On Radon's recipe for choosing correct sites for multivariate polynomial interpolation

Dominik Stahl
dominik.stahl@itwm.fhg.de
Fraunhofer ITWM, Fraunhofer-Platz 1, 67663 Kaiserslautern, Germany

and

Carl de Boor
deboor@cs.wisc.edu; http://pages.cs.wisc.edu/∼deboor
Univ. Wisconsin-Madison, Department of Computer Sciences, POB 1076, Eastsound, WA 98245, US

28aug11

Abstract. A class of sets correct for multivariate polynomial interpolation is defined and verified, and shown to coincide with the collection of all correct sets constructible by the recursive application of Radon's recipe, and a recent concrete recipe for correct sets is shown to produce elements in that class.

Keywords: Interpolation; Polynomial; Multivariate

For $X \subset \mathbb{F}^s$ for some natural number s and with $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$, denote by

$$\flat(X) := \{\sum_{x\in X} x w(x) : \sum_{x\in X} w(x) = 1, \quad \#\operatorname{supp} w < \infty\}$$

the **affine hull of**, or the **flat spanned by**, X. Its dimension,

$$d_X := \dim \flat(X),$$

is the **affine dimension** of X and equals the dimension of the subspace $\flat(X) - x$ for any $x \in \flat(X)$.

(1) Definition. *Call the set $X \subset \mathbb{F}^s$ n-**correct** if there is, for every $a : X \to \mathbb{F}$, exactly one polynomial p of degree $\le n$ on $\flat(X)$ that agrees with a on X, i.e., satisfies $p(x) = a(x)$ for all $x \in X$. Equivalently, the map*

$$\Pi_{\le n}(\flat(X)) \to \mathbb{F}^X : p \mapsto p|_X := (p(x) : x \in X)$$

*is invertible. More explicitly, we call such a set (n, d_X)-**correct**, and observe that, provided $\#X \le \dim \Pi_{\le n}(\flat(X)) = \binom{n+d_X}{n}$, this is equivalent to the map $p \mapsto p|_X$ being 1-1 on $\Pi_{\le n}(\flat(X))$, i.e., $p \in \Pi_{\le n}(\flat(X))$ and $p|_X = 0$ implying $p = 0$.*

In [R48], Radon proposes (albeit only for the bivariate case) the following recipe for the construction of an (n, d)-correct set: In $\mathbb{F}^d$, choose an $(n, d-1)$-correct set Y and an $(n-1, d)$-correct set Z that has no intersection with the hyperplane $\flat(Y)$; then $Y \cup Z$ is (n, d)-correct. This recipe even works for $n = 1$ and arbitrary d as long as we interpret $(0, d)$-correctness to mean $(0, 0)$-correctness as we will do from now on.

In the present note, we present a characterization of (n, d)-correct sets obtained by the recursive application of Radon's recipe.

(2) Definition. *Denote by*

$$R_{n,d}$$

the collection of all $X \subset \mathbb{F}^s$ whose affine dimension is bounded by d and for which there is a map $\alpha \mapsto x_\alpha$ onto X from the set

$$\mathrm{A}_{n,d} := \{\alpha \in \mathbb{Z}_+^d : |\alpha| := \sum_j \alpha_j \le n\}$$

of multi-indices such that, for each

$$j \in 1{:}d \ := \ \{1, 2, \ldots, d\}$$

and each $\gamma \in \mathrm{A}_{n-1,j}$, $X = \{x_\alpha : \alpha \in \mathrm{A}_{n,d}\}$ *satisfies the following condition.*

Condition(γ, j): *The affine hull of*

$$Y_\gamma^j := \{x_\alpha \in X : \alpha_i = \gamma_i \ \text{ for } \ 0 < i \leq j\} \tag{3}$$

has only Y_γ^j *in common with*

$$X_\gamma^j := \{x_\alpha \in X : \alpha_i = \gamma_i \ \text{ for } \ 0 < i < j; \ \alpha_j \geq \gamma_j\}. \tag{4}$$

