example, the upper bound of (a) is majorised by the sup-norm of

$$\sum_{\widetilde{k}\in\widetilde{K}, l\in L} |\widehat{\varphi}(\cdot + \widetilde{k} + l)\widehat{\varphi}(\cdot + l)| = \sum_{l\in L} |\widehat{\varphi}(\cdot + l)| \sum_{\widetilde{k}\in\widetilde{K}} |\widehat{\varphi}(\cdot + \widetilde{k} + l)|, \tag{3.22}$$

while the upper bound of (b) is majorised by

$$\sum_{\widetilde{k}\in\widetilde{K}} |\widehat{\varphi}(\cdot + \widetilde{k})| \sum_{l\in L} |\widehat{\varphi}(\cdot + \widetilde{k} + l)|. \tag{3.23}$$

Assume, for example, that $L \subset \widetilde{K}$. Then, (3.22) becomes

$$\sum_{k\in\widetilde{K}} |\widehat{\varphi}(\cdot + k)| \sum_{l\in L} |\widehat{\varphi}(\cdot + l)|. \tag{3.24}$$

As to (3.23), identifying $\widetilde{K}/L$ with some finite $\Delta \subset \widetilde{K}$, that expression can be written as

$$\sum_{\delta\in\Delta} (\sum_{l\in L} |\widehat{\varphi}(\cdot + l + \delta)|)^2. \tag{3.25}$$

While *pointwise* the two estimates are not comparable, it is easily seen that the sup-norm of (3.25) is $\leq$ the sup-norm of (3.24). Since the second estimate is based on the Gramian, and the example just studied is of a low-density system, this suggests that the Gramian upper bound estimates ((b) in the theorem) are better (i.e., tighter) for low-density systems, hence that the dual Gramian upper bound estimates (i.e., (a) of the theorem) yield better results for high-density systems.

In any event, the above analysis shows that T is bounded whenever $\widehat{\varphi}$ decays fast enough, or equivalently, whenever φ is smooth enough:

Corollary 3.26. *Let X be a PWH system generated by φ. Then X is a Bessel system if*

$$\sum_{\alpha\in\mathcal{L}} |\widehat{\varphi}(\cdot + \alpha)|$$

is essentially bounded, for $\mathcal{L} = L$, $\widetilde{K}$.

Proof: From the above discussion, it follows that

$$\sum_{k\in\widetilde{K}} |\mathcal{A}_L \widehat{\varphi}(k, w)| \leq \|\sum_{l\in L} |\widehat{\varphi}(\cdot + l)|\|_{L_\infty(\mathbb{R}^d)} \|\sum_{k\in\widetilde{K}} |\widehat{\varphi}(\cdot + k)|\|_{L_\infty(\mathbb{R}^d)}.$$

Theorem 3.21 (a) then implies the claim. □

Discussion cont'ed. Analogous estimates are available in terms of φ (rather than $\widehat{\varphi}$) with the lattices $L, \widetilde{K}$ replaced by their duals $K, \widetilde{L}$. Consequently, X is a Bessel system if *either φ is sufficiently smooth or decays sufficiently fast.*

Literature Discussion. We compare the f-estimate in (a) of Theorem 3.21 to Theorem 2.6 of [D1] and Theorem 2 of [TO2]. Theorem 2.6 of [D1] contains an estimate of the upper frame bound for a univariate X in terms of φ rather than $\widehat{\varphi}$. After making that necessary switch, we find that it employs quantities of the form $\| \sum_{l\in L} |E^{s+l} f\, \overline{E^l f}|\, \|_\infty$. However, these norms are not summed up directly, but rather are grouped into pairs, with respect to which a geometric average is computed, hence altogether Theorem 2.6 of [D1] is *not comparable* to our results here. As to Theorem 2 of [TO2], rewriting it in terms of $\widehat{\varphi}$ rather than φ, its sufficient condition for the Bessel property of the univariate X is the finiteness of $\sum_{l\in L} \|a_l\|_{\ell_1}$, with a_l the Fourier coefficients of $\mathcal{A}_L\widehat{\varphi}(l,\cdot)$. That boundedness condition clearly implies the boundedness of f in Theorem 3.21.

The derivation of sufficient conditions for X to be a Bessel system is much simpler than guaranteeing X to be a frame. Unless we are willing to settle for conditions for a fundamental frame X or a Riesz basis X, it is virtually impossible to derive feasible sufficient conditions for frames that are based on the magnitude of various entries of $\widetilde{G}$ or G. However, in order for X to be a *fundamental* frame, $\widetilde{G}(w)$ should be invertible, and that can be observed in case $\widetilde{G}(w)$ is *diagonally dominant*. Sufficient conditions of this type are discussed in the next theorem.

