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# Characterizations of linear independence and stability of the shifts of a univariate refinable function in terms of its refinement mask

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**Abstract:** Characterizations of the linear independence and stability properties of the integer translates of a compactly supported univariate refinable function in terms of its mask are established. The results extend analogous ones of Jia and Wang which were derived for dyadic refinements and finite masks.

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# Characterizations of linear independence and stability of the shifts of a univariate refinable function in terms of its refinement mask

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## 1. The problem

Let $\phi$ be a compactly supported function in $L_2(\mathbb{R})$, $\widehat{\phi}$ being its Fourier transform. We say that $\phi$ *is refinable* (or more precisely 2-refinable) if there exists a $4\pi$-periodic function $A$ that satisfies

$$\widehat{\phi} = A\,\widehat{\phi}(\cdot/2). \tag{1.1}$$

The equation (1.1) is usually termed the **refinement equation**, and the function $A$ there is known as the **mask function**. Refinable functions are exploited in stationary wavelet decompositions of $L_2(\mathbb{R})$ via the idea of multiresolution as originated in [Ma] and [Me]; see also [CW2], [JM2] and the references therein. The definition given here is taken from [BDR3] (see also [JS]) and frees the mask function from various restrictions imposed on it in earlier definitions. The adjective "stationary" follows the terminology of [CDM] and distinguishes (1.1) from more general choices [BDR3] in which the two functions in the refinement equation need not be a scale of each other. The two main compactly supported families of refinable functions are the cardinal polynomial B-splines [CW1], and Daubechies' orthogonal scaling functions [D1].

For a given positive $n$, we consider in this paper **principal $n$-shift-invariant** (PSI$_n$ for short) subspaces of $L_2(\mathbb{R})$. By a **shift** we mean "an integer translate", and more generally, by $n$-shift we mean a $\mathbb{Z}/n$-translate. A closed subspace $S$ of $L_2(\mathbb{R})$ is **$n$-shift-invariant** if it is invariant under all possible $n$-shifts. Finally, such a space $S$ is a PSI$_n$ space if there exists $\phi \in L_2(\mathbb{R})$ whose $n$-shifts form a fundamental set in $S$ (i.e., the finite linear combinations of the $\mathbb{Z}/n$-translates of $\phi$ are dense in $S$). Such $\phi$ is termed an **$n$-generator** of $S$. In case $n = 1$, it is suppressed from all the terminology and notations above.

Given a refinable $\phi$, one considers the PSI space $S := S(\phi)$ generated by (the shifts of) $\phi$. While $\phi$ generates $S$ by definition, there are many other functions in $S$ which generate it. For example, every non-zero compactly supported $\phi_0 \in S$ generates $S$ (cf. [BDR2] for more details). For computational purposes, it is essential to find a favourable generator $\phi$ for $S$, where "favourable" is meant in the sense of satisfying any of the following three properties, which are listed in increasing difficulty of attainment.

**Definition 1.2.** *We say that the shifts of $\phi$ are:*

*(a)* **stable**, *if there exists $C > 0$ such that*

$$\| \sum_{j\in\mathbb{Z}} c(j)\phi(\cdot - j)\|_{L_2(\mathbb{R})} \ge C\|c\|_{\ell_2(\mathbb{Z})} \tag{1.3}$$

*for every finitely supported $c : \mathbb{Z} \to \mathbb{C}$;*

*(b)* **linearly independent**, *if the map*

$$\mathbb{C}^{\mathbb{Z}} \ni c \mapsto \sum_{j\in\mathbb{Z}} c(j)\phi(\cdot - j)$$

*is injective;*

*(c)* **orthogonal** *if the shifts of $\phi$ form an $L_2(\mathbb{R})$-orthogonal system.*

Somewhat loosely, we sometimes refer to $\phi$ as a stable (linearly independent, orthogonal) generator of $S$, and mean by that that the shifts of the generator $\phi$ for $S$ are stable (linearly independent, orthogonal).