Note that Condition(γ, j) is satisfied in case there is a hyperplane containing Y_γ^j whose intersection with X_γ^j is Y_γ^j. Note also that there is no assumption that the map $\alpha \mapsto x_\alpha$ be 1-1, though this readily follows directly from the Condition(γ, j). Indeed, if $\alpha, \beta \in \mathrm{A}_{n,d}$ with $\alpha \neq \beta$, then there is a smallest j for which $\alpha_j \neq \beta_j$ and, assuming wlog that $\alpha_j < \beta_j$, then $\gamma := (\alpha_1, \ldots, \alpha_j)$ satisfies $|\gamma| < n$ and so, by Condition(γ, j), x_α must lie in some flat that does not contain x_β, therefore $x_\alpha \neq x_\beta$. Note finally that we require the affine dimension d_X of $X \in R_{n,d}$ to be bounded by d, yet it will follow from the definition that, for $n > 0$, necessarily $d_X = d$. In fact, the definition of $R_{n,d}$ is tailor-made for an inductive proof of the following claim.

(5) Proposition. *For* $n, d > 0$, $X \subset \mathbb{F}^s$ *is in* $R_{n,d}$ *if and only if* X *is constructible by recursive application of the Radon recipe. In particular,* $X \in R_{n,d}$ *is* (n, d)*-correct for* $n, d \geq 0$, *and* $d_X = d$ *for* $n > 0$.

Proof: The proof is by induction on n and d. For $n = 0$ or $d = 0$, any $X \in R_{n,d}$ consists of exactly one point, hence is evidently (n, d)-correct. Now assume $n, d > 0$ and let $X \in R_{n,d}$. Then X is the disjoint union of the two sets

$$Y := \{x_\alpha : \alpha_1 = 0, \alpha \in \mathrm{A}_{n,d}\}$$

and

$$Z := \{x_\alpha : \alpha_1 > 0, \alpha \in \mathrm{A}_{n,d}\}$$

with

$$\flat(Y) \cap X = Y,$$

hence $d_Y \leq d_X - 1 \leq d - 1$, while $d_Z \leq d_X \leq d$. Thus we know that X is obtainable by the recursive application of the Radon recipe once we know that each of Y and Z is so obtainable (or, else, contains just one point), and this we know by induction hypothesis once we show that $Y \in R_{n,d-1}$ and $Z \in R_{n-1,d}$. For this, we observe that Y satisfies the other requirements of being an $R_{n,d-1}$-set with the assignment

$$y_\alpha \leftarrow x_{(0,\alpha)}, \quad \alpha \in \mathrm{A}_{n,d-1},$$

while Z satisfies the other requirements for being in $R_{n-1,d}$ with the assignment

$$z_\alpha \leftarrow x_{\alpha+\epsilon}, \quad \alpha \in \mathrm{A}_{n-1,d},$$

with $\epsilon := (1, 0, 0, \ldots)$ of the appropriate length. Hence, by induction hypothesis, Y is $(n, d-1)$-correct, and $d_Y = d - 1$, therefore $d_X = d$, hence $\dim \Pi_{\leq n}(\flat(X)) = \#\mathrm{A}_{n,d} = \dim \mathbb{F}^X$. Therefore, we know that X is n-correct as soon as we have shown that the linear map $p \mapsto p|_X$ is 1-1 on $\Pi_{\leq n}(\flat(X))$. For this, if $p \in \Pi_{\leq n}(\flat(X))$ vanishes on X, therefore vanishes on Y, then it must vanish on all of $\flat(Y)$ by induction hypothesis, therefore, with h any polynomial of degree 1 on $\flat(X)$ vanishing on $\flat(Y)$, h must be a factor of p, i.e., $p = hq$ for some $q \in \Pi_{<n}(\flat(X))$ and since, by assumption, h fails to vanish anywhere on Z, therefore q must vanish on Z, hence must be identically zero by the induction hypothesis, therefore, finally, $p = 0$, showing that the linear map $\Pi_{\leq n}(\flat(X)) \to \mathbb{F}^X : p \mapsto p|_X$ is 1-1, hence invertible, therefore X is, indeed, (n, d)-correct, and obtainable by the recursive application of the Radon recipe, thus advancing the induction hypothesis.