Theorem 3.27. *Let $X = (K, L)_\varphi$ be a Bessel PWH system.*

(a) If the function

$$f := \mathcal{A}_L\widehat{\varphi}(0,\cdot) - \sum_{k\in\widetilde{K}\backslash 0} |\mathcal{A}_L\widehat{\varphi}(k,\cdot)|$$

is positive a.e. and is (essentially) bounded away from 0, then X is a fundamental frame for $L_2(\mathbb{R}^d)$ and

$$\|T^{*-1}\|^2 \le |K|\|1/f\|_{L_\infty(\mathbb{R}^d)}. \tag{3.28}$$

(b) If the function

$$g := \mathcal{A}_{\widetilde{K}}\widehat{\varphi}(0,\cdot) - \sum_{l\in L\backslash 0} |\mathcal{A}_{\widetilde{K}}\widehat{\varphi}(l,\cdot)|$$

is positive a.e., and is (essentially) bounded away from 0, then X is a Riesz basis, and

$$\|T^{-1}\|^2 \le |K|\|1/g\|_{L_\infty(\mathbb{R}^d)}.$$

Proof: Part (b) follows from (a) and the duality principle: to prove that X is Riesz basis, it suffices, by that principle, to prove that X^* is a fundamental frame. A conversion of the condition in (a) from $(K, L)_\varphi$ to $(\widetilde{L}, \widetilde{K})_\varphi$ then yields the condition in (b).

We now prove (a): by Theorem 2.12, we need to show that each $\mathcal{G}(w)^{*-}$ is bounded. Note that the entries of each row of the dual Gramian $\widetilde{G}(w)$ are of the form

$$|K|^{-1}\mathcal{A}_L\widehat{\varphi}(k, w'), \quad k \in \widetilde{K},$$

for an appropriately chosen $w' \in \mathbb{R}^d$. The condition

$$\mathcal{A}_L\widehat{\varphi}(0,w) - \sum_{k\in\widetilde{K}\backslash 0} |\mathcal{A}_L\widehat{\varphi}(k,w)| \geq \varepsilon, \quad \text{a.e. } w \in \mathbb{R}^d$$

thus implies that, for almost every $w \in \mathbb{R}^d$, and for every row of $\widetilde{G}(w)$, the diagonal entry $\mathcal{A}_L\widehat{\varphi}(0,w')$ is ε greater than the ℓ_1-norm of the other entries of this row. Thus, as soon as this condition holds, the self-adjoint operator $\widetilde{G}(w)$ is diagonally dominant, hence invertible, for almost every w, and $\mathcal{G}^{*-}(w) = \|\widetilde{G}(w)^{-1}\| \leq |K|/\varepsilon$. By Theorem 2.12, X is a fundamental frame, and the estimate (3.28) holds. □

Remark. A slightly different variant of the above theorem is as follows: if f in the theorem is $\geq \varepsilon$ *on the support* Ω *of* $\mathcal{A}_L\widehat{\varphi}(0,\cdot)$, then X is a frame (not necessarily fundamental) and the estimate (3.28) holds with $\|1/f\|_{L_\infty(\Omega)}$ replacing $\|1/f\|_{L_\infty(\mathbb{R}^d)}$.

Literature Discussion cont'ed. Theorem 2.5 of [D1] provides a sufficient condition for a univariate system to be a fundamental frame. The condition is based on the positivity of (a slightly coarser expression than) the quantity

$$\inf \mathcal{A}_L\widehat{\varphi}(0,\cdot) - \sum_{k\in\widetilde{K}\backslash 0} \|\mathcal{A}_L\widehat{\varphi}(k,\cdot)\|_\infty,$$

and thus the condition given in Theorem 3.27 is somewhat better. In any event, both conditions are based on very coarse estimates. The comparison of Theorem 3.27 to the lower bound estimate provided in Theorem 7 of [TO2] is similar.

As stated several times before, the "basis" $(K,L)_\varphi$ satisfies the same properties as its Fourier transform set $(L,K)_{\widehat{\varphi}}$. This means that all the above estimates are valid with φ replacing $\widehat{\varphi}$ and K, L interchanging roles.

4. Detailed Zak transform analysis

4.1. Zak transform analysis of compressible WH systems

In §2.4, it was shown that, for a sup-adjoint WH systems X, the study of T_X^* is reduced to studying the behaviour of the single function $\widetilde{\mathcal{Z}}_X$. Though the same does not hold for other WH systems, there are more general situations when the analysis of X can be reduced to the study of *finite-order* Hermitian matrices. In fact, the situation here is analogous to that that occurs in the study of shift-invariant spaces. While, as discussed in §2.4, the PSI space and the sub-adjoint WH space have one-dimensional fiber spaces, the more general FSI (=finitely generated shift-invariant) spaces are similar to the present compressible WH systems: the fiberization of both lead to fiber spaces of *finite dimension*. In the latter case, each fiber space $S_{w,t}$ is spanned by the columns of $PZ_X(w,t)$, hence is considered a subspace of $\ell_2(\Delta)$ (see below for definitions and details). The critical information required is that basic operations (such as finding image and kernel of operators, describing orthogonal projectors, computing dual system, etc.) can be performed fiber by fiber. For FSI spaces, this was done in §2 of [RS1]; the techniques, however, apply to general fiberization with finite-dimensional fibers, hence to our present case of interest.