While there are satisfactory characterizations of each of the above properties of $\phi$ in terms of $\widehat{\phi}$, it is desirable also to characterize the stability, independence and orthogonality of the generator in terms of the mask $A$: in many instances the mask is the readily available data, while $\widehat{\phi}$ is only known as an infinite product of the mask and its dilates.

**Problem 1.4.** *Given the mask $A$ of a compactly supported 2-refinable function $\phi$ characterize the properties of stability, linear independence, and orthogonality of the shifts of $\phi$ in terms of the mask $A$.*

The interesting paper [JW] solves the above problem completely, and two of its three main results are quoted in §5 of the present paper. Furthermore, [JW] contains a carefully detailed discussion of earlier contributions to the orthogonality problem made by Meyer [Me], Daubechies [D1,2], Cohen [C] and Mallat [Ma] (cf. §1 of [JW] for more details). My interest in this problem was initiated with the reading [JW] and was stimulated by a surprising asymmetry in the Jia-Wang results. A discussion of this point together with an explanation for the source of that phenomenon is contained in §5.

A main tool employed by Jia and Wang in their analysis was part (a) of Result 3.2 below. Alternatively, I tried to understand their characterizations with the aid of part (b) of that result. This, in turn, led to seemingly new characterizations of the stability and linear independence, and the attempt to draw the connection between the characterizations of [JW] and these here sheds new light on former results. Also, the characterizations here are proved to be valid in a setting more general than the one considered in [JW]. The generalization is done in two different directions: (i) while [JW] preassumes the mask $A$ to be a trigonometric polynomial, we will not impose any a-priori restriction on $A$; (ii) the new characterizations immediately extend to an $n$-refinable $\phi$. An analogous extension of the Jia-Wang characterization (given in §5, too) is shown to be more involved.

Our two main results are stated in §2, and various (mostly known) facts concerning shift-invariant spaces and the stability and linear independence properties are presented in §3. The proofs of the main results are then provided in §4, while in §5 we show how to derive Jia-Wang characterizations from our characterizations, and how to extend the former characterizations to $n$-refinable functions. We finally provide in §6 an algorithm that, given an arbitrary mask of a compactly supported $n$-refinable (unknown) function $\phi$, finds the mask $A'$ of the linearly independent generator of $S(\phi)$ (whose existence is guaranteed by Result 3.2).

It is worth noting (as was observed by one of the referees), that the theorems of section 2 could have equally derived from their counterparts in section 5, when combined with the tools of section 3 (i.e., the course of development in this paper could have been reversed). However, it is my opinion that the approach taken here towards Problem 1.4 is the most natural one, and that the novelty in that approach is the main justification for the existence of the present paper.

I take this opportunity to thank the referees for their constructive suggestions concerning this article. In particular, they brought to my attention the two papers [Z] and [Zh], which (independently) consider questions similar to those of the present paper. Specifically, Theorem 1 in [Z] establishes a result similar to that of Theorem 5.5 here, but for exponentially decaying functions and not only for compactly supported ones. Further, Theorem 5 of [Z] is similar to Theorem 5.4 here, only that it assumes the mask $A$ to be a trigonometric polynomial, while we allow here a rational trigonometric polynomial. The arguments in [Z] follow in general those of [JW].

We emphasize that the terminology "compactly supported" should always be interpreted in this paper as "compactly supported non-zero". $\mathbb{T}$ stands for the unit circle. The notion of a "trigonometric polynomial" is used here in a broad sense: it means a finite linear combination of pure exponentials whose frequences are chosen from some lattice $\mathbb{Z}^d/n$.

## 2. Main Results

Let $\phi \in L_2(\mathbb{R})$ be compactly supported, and let $n > 1$ be an integer. $\phi$ is said to be **$n$-refinable** if there exists a $2\pi n$-periodic function $A$ such that

$$\widehat{\phi} = A\,\widehat{\phi}(\cdot/n). \tag{2.1}$$

No restriction on $A$ is imposed here, but, still, $A$ must be a rational trigonometric polynomial (see Proposition 3.8).