If, on the other hand, X is an (n,d)-correct set obtainable by the recursive application of the Radon recipe, then $d_X = d$ and X must be the disjoint union of two sets Y and Z, with Y an $(n,d-1)$-correct set and Z an $(n-1,d)$-correct set, both obtainable by recursive application of the Radon recipe, and $\flat(Y)\cap Z = \emptyset$. By induction hypothesis, $Y \in R_{n,d-1}$ and $Z \in R_{n-1,d}$, hence, in terms of the appropriate indexing of the elements of Y and Z, we may index the elements of X thusly

$$x_\alpha := \begin{cases} y_{\alpha_{2:d}}, & \alpha_1 = 0; \\ z_{\alpha-\epsilon}, & \alpha_1 > 0; \end{cases}, \quad \alpha \in \mathrm{A}_{n,d},$$

(using the facts that $X = Y \cup Z$, $Y \cap Z = \emptyset$ to be sure that the resulting map $\mathrm{A}_{n,d} \to X : \alpha \mapsto x_\alpha$ is 1-1 and onto), and observe that X, so indexed, satisfies

(i) Condition$(0,1)$ by the Radon recipe;
(ii) Condition$((0,\gamma),j)$ for $1 < j \le d$ and $\gamma \in \mathrm{A}_{n-1,j-1}$ since that corresponds to the Condition$(\gamma, j-1)$ satisfied by Y;
(iii) Condition$(\gamma+\epsilon, j)$ for $1 \le j \le d$ and $\gamma \in \mathrm{A}_{n-2,j}$ since that corresponds to the Condition(γ, j) satisfied by Z.

In short, then $X \in R_{n,d}$, thus advancing the induction hypothesis. □

Since any affine map carries flats to flats, the set $R_{n,d}$ is closed under invertible affine maps of $\mathbb{F}^s$. The index set $\mathrm{A}_{n,d}$ as a subset of $\mathbb{F}^d$ is evidently in $R_{n,d}$ since, for any $j \in 1{:}d$ and $\gamma \in A_{n,j}$, $\{\alpha \in \mathrm{A}_{n,d} : \alpha_i = \gamma_i, i \in 1{:}j\}$ lies in the hyperplane $\{x \in \mathbb{F}^d : x_j = \gamma_j\}$ which does not contain any $\beta \in \mathrm{A}_{n,d}$ with $\beta_j > \gamma_j$.

More than that, any fully generalized principal lattice is in $R_{n,d}$. To see this, recall from [B09] the following definition.

(6) Definition. *A* **fully generalized principal lattice of degree** n *(or,* FGPL$_n$**-set** *for short) is a set* X *in* $\mathbb{F}^d$ *that can be so indexed as* $X = \{x_{(n-|\alpha|,\alpha)} : \alpha \in \mathrm{A}_{n,d}\}$ *that*

(7) $$\beta_r < n \quad \Longrightarrow \quad x_\beta \in H^r_{\beta_r}$$

and

(8) $$x_\beta \in H^j_i \quad \Longrightarrow \quad \beta_j \le i$$

hold for some collection $(H^j_i : i \in 0{:}(n-1), j \in 0{:}d)$ *of hyperplanes and all applicable* β, r, *and* i.

Let X be such an FGPL$_n$-set and let $x_\alpha := x_{(n-|\alpha|,\alpha)}$ for $\alpha \in \mathrm{A}_{n,d}$. Then, as a subset of $\mathbb{F}^d$, its affine dimension is bounded by d. Further, for any $j \in 1{:}d$ and $\gamma \in A_{n-1,j}$, the hyperplane $H^j_{\gamma_j}$ mentioned in the above definition (6) contains, according to (7), the set Y^j_γ defined in (3), since $H^j_{\gamma_j}$ contains every x_β with $\beta_j = \gamma_j$, hence also contains $\flat(Y^j_\gamma)$, but, according to (8), fails to contain any x_β with $\beta_j > \gamma_j$, therefore Condition(γ, j) holds. Thus, $X \in R_{n,d}$.