We introduce a new type of "inner product", the *Zak product*, of which the function $\widetilde{\mathcal{Z}}_X$ that was used in the context of the sup-adjoint case is a special "diagonal" case. We will then form finite-order matrices (the *Zak matrix* and the *dual Zak matrix*) whose entries are such Zak products. These matrices may replace the infinite order Gramian matrices, and we reach that reduction by employing either of the two arguments: (a) direct adaptation of the FSI techniques of [RS1] to WH setup here; or (b) adaptation of the FSI techniques in order to fiberize further the (already partially fiberized) Gramian operator, and then invoking Theorem 2.12.

We call a lattice pair (K, L) **compressible** if the group $\widetilde{K} \cap L$ has a finite index in L (equivalently, in $\widetilde{K}$), which is the case exactly when $\widetilde{K} \cap L$ is d-dimensional. We define then two quotient groups:

$$\Gamma := L/(L \cap \widetilde{K}),$$

and

$$\Delta := \widetilde{K}/(L \cap \widetilde{K}).$$

We also think about Γ, Δ as any set of representers for the above groups. We refer to the order of Γ as the **compression factor** of X, and to the order of Δ is the **decomposition factor** of X. (The terminology reflects the fact that we are interested primarily in high-density systems X. For low-density systems, the notions of compression and decompositions factors should be interchanged). Note that X is self-adjoint if and only if both Γ and Δ are trivial, X is sup-adjoint if and only if Δ is trivial, and X is sub-adjoint if and only if Γ is trivial. Note also that the notations Γ, Δ retain their meaning from the last section in case of a sup/sub-adjoint system.

Example . Suppose that $X = (2\pi p\mathbb{Z}, q\mathbb{Z})_\varphi$. Then, $\widetilde{K} \cap L = \mathbb{Z}/p \cap q\mathbb{Z}$. Clearly, $\widetilde{K} \cap L$ is non-trivial if and only if pq is rational. Since $pq = (\operatorname{den} X)^{-1}$, we conclude that a *univariate* (K, L) is compressible if and only if its density is rational.

In the case of a sup-adjoint X, we identified the pre-Gramian with a vector-valued convolution operator. It is not hard to see (as we are going to show), that in the present case the pre-Gramian is a *matrix*-valued convolution operator.

Here are the details: we let $X = (K, L)_\Phi$, Φ finite, be a compressible WH system. The rows of the pre-Gramian J_X^* are indexed by $L \times \Phi = (\widetilde{K} \cap L) \times \Gamma \times \Phi$, and we re-organize them in blocks $((\widetilde{K} \cap L), \gamma, \varphi)$, $\varphi \in \Phi$, $\gamma \in \Gamma$. The columns of J_X^* are indexed $\widetilde{K} = (\widetilde{K} \cap L) \times \Delta$, which we also organize in blocks $((\widetilde{K} \cap L), \delta)$, $\delta \in \Delta$. This induces a block re-organization of the entire J_X^*, with a typical block indexed by $((\gamma, \varphi), \delta)$. As in the sup-adjoint case, one observes that the block $J^*_{(\gamma,\varphi),\delta}(w)$ is a convolution operator

$$a_{\gamma,\varphi,\delta,w} : \ell_2(\widetilde{K} \cap L) \to \ell_2(\widetilde{K} \cap L) : c \mapsto a_{\gamma,\varphi,\delta,w} * c.$$

Here, the sequence $a_{\gamma,\varphi,\delta,w}$ is defined by

$$a_{\gamma,\varphi,\delta,w}(j) := |K|^{-1/2}\overline{\widehat{\varphi}}(w + \gamma + \delta + j). \tag{4.1}$$

The operator $J_X^*(w)$ acts from $\ell_2(\widetilde{K}) = \ell_2((\widetilde{K} \cap L) \times \Delta)$ into $\ell_2(L \times \Phi) = \ell_2((\widetilde{K} \cap L) \times \Gamma \times \Phi)$, and, by the above, acts, indeed, as a matrix-valued convolution operator:

$$J_X^*(w) : (c_\delta)_{\delta\in\Delta} \mapsto (\sum_{\delta\in\Delta} a_{\gamma,\varphi,\delta,w} * c_\delta)_{\gamma\in\Gamma,\, \varphi\in\Phi}.$$