The stability and linear independence properties rely on the distribution of the zeros of $A$. To avoid the redundancy induced by the periodicity of $A$, one may consider, alternatively, the **symbol** $a$ of the refinement equation defined by the equation

$$a(e^{i\omega/n}) := A(\omega). \tag{2.2}$$

By the above discussion $a$ is a rational algebraic polynomial.

The main results of this paper characterize the stability and linear independence of the shifts of an $n$-refinable $\phi$ in terms of the symbol $a$. While the theorems below are stated for $\phi \in L_2(\mathbb{R})$, we note that the results are valid for a compactly supported distribution $\phi$ (with the same proofs) if we modify appropriately the definition of stability (i.e., if we *define* stability to mean that $\widehat{\phi}$ does not have a $2\pi n$-periodic zero on $\mathbb{R}^d$).

**Theorem 2.3.** *Suppose that $\phi \in L_2(\mathbb{R})$ is compactly supported and $n$-refinable with mask $A$ and symbol $a$. Then, the shifts of $\phi$ are linearly independent if and only if the following two conditions hold:*
*(a)* $a(1) = 1$.
*(b) If $a$ can be written in the form*

$$a(z) = a_0(z)\frac{b(z^n)}{b(z)}, \tag{2.4}$$

*for some Laurent polynomials $a_0, b$, then the only possible zeros of $b$ are 0 and 1.*

**Theorem 2.5.** *Let $\phi$, $A$ and $a$ be as in the previous theorem. Then, the shifts of $\phi$ are $L_2$-stable if and only if the following two conditions hold:*
*(a)* $a(1) = 1$.
*(b) If $a$ can be written in the form*

$$a(z) = a_0(z)\frac{b(z^n)}{b(z)}, \tag{2.6}$$

*for some Laurent polynomials $a_0, b$, then $b$ vanishes nowhere on $\mathbb{T}\backslash\{1\}$.*

## 3. Background

We collect here several results which are relevant to our topic. Some of these results are needed for proofs and discussions in subsequent sections and the rest can be regarded as general background. For the readers' benefit, results which are not specifically univariate are stated in their natural multivariate setting. (The multivariate extension of the notions and definitions from the introduction is, by large, self-understood, and can be found e.g., in [JM2] and [BDR2,3].)

We start the discussion with a quick overview of stability and linear independence. Generally speaking, the issues of stability and linear independence of the shifts of one (or several) compactly supported (or not) function(s) received close attention in the spline and wavelet literature (e.g., there are at least 15 papers that discuss the linear independence problem for box splines). Results that deal with special choices of generators are not needed here, hence are not mentioned. We do mention in passing that general discussion of the non-compactly supported case can be found in [JM2] and [BDR2], and that results concerning the case of several compactly supported generators are obtained in [JM1].

The basic result concerning the independence of the shifts of a compactly supported $\phi$ was obtained in [R1] (see also [DM1]):

**Result 3.1.** *Let $\phi$ be a compactly supported distribution defined on $\mathbb{R}^d$; let $\widehat{\phi}$ be the analytic extension of its Fourier transform. Then, the $n$-shifts of $\phi$ are linearly independent if and only if $\widehat{\phi}$ does not possess in $\mathbb{C}^d$ any $2\pi n$-periodic zeros, i.e., $\widehat{\phi}$ vanishes identically on no set of the form $\theta + 2\pi n\mathbb{Z}^d$, $\theta \in \mathbb{C}^d$.*

Stronger results concerning linear independence are valid for $d = 1$. We record two of them (which are taken from [R1] and [R2] respectively) in the next statement. The first of which was an important tool in the approach taken by Jia and Wang in [JW]. The second one is our main tool in this paper.