[CY77] introduced the more general notion of a GC$_n$**-set** as an n-correct set X for which, for each $x \in X$, the subset $X\backslash\{x\}$ is contained in the union of $\le n$ hyperplanes. It is known that every FGPL$_n$-set is a GC$_n$-set. If we knew that every GC$_n$-set were in $R_{n,d}$, then we would know that the outstanding Gasca-Maeztu conjecture from [GM82] were true which asserts that, in the bivariate case, there is, for every GC$_n$-set X, a straight line containing $n+1$ points from X. See [B07] for the state of this challenging conjecture as of 2006.

Here is a simple $R_{2,2}$-set X that fails to be a GC$_2$-set:

$$x_\alpha = \begin{cases} \alpha, & \text{if } \alpha_1 < 2 \\ (2,2), & \text{if } \alpha_1 = 2 \end{cases}, \quad \alpha \in \mathrm{A}_{2,2}.$$

Indeed, $X\backslash\{(0,0)\}$ fails to be contained in the union of two straight lines.

In [S10], a recipe for an (n,d)-correct set is given which can be described in the following terms.

(9) Definition. *Denote by*

$$S_{n,d}$$

the collection of all subsets X *of* $\mathbb{F}^d$ *that can be so indexed by* $\mathrm{A}_{n,d}$ *that, for every* $j \in 1{:}d$ *and every* $\alpha, \beta \in \mathrm{A}_{n,d}$*, if* $\alpha_i = \beta_i$ *for* $i < j$*, then* $(x_\alpha)_j = (x_\beta)_j$ *if and only if* $\alpha_j = \beta_j$*.*

Any $X \in S_{n,d}$ is shown in [S10] to be n-correct by an argument involving elimination and determinants. We show it here by the following

(10) Observation. $S_{n,d} \subset R_{n,d}$*. In particular, any* $S_{n,d}$*-set is* (n,d)*-correct.*

Proof: Let $X \in S_{n,d}$. Since $X \subset \mathbb{F}^d$, $d_X \le d$. Also, for $j \in 1{:}d$ and $\gamma \in \mathrm{A}_{n-1,j}$, the hyperplane $\{x \in \mathbb{F}^d : x_j = (x_{(\gamma,\beta)})_j\}$ with $\beta := 0 \in \mathbb{F}^{d-j}$ contains $x_\alpha \in X$ with $\alpha_i = \gamma_i$ for $i < j$ and $\alpha_j \ge \gamma_j$ if and only if $\alpha_j = \gamma_j$, hence Condition(γ, j) holds. Thus, $X \in R_{n,d}$. $\square$

References

[B07] C. de Boor (2007), "Multivariate polynomial interpolation: conjectures concerning GC-sets", *Numer. Algorithms* **45**, 113–125.

[B09] C. de Boor (2009), "Multivariate polynomial interpolation: Aitken-Neville sets and generalized principal lattices", *J. Approx. Theory* **161(1)**, 411–420.

[CY77] K. C. Chung and T. H. Yao (1977), "On lattices admitting unique Lagrange interpolations", *SIAM J. Numer. Anal.* **14(3)**, 735–743.

[GM82] M. Gasca and J. I. Maeztu (1982), "On Lagrange and Hermite interpolation in $\mathbb{R}^k$", *Numer. Math.* **39**, 1–14.

[R48] J. Radon (1948), "Zur mechanischen Kubatur", *Monatshefte der Math. Physik* **52(4)**, 286–300.

[S10] D. Stahl (2010), "Sets of interpolation points yielding unique filter coefficients for the lifting scheme in arbitrary dimensions", Fraunhofer ITWM, ms.