This observation allows us to replace the pre-Gramian by the matrix of the symbols of the various $a_{\gamma,\varphi,\delta,w}$. As in the sup-adjoint case, the symbol of the sequence $a_{\gamma,\varphi,\delta,w}$ is the Zak transform

$$|K|^{-1/2} Z^{\widetilde{K}\cap L}(E^{\gamma+\delta}\overline{\widehat{\varphi}})(w,\cdot).$$

Thus, we obtained a representation of the pre-Gramian $J_X^*(w)$ as a finite order matrix whose rows are indexed by $\Gamma\times\Phi$, whose columns are indexed by Δ, and whose structure is as follows:

$$PZ_X^*(w,\cdot) := \bigl(|K|^{-1/2}\, Z^{\widetilde{K}\cap L}\overline{\widehat{\varphi}}(w+\gamma+\delta,\cdot)\bigr)_{(\gamma,\varphi)\in(\Gamma\times\Phi),\,\delta\in\Delta}.$$

It is quite obvious that the analogous representer of $J_X(w)$ is the adjoint $PZ_X(w,\cdot)$ of the above $PZ_X^*(w,\cdot)$. We refer to both of these matrices as **pre-Zak matrices**. The **spectrum** $\sigma(X)$ of X is the set of all (w,t) where $PZ_X(w,t)$ is not the zero matrix.

The development now becomes quite transparent. Using the pre-Zak matrices, we construct finite-order analogs of the Gramian and the dual Gramian. The analog of $G(w)=J_X^*J_X(w)$ is the square non-negative $(\Gamma\times\Phi)$-order matrix $\mathcal{Z}_X(w,\cdot)$, whose $((\gamma,\varphi),(\gamma',\psi))$-entry is

$$|K|^{-1}\,\mathbf{Z}_\Delta^{\widetilde{K}\cap L}(E^{\gamma'}\widehat{\psi}, E^{\gamma}\widehat{\varphi})(w,\cdot),$$

where, for any lattice $\mathcal{L}$, finite set $\Delta\subset\mathbb{R}^d$, and $f,g\in L_2$, $\mathbf{Z}_\Delta^{\mathcal{L}}(f,g)$ is the **Zak product** of f and g defined as

$$\mathbf{Z}_\Delta(f,g) := \mathbf{Z}_\Delta^{\mathcal{L}}(f,g) := \sum_{\delta\in\Delta}(Z^{\mathcal{L}}f\,\overline{Z^{\mathcal{L}}g})(\cdot+\delta,\cdot). \tag{4.2}$$

In analogy with the Gramian matrix, we call the matrix $\mathcal{Z}_X$ **the Zak matrix**. The analog of the dual Gramian is the **dual Zak matrix** obtained as $\widetilde{\mathcal{Z}}_X := PZ_X PZ_X^*$. Thus, it is a square non-negative Δ-order matrix, whose (δ,δ')-entry is

$$|K|^{-1}\sum_{\varphi\in\Phi}\mathbf{Z}_\Gamma^{\widetilde{K}\cap L}(E^{\delta}\widehat{\varphi}, E^{\delta'}\widehat{\varphi}).$$

Example. Let m,n be positive integers, g.c.d.$(m,n)=1$. Let X be the univariate WH system $(2\pi\mathbb{Z}/m, n\mathbb{Z})_\Phi$ (with Φ finite, otherwise arbitrary). Then $\widetilde{K}=m\mathbb{Z}$, hence $\widetilde{K}\cap L = mn\mathbb{Z}$. We may thus take $\Gamma = L/(\widetilde{K}\cap L) = (n,2n,...,mn)$, and $\Delta = \widetilde{K}/(\widetilde{K}\cap L) = (m,2m,...,nm)$. The relevant Zak transform here is $Z := Z^{mn\mathbb{Z}}$ which can be defined as

$$Zf(w,t) := \sum_{j\in\mathbb{Z}} f(w+mnj)e_{2\pi j/mn}(t).$$

The pre-Zak matrix PZ_X has the form

$$(m/2\pi)^{1/2}\bigl(Z\widehat{\varphi}(\cdot+jn+j'm,\cdot)\bigr)_{j=1}^{m}{}_{j'=1}^{n}{}_{\varphi\in\Phi},$$

(where rows are indexed by j' and columns by (j,φ)). Consequently, the dual Zak matrix is an $n\times n$ matrix whose (j,j')-entry is

$$(m/2\pi)\sum_{\varphi\in\Phi}\sum_{i=1}^{m}\overline{(Z\widehat{\varphi})(w+in+jm,t)}(Z\widehat{\varphi})(w+in+j'm,t).$$