**Result 3.2.** *Let $\phi$ and $\widehat{\phi}$ be as in Result 3.1, assume that $d = 1$, and let $n > 0$ be given. Then:*

*(a) $\widehat{\phi}$ has only a finite number of $2\pi n$-periodic zeros.*

*(b) There exists a compactly supported distribution $\phi_0$ whose $n$-shifts are linearly independent, such that every compactly supported distribution of the form $\sum_{j\in\mathbb{Z}} c(j)\phi(\cdot - j)$ (in particular $\phi$ itself) is a finite linear combination of the $n$-shifts of $\phi_0$. Furthermore, $\phi_0 \in L_2(\mathbb{R})$ if $\phi$ is so, and in that case, the PSI$_n$ space $S$ that the $n$-shifts of $\phi$ generate, is identical with the one generated by the $n$-shifts of $\phi_0$.*

We refer to [BDR2] for a further elaboration of (b) above. For the purposes here, we need to convert (b) of Result 3.2 to the Fourier domain. The fact that $\phi$ is finitely generated by $n$-shifts of $\phi_0$ is equivalent to the representation

$$\widehat{\phi} = B\widehat{\phi}_0, \tag{3.3}$$

for some $2\pi n$-periodic trigonometric polynomial. Because of the linear independence property of $\phi_0$, $\widehat{\phi}_0$ does not have any $2\pi n$-periodic zeros, and the $2\pi n$-periodicity of $B$ then implies the following:

**Corollary 3.4.** *Let $\phi$ be a univariate compactly supported distribution. Then, for every positive $k$, there exists a compactly supported distribution $\phi_0$ and a $2\pi k$-periodic trigonometric polynomial $B$ such that*

*(i)* *The $k$-shifts of $\phi_0$ are linearly independent;*
*(ii)* $\widehat{\phi} = B\widehat{\phi_0}$*;*
*(iii)* *the $2\pi k$-periodic zeros of $\widehat{\phi}$ coincide with the zeros of $B$.*

Note that, incidently, the last corollary show how (a) of Result 3.2 can be derived from (b) there.

The next result collects several characterizations of stability of a compactly supported generator $\phi$.

**Result 3.5.** *Let $\phi$ be a compactly supported $L_2(\mathbb{R}^d)$-function. Then the following conditions are equivalent*
*(a)* *The $n$-shifts of $\phi$ are stable.*
*(b)* $\widehat{\phi}$ *has no real $2\pi n$-periodic zero.*
*(c)* *The kernel of the map*

$$\mathbb{C}^{\mathbb{Z}^d} \ni c \mapsto \sum_{j\in\mathbb{Z}^d} c(j)\phi(\cdot - j)$$

*contains no bounded non-zero sequences.*
*(d)* *The above kernel contains no tempered non-zero sequences.*

The equivalence of (a) and (b) is mentioned in [SF] and proved in [DM2]. A proof of the equivalence of (b), (c), and (d) can be found in [R1].

For univariate functions, the understanding of the stability issue is facilitated by (b) of Result 3.2:

**Corollary 3.6.** *Let $\phi$, $n$, $\phi_0$, and $B$ be as in Corollary 3.4. Assume further that $\phi \in L_2(\mathbb{R})$. Then the $n$-shifts of $\phi$ are stable if and only if $B$ has no real zero.*

**Proof:** By Corollary 3.4, the $2\pi n$-periodic zeros of $\widehat{\phi}$ coincide with the zeros of $B$, and hence $B$ has no real zeros if and only if $\widehat{\phi}$ has no $2\pi n$-periodic real zeros. Now apply the equivalence of (a) and (b) in Result 3.5. □

In the second part of this section we present selected results on PSI spaces.

We first recall the following result from [BDR1]:

**Result 3.7.** *Let $S$ be a PSI$_n$ space, and let $\phi$ be a generator of $S$. Then, for $f \in L_2(\mathbb{R}^d)$, $f \in S$ if and only if there exists a $2\pi n$-periodic function $A$ such*

$$\widehat{f} = A\widehat{\phi}, \quad \text{a.e.}$$

Our next interest is in the nature of $A$ in Result 3.7. We show below that $A$ is always a rational trigonometric polynomial provided that $\phi$ and $f$ are compactly supported, and that, further, $A$, in times, is guaranteed to be a polynomial.