The Zak matrix, say in case Φ is a singleton φ, is an $m\times m$ matrix whose (j,j')-entry is

$$(m/2\pi)\sum_{i=1}^{n}(Z\widehat{\varphi})(w+im+jn,t)\overline{(Z\widehat{\varphi})(w+im+j'n,t)}.$$

Finally, the following variant of the duality principle, though is not an issue here, arises in a natural way: interchanging between m and n results at a similar interchange between the Zak matrix and the dual Zak matrix. At the same time, such an interchange amounts to passing from X to its adjoint X^*. Indeed, the fact that passing to the adjoint amounts, up to a multiplicative constant, to an interchange between the two basic matrices is the essence of the duality principle. $\square$

Remark. Zibulski and Zeevi, [ZZ], as well as, implicitly, Daubechies, [D1], employ the Zak transform in order to decompose univariate systems of rational density. In particular, [ZZ] employs two matrices, S and G, which correspond to, yet look quite different from, our dual Zak matrix, and pre-Zak matrix. However, the difference can be simply attributed to a different choice of a basis for $\ell_2(\Delta)$ or $\ell_2(\Gamma)$. Our Zak matrices here are related to the standard bases for these spaces (made of functions of one-point-support). On the other hand, the [ZZ]-matrices are related to choosing the characters of the dual group as the relevant basis. Since both choices are orthonormal, our dual Zak matrix here is unitarily equivalent to the S matrix of [ZZ].

In order to convert the analysis of the "basis set" X from the infinite-order Gramian matrices to the finite order Zak matrices, we need to know how to relate the functions that appear in Theorem 2.12 ($\mathcal{G}$, $\mathcal{G}^*$, etc.) to the Zak matrices. However, as was already explained before, the study of the connection between $G(w)$ and $\mathcal{Z}_X(w,\cdot)$ here is entirely analogous to the connection, in the case of an FSI set X, between the standard infinite-order Gramian $(\langle x,y\rangle)_{x,y\in X}$ and the compressed Gramians $(G(w))_{w\in\mathbb{R}^d}$. These latter relations were analysed in detail in several references, with the most comprehensive results contained in [BDR] and [RS1]. In particular, the arguments used to establish Theorem 2.3.6 of [RS1], when transported to the new setup here, lead to the following result.

Lemma 4.3. *Given a compressible WH system $X:=(K,L)_\Phi$ and its associated Zak matrix $\mathcal{Z}_X$, consider, for fixed $w,t\in\mathbb{R}^d$, the matrix $\mathcal{Z}_X(w,t)$ as a map from $\ell_2(\Gamma\times\Phi)$ into itself. Let $\zeta_X(w,t)$ be the norm of this map, $\zeta_X^-(w,t)$ the norm of the inverse (defined as ∞ if $\mathcal{Z}_X(w,t)$ is singular), and $\zeta_{X|}^-(w,t)$ the norm of the pseudo-inverse. Then, in the notations of Theorem 2.12, the following relations hold:*

(a) For a.e. $w\in\mathbb{R}^d$, $\mathcal{G}_X(w)=\|\zeta_X(w,\cdot)\|_{L_\infty}$.
(b) For a.e. $w\in\mathbb{R}^d$, $\mathcal{G}_X^-(w)=\|\zeta_X^-(w,\cdot)\|_{L_\infty}$.
(c) For a.e. $w\in\mathbb{R}^d$, $\mathcal{G}_{X|}^-(w)=\|\zeta_{X|}^-(w,\cdot)\|_{L_\infty(\sigma_w)}$, with σ_w the support of $\zeta_X(w,\cdot)$.

Invoking, thus, Theorem 2.12, we obtain the following characterization of the "basis" properties of the compressible X in terms of the Zak matrix.

Theorem 4.4. *Let $X = (K, L)_\Phi$ be compressible, and let $\mathcal{Z}_X$ be its associated Zak matrix. Consider, for each $(w, t) \in \mathbb{R}^d \times \mathbb{R}^d$, the matrix $\mathcal{Z}_X(w, t)$ as a map from $\ell_2(\Gamma \times \Phi)$ into itself, and let $\zeta_X(w, t)$, $\zeta_X^-(w, t)$ and $\zeta_{X|}^-(w, t)$ be the associated norm functions defined in Lemma 4.3. Then:*

(a) *X is a Bessel system if and only if $\zeta_X \in L_\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Furthermore, $\|T_X\|^2 = \|T_X^*\|^2 = \|\zeta_X\|_{L_\infty}$.*

(b) *Assume X is a Bessel system. Then X is a Riesz basis if and only if $\zeta_X^- \in L_\infty$. Furthermore, $\|T_X^{-1}\|^2 = \|\zeta_X^-\|_{L_\infty}$.*

(c) *Assume X is Bessel. Then X is a frame if and only if $\zeta_{X|}^- \in L_\infty(\sigma(X))$, with $\sigma(X)$ the spectrum of X. Furthermore, $\|T_{X|}{}^{-1}\|^2 = \|\zeta_{X|}^-\|_{L_\infty(\sigma(X))}$.*

While the Zak matrix is useful for computing the important functions $\mathcal{G}$, $\mathcal{G}^-$ and $\mathcal{G}_|^-$ associated with the Gramian of X, the dual Zak matrix is useful for computing the complementary quantities $\mathcal{G}^*$, $\mathcal{G}^{*-}$ and $\mathcal{G}_|^{*-}$ associated with the dual Gramian $\widetilde{G}$. For that, we consider, for each $(w, t) \in \mathbb{R}^d$, the matrix $\widetilde{\mathcal{Z}}_X(w, t)$ as a map from $\ell_2(\Delta)$ into itself, and denote by $\zeta_X^*(w, t)$ ($\zeta_X^{*-}(w, t)$, $\zeta_{X|}^{*-}(w, t)$) the norm-function (respectively, the inverse-norm function, and the pseudo-inverse-norm function) of $\widetilde{\mathcal{Z}}_X$. Again, the arguments used in Theorem 2.3.6 of [RS1] apply here, and lead to a result analogous to Lemma 4.3, that connects now the ζ^*-functions to the $\mathcal{G}^*$-functions. This, in view of Theorem 2.12, leads to following theorem.

Theorem 4.5. *Let X be a compressible WH system $(K, L)_\Phi$. Let $\widetilde{\mathcal{Z}}_X$ be its associated dual Zak matrix, and let ζ_X^*, ζ_X^{*-} and $\zeta_{X|}^{*-}$ be the corresponding norm-functions. Then:*

(a) *X is a Bessel system if and only if $\zeta_X^* \in L_\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Furthermore, $\|T_X\|^2 = \|T_X^*\|^2 = \|\zeta_X^*\|_{L_\infty}$.*

(b) *Assume X is Bessel. Then X is a fundamental frame if and only if $\zeta_X^{*-} \in L_\infty$. Furthermore, $\|T_X^*{}^{-1}\|^2 = \|\zeta_X^{*-}\|_{L_\infty}$.*

(c) *Assume X is Bessel. Then X is a frame if and only if $\zeta_{X|}^{*-} \in L_\infty(\sigma(X))$. Furthermore, $\|T_{X|}^*{}^{-1}\|^2 = \|\zeta_{X|}^{*-}\|_{L_\infty(\sigma(X))}$.*

Remark. For a *univariate* X, parts (a) and (b) of the above result, as well as part (b) of the following corollary, are due to [ZZ] (Theorem 2 and Proposition 1 there). Though Theorem 2 in [ZZ] is stated without proof, the supporting discussion there provides an almost complete argument. The only missing ingredient in the approach of [ZZ] is its sought-for unitary diagonalization (14), which, incidentally, is proved in [RS1] as Lemma 2.3.5.

Corollary 4.6. *Let X be a compressible WH system with Zak matrix $\mathcal{Z}_X$ and dual Zak matrix $\widetilde{\mathcal{Z}}_X$. Then:*

(a) *X is a tight frame if and only if, after normalizing the generator of X, either $\widetilde{\mathcal{Z}}_X$ or $\mathcal{Z}_X$ (or both) is an orthogonal projector a.e. (equivalently, if the matrix spectrum of almost every fiber of the Zak or dual Zak matrix consists of at most 0 and 1).*

(b) *X is a fundamental tight frame if and only if $\widetilde{\mathcal{Z}}_X$ is a constant multiple of the identity a.e. on $\mathbb{R}^d \times \mathbb{R}^d$.*

(c) *X is orthonormal if and only if $\mathcal{Z}_X$ is the identity a.e. on $\mathbb{R}^d \times \mathbb{R}^d$.*

Proof: Since X is a tight frame if and only if $\|T_X\| = \|T_{X|}{}^{-1}\|^{-1}$, we obtain from Theorem 4.4 that the tightness of X is equivalent to the equality $\|\zeta_X\|_{L_\infty} = \|\zeta^-_{X|}\|_{L_\infty}$, which clearly amounts to the fact that $\mathcal{Z}_X(w,t)$ has only one eigenvalue, independent of (w,t), other than 0. This eigenvalue is the normalization constant appearing in (a). The statement with respect to $\widetilde{\mathcal{Z}}_X$ is proved similarly, by invoking Theorem 4.5, and recalling that tightness is also equivalent to the equality $\|T^*_X\| = \|T^*_{X|}{}^{-1}\|^{-1}$.

Statement (b) follows from (a), when combined with (b) of Theorem 4.5.