**Proposition 3.8.** *Assume that $\phi$ and $f$ are compactly supported distributions, and that $\widehat{f} = A\widehat{\phi}$ for some $2\pi n$-periodic $A$. Then $A$ is a rational trigonometric polynomial. If, further, the $n$-shifts of $\phi$ are linearly independent, $A$ is necessarily a trigonometric polynomial.*

**Proof:** The second assertion of the proposition is equivalent to the statement that $f$ is a finite linear combination of the $n$-shifts of $\phi$, with the latter statement implied by Theorem 1.3 of [BR], as observed in [JM2].

To prove the first assertion, we multiply both sides of the equation $\widehat{f} = A\widehat{\phi}$ by a band-limited rapidly decaying function $\sigma$ and sum all the $2\pi n\mathbb{Z}^d$-translates of the resulting equation to obtain

$$\sum_{\alpha\in 2\pi n\mathbb{Z}^d} (\widehat{f}\sigma)(\cdot+\alpha) = A \sum_{\alpha\in 2\pi n\mathbb{Z}^d} (\widehat{\phi}\sigma)(\cdot+\alpha). \tag{3.9}$$

We observe that both $\widehat{f}\sigma$ and $\widehat{\phi}\sigma$ are in $L_1(\mathbb{R}^d)$ and are Fourier transforms of compactly supported functions. At the same time, a standard application of Poisson's summation formula shows that $\sum_{\alpha\in 2\pi n\mathbb{Z}^d} \widehat{g}(\cdot+\alpha)$ is $L_1$-convergent to a trigonometric polynomial, whenever $g$ is compactly supported and $\widehat{g} \in L_1(\mathbb{R}^d)$. Thus, the two sums in (3.9) are trigonometric polynomials, and hence $A$ is a rational trigonometric polynomial. $\square$

Finally, we invoke Theorem 2.4 of [JM2] to conclude the following:

**Proposition 3.10.** *Assume that the compactly supported $L_2(\mathbb{R}^d)$-function $\phi$ satisfies the $n$-refinement equation*

$$\widehat{\phi} = A\,\widehat{\phi}(\cdot/n).$$

*Assume further, that $A$ is continuous at the origin. Then either $A(0) = 1$, or $\widehat{\phi}$ vanishes on $2\pi\mathbb{Z}^d$ (and hence, by Result 3.5, its shifts are not stable).*

**Proof:** We want to use Theorem 2.4 of [JM2]. This theorem, which is stated for a 2-refinable $\phi$, extends *verbatim* to an $n$-refinable $\phi$. It assumes that $\phi \in L_1(\mathbb{R}^d)$, an assumption that our $\phi$, being compactly supported and square-summable, satisfies. It also assumes that the Fourier coefficients of $A$ are summable, but a closer look at the proof there reveals that the only property of $A$ which is used is its continuity at the origin. Thus we are entitled to invoke that theorem.

The theorem tells us that $\widehat{\phi}$ vanishes on the lattice $2\pi\mathbb{Z}^d$, with the possible exception of the origin. However, if $A(0) \neq 1$, then, substituting 0 into the refinement equation, we obtain that $\widehat{\phi}(0) = 0$, making $\widehat{\phi}$ vanishing on the full lattice $2\pi\mathbb{Z}^d$. $\square$

## 4. Proofs of main results

The following result is straightforward, but, nonetheless, is the key for the proof of Theorems 2.3 and 2.5.

**Theorem 4.1.** *Let $\phi$ be an $n$-refinable compactly supported $L_2(\mathrm{I\!R})$-function, with mask $A$ and symbol $a$. Let $\phi_0$ and $B$ be as in Corollary 3.4 for $k = 1$. Then $\phi_0$ is also $n$-refinable, its mask $A_0$ is a trigonometric polynomial, and its symbol $a_0$ is a Laurent polynomial that satisfies*

$$a_0(z) = a(z)\frac{b(z)}{b(z^n)}, \tag{4.2}$$

*with the Laurent polynomial $b$ being the symbol of $B(\cdot/n)$, i.e.,*

$$b(e^{i\omega}) := B(\omega).$$

**Proof:** From Corollary 3.4, we know that

$$\widehat{\phi} = B\widehat{\phi}_0,$$

hence also

$$\widehat{\phi}(\cdot/n) = B(\cdot/n)\widehat{\phi}_0(\cdot/n).$$

Note that $B$ here is $2\pi$-periodic, hence that $B(\cdot/n)$ is $2\pi n$-periodic. A combination of the above two equations with the refinement equation $\widehat{\phi} = A\widehat{\phi}(\cdot/n)$, provides us the relation

$$A\,B(\cdot/n)\,\widehat{\phi}_0(\cdot/n) = B\,\widehat{\phi}_0,$$

which implies that $\widehat{\phi}_0$ is refinable with $2\pi n$-periodic mask

$$A_0 := A\frac{B(\cdot/n)}{B},$$

an equality which is equivalent to (4.2).

It remains to show that $a_0$ is a Laurent polynomial or equivalently that $A_0$ is a trigonometric polynomial. The $n$-refinability of $\phi_0$ implies (by Result 3.7) that $\phi_0$ lies in the space generated by the $n$-shifts of $\phi(n\cdot)$ (we are using here the fact that, up to a constant, $\widehat{\phi}(\cdot/n)$ is the Fourier transform of $\phi(n\cdot)$.) On the other hand, the linear independence of the shifts of $\phi_0$ trivially implies the linear independence of the $n$-shifts of $\phi_0(n\cdot)$. Thus, in the equation

$$\widehat{\phi}_0 = A_0\,\widehat{\phi}_0(\cdot/n),$$

$\widehat{\phi}_0$ is the Fourier transform of a compactly supported function, and $\widehat{\phi}_0(\cdot/n)$ is the Fourier transform of a compactly supported function whose $n$-shifts are linearly independent. Result 3.8 (and also Result 3.2 (b)) then implies that $A_0$ is a polynomial. $\square$

**Proof of Theorem 2.3.** We first prove the sufficiency claim of the theorem. Letting $\phi_0$, $A_0$, $a_0$, and $b$ be as in Theorem 4.1, that theorem implies that

$$a(z) = a_0(z)\frac{b(z^n)}{b(z)}, \tag{4.3}$$

and that $a_0$ and $b$ are Laurent polynomials. Since the shifts of $\phi_0$ are linearly independent, Proposition 3.10 implies that $A_0(0) = 1$, or equivalently, $a_0(1) = 1$. Since we assume (a), it follows that

$$\lim_{z\to 1} \frac{b(z^n)}{b(z)} = 1.$$

Since the above limit is $n^k$, with $k$ being the multiplicity of the root 1 of $b$, we conclude that $b(1) \neq 0$.

Now, if the shifts of $\phi$ are linearly dependent, then, by Corollary 3.4, $B$ above must vanish somewhere, or, equivalently, $b$ vanishes at some point $\theta \in \mathbb{C}\backslash 0$. Since $b(1) \neq 0$, we must have $\theta \neq 0, 1$, and thus (4.3) is a factorization of $a$ that violates condition (b). The sufficiency is thus established.

The fact that condition (a) is necessary follows from Proposition 3.10. It remains to show that condition (b) is necessary as well. For this, we assume that we are given a factorization of $a$ that violates condition (b). We will show that the shifts of $\phi$ are linearly dependent. Such a factorization gives rise to a factorization of the mask $A$ of the form

$$A = A_0 \frac{B}{B(\cdot/n)}, \tag{4.4}$$

with $B$ (respectively $A_0$) being $2\pi$-periodic (respectively $2\pi n$-periodic) trigonometric polynomial. Since $b$ vanishes at some $z \neq 0, 1$, $B$ vanishes at a point $\theta \notin 2\pi\mathbb{Z}$. We will show that $\widehat{\phi}$ vanishes on $\theta + 2\pi\mathbb{Z}$. From this, the linear dependence will follow by Result 3.1.