Statement (c) follows easily from the last part of Theorem 2.12. □

Theorem 4.5 , in particular, shows that, in order for X to be a fundamental frame, almost all dual Zak matrices $\widetilde{\mathcal{Z}}_X(w,t)$ must be invertible. In fact, as is already established in Theorem 1 of [ZZ], the frame property of X is irrelevant here: By invoking Lemma 2.3.5 of [RS1], we may unitarily diagonalize $\widetilde{\mathcal{Z}}_X$, using a unitary matrix with *measurable* entries. Thus, if the fibers $\{\widetilde{\mathcal{Z}}_X(w,t)\}_{(w,t)}$ of $\widetilde{\mathcal{Z}}_X$ are singular on a set of positive measure, one can easily construct a function f such that $T_X T^*_X f = 0$ (the fibers of f, $(Z^{\widetilde{K}\cap L}\widehat{f}(w+\delta,t))_{\delta\in\Delta}$ should each lie in the kernel of $\widetilde{\mathcal{Z}}_X(w,t)$ for a.e. (w,t), and the only reason we need Lemma 2.3.5 here is in order to synthesize the fiber kernels into a *measurable* function). However, $\widetilde{\mathcal{Z}}_X(w,t)$ is non-singular if and only if the pre-Gramian $PZ_X(w,t)$ is of rank $\#\Delta$, the later being possible only if

$$\#\Delta \le \#\Gamma \times \#\Phi.$$

Since $\#\Delta$ is the index of $\widetilde{K}\cap L$ in $\widetilde{K}$, and $\#\Gamma$ is the index of $\widetilde{K}\cap L$ in L, we see that

$$\frac{\#\Delta}{\#\Gamma} = \frac{|\widetilde{K}\cap L|}{|\widetilde{K}|}\frac{|L|}{|\widetilde{K}\cap L|} = \frac{|L|}{|\widetilde{K}|} = \frac{1}{\mathrm{den}\,(K,L)}.$$

Defining, naturally, the **density of** X as

$$\#\Phi \times \mathrm{den}\,(K,L),$$

we obtain the following result. The univariate case of this result can be found in [D1], and was proved there by a similar argument.

Corollary 4.7. *Let $X = (K,L)_\Phi$ be a compressible WH system. Then X is fundamental only if* $\mathrm{den}\, X \ge 1$.

Finally, we consider the problem of computing the dual frame of $X = (K,L)_\Phi$. We first study this problem when X is a *Riesz basis*, say, for $H \subset L_2(\mathrm{I\!R}^d)$. Let R be a map from Φ to $L_2(\mathrm{I\!R}^d)$. In order for $Y := (K,L)_{\mathrm{R}\Phi}$ to be bi-orthogonal with X, we need $T^*_Y T_X$ to be the identity, hence that $PZ^*_Y PZ_X$ is the identity matrix a.e. Furthermore, for a dual basis Y, the column-space of (almost) each $PZ_Y(w,t)$ should be the same as that of $PZ_X(w,t)$ (follows from the fact that $\ker T^*_X = \ker T^*_Y$). Thus, finding $PZ_Y(w,t)$ here is a standard finite-dimensional least squares problem, and we obtain that

$$PZ_Y(w,t) = \mathcal{Z}_X^{-1}(w,t) PZ_X(w,t), \qquad \text{for a.e. } (w,t); \tag{4.8}$$

(compare with Theorem 2.4.7 of [RS1]).

The situation is even simpler for a fundamental frame: with the understanding that inverting $\widetilde{\mathcal{Z}}_X$ amounts to the pointwise inversion of each of its fibers $\widetilde{\mathcal{Z}}_X(w,t)$, we realize that $\widetilde{\mathcal{Z}}_X^{-1}$ represents the inverse of the dual frame operator $T_X T_X^*$. Thus the Zak transform representation of the generators RΦ of the frame dual to X can be found by applying $\widetilde{\mathcal{Z}}_X^{-1}$ to the representation of Φ. Since the action $T_X T_X^*$ on f is represented in the form

$$\widetilde{\mathcal{Z}}_X(Z^{\widetilde{K}\cap L}\widehat{f}(\cdot+\delta,\cdot))_{\delta\in\Delta},$$

we see that (cf. (29) of [ZZ])

$$PZ_Y(w,t)=\widetilde{\mathcal{Z}}_X^{-1}(w,t)PZ_X(w,t),\quad \text{a.e.}. \tag{4.9}$$

In the case of a non-fundamental frame X, the same argument can be employed, but, alas, the pseudo-inverse of $\widetilde{\mathcal{Z}}_X$ should be computed. In the special case when $\widetilde{\mathcal{Z}}_X(w,t)$ is either non-singular or 0, the computation of the pseudo-inverse is avoided, and this explains why we do not have to tackle this problem in the sup-adjoint case.

4.2. Zak transform estimates

The Zak transform analysis had led us to the decomposition of the operators $T_X^*T_X$ and $T_X T_X^*$, for a compressible WH set, into the fibers $(\mathcal{Z}_X(w,t))_{w,t\in\mathbb{R}^d}$ and $(\widetilde{\mathcal{Z}}_X(w,t))_{w,t\in\mathbb{R}^d}$. In §3, we had exploited the Gramian fiberization in order to estimate the frame (Riesz) bounds. In the same manner, we may exploit here the Zak transform fiberization is order to derive analogous estimates. These estimates are collected below without further explanation (other than pointing to their §3 counterparts). One should observe that despite of the seemingly close relation between the Gramian estimates and the Zak transform estimates, there is practically substantial difference between the two: the information required for the Gramian estimates is readily available (i.e., values of either $\widehat{\varphi}$ or φ; we are tacitly assuming that φ is explicitly known, where "explicitly" might mean, e.g., that φ or $\widehat{\varphi}$ are given analytically); at the same time, the Gramian estimates are crude. In contrast, the Zak transform estimates require finer information (certain Zak transforms and subsequently Zak products), but award us with better estimates (particularly when the relevant Zak matrix is of small order).

As we did in §3.4, we assume here that X is the *principal compressible* $(K,L)_\varphi$. The compressibility of X is essential (otherwise, the Zak matrices remain of infinite order). The extension to non-principal systems is primarily notational. All Zak transforms of this subsection are computed with respect to the lattice $\widetilde{K}\cap L$. In fact, the only Zak transform which is required is that of $\widehat{\varphi}$, hence we set

$$g:=Z^{\widetilde{K}\cap L}\widehat{\varphi}.$$

As before, Γ and Δ are the quotient groups

$$\Gamma=L/(\widetilde{K}\cap L),\quad \Delta=\widetilde{K}/(\widetilde{K}\cap L).$$

It is, perhaps, worth noting that the Zak matrix and the dual Zak matrix are invariant under shifts by $k\in\widetilde{K}\cap L$ (in both variables); this observation entitles us to represent any $\gamma\in\Gamma$ or $\delta\in\Delta$ in any convenient way, as well as to switch between representers, as the arguments for proving the estimates below require.

Theorem 4.10. *Let $X = (K, L)_\varphi$ be a compressible PWH system. Let g, Γ, and Δ be as in the second paragraph of this subsection. Then X is a Bessel system if and only if g is essentially bounded. Furthermore, let $g_{\Delta,\gamma}$ be the function*

$$g_{\Delta,\gamma} := \sum_{\delta\in\Delta} g(\cdot + \gamma + \delta, \cdot)\overline{g}(\cdot + \delta, \cdot),$$

and let $g_{\Gamma,\delta}$ be the function

$$g_{\Gamma,\delta} := \sum_{\gamma\in\Gamma} g(\cdot + \delta + \gamma, \cdot)\overline{g}(\cdot + \gamma, \cdot).$$

Then:

(a)

$$|K|^{-1}\|(\sum_{\gamma\in\Gamma} |g_{\Delta,\gamma}|^2)\|_{L_\infty}^{1/2} \le \|T_X\|^2 = \|T_X^*\|^2 \le |K|^{-1}\|\sum_{\gamma\in\Gamma} |g_{\Delta,\gamma}|\|_{L_\infty}.$$

(b)

$$|K|^{-1}\|(\sum_{\delta\in\Delta} |g_{\Gamma,\delta}|^2)\|_{L_\infty}^{1/2} \le \|T_X\|^2 = \|T_X^*\|^2 \le |K|^{-1}\|\sum_{\delta\in\Delta} |g_{\Gamma,\delta}|\|_{L_\infty}.$$

Proof: We note that $|K|^{-1/2}g(w,t)$ is one of the entries of the pre-Zak matrix $PZ_X(w,t)$, and that, further, every entry of $PZ_X(w,t)$ is, up to a unit multiplicative constant, of the form $|K|^{-1/2}g(w',t')$ for (possibly other) w', t'. Since the pre-Zak matrix is of finite order, this proves, in view of Theorem 4.4, the fact that X is Bessel if and only if g is bounded.

The bounds asserted in parts (a) and (b) are proved by an argument analogous to that used in the proof of Theorem 3.21, after observing that $|K|^{-1}(g_{\Delta,\gamma})_{\gamma\in\gamma}$ comprise the entries of a typical row of $\mathcal{Z}_X$, while $|K|^{-1}(g_{\Gamma,\delta})_{\delta\in\Delta}$ comprise the entries of a typical row of $\widetilde{\mathcal{Z}}_X$. □

"Diagonal dominance" arguments similar to the ones employed in the proof of Theorem 3.27, when combined with the observations just made in the proof of Theorem 4.10, lead to sufficient conditions for X being a Riesz basis or a fundamental frame. We omit these details.

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