Iterating with the refinement equation, we see that, for any positive integer $k$,

$$\widehat{\phi} = \prod_{j=0}^{k-1} A(\cdot/n^j)\, \widehat{\phi}(\cdot/n^k),$$

and substituting (4.4) here, we obtain that

$$\widehat{\phi} = \tau_k \frac{B}{B(\cdot/n^k)}\, \widehat{\phi}(\cdot/n^k), \tag{4.5}$$

with $\tau_k$ a trigonometric polynomial. Since $B$ is $2\pi$-periodic and vanishes at $\theta$, it also vanishes everywhere on $\theta + 2\pi\mathbb{Z}$. Thus, given any $\alpha \in \theta + 2\pi\mathbb{Z}$, (4.5) proves that $\widehat{\phi}(\alpha) = 0$, unless $B(\alpha/n^k) = 0$. But since $k$ is arbitrary, the assumption $\widehat{\phi}(\alpha) \neq 0$ implies that $B(\alpha/n^k) = 0$, for *every* positive integer $k$, which is impossible, since by the choice of $\theta$ we know that $\alpha \neq 0$. Thus, indeed, $\widehat{\phi}$ vanishes everywhere on $\theta + 2\pi\mathbb{Z}$. This completes the proof of the necessity part, thereby the proof of the entire theorem. □

**Proof of Theorem 2.5.** The proof closely follows that of the previous one, and all the major steps needed here were already prepared there.

We first prove the sufficiency part. For that, we let $\phi_0$, $A_0$, $B$, $a_0$ and $b$ be as in the first part of the proof of Theorem 2.3. The proof there provides us with the factorization $a(z) = a_0(z)b(z^n)/b(z)$, and invokes (a) to prove that $b(1) \neq 0$. Further, condition (b) asserts that $b$ cannot vanish on $\mathbb{T}\backslash 1$, and, consequently, $b$ vanishes nowhere on $\mathbb{T}$. This last condition is equivalent to $B$ having no real zeros, and an application of Corollary 3.6 shows that the shifts of $\phi$ are stable.

Assuming that the shifts of $\phi$ are stable, condition (a) follows (as before) from Proposition 3.10. We prove the validity of condition (b) by contradiction: we assume that condition (b) is invalid, and prove that the shifts of $\phi$ are unstable. Let $a(z) = a_0(z)\frac{b(z^n)}{b(z)}$ be a factorization the violates condition (b). Following the argument used in the second part of the proof of the previous theorem, we see that every zero $e^{i\theta} \neq 1$ that $b$ has, is translated to the existence of a $2\pi$-periodic zero $\theta$ for $\widehat{\phi}$. Since $b$ is known to vanish at some $z \in \mathbb{T}\backslash 1$, $\widehat{\phi}$ had a real $2\pi$-periodic zero, and hence, by Result 3.5, the shifts of $\phi$ are unstable. □

## 5. Jia-Wang characterizations revisited and extended

The following two theorems were proved in [JW]:

**Result 5.1.** *Assume that $\phi \in L_2(\mathbb{R})$ is compactly supported and 2-refinable with respect to a trigonometric polynomial mask $A$ and corresponding symbol $a$. Assume also that $a(1) = 1$. Then the shifts of $\phi$ are linearly independent if and only if the following two conditions hold:*

*(a) $a$ does not have symmetric zeros, i.e., if $a(r) = 0$ then $a(-r) \neq 0$, all $r \in \mathbb{C}\backslash 0$.*
*(b) For any odd integer $m > 1$ and a primitive $m$th root $\xi$ of unity, there exists an integer $d$, such that $a(-\xi^{2^d}) \neq 0$.*

**Result 5.2.** *Let $\phi$, $A$, and $a$ be as in the previous theorem. Then the shifts of $\phi$ are stable if and only if the following two conditions hold:*

*(a) $a$ does not have symmetric zeros on the unit circle.*
*(b) Same condition as (b) of the previous theorem.*

They also invoked results of Cohen and Daubechies to derive from Result 5.2 a characterization of the orthogonality property of the shifts.

In the first part of the section we show how Result 5.1 can be explained via the result of Theorem 2.3. Result 5.2 can be approached very similarly. In the second part of the section, we discuss suitable generalizations of the [JW] results to $n$-refinable functions.

One should notice in the two theorems an apparent asymmetry between symmetric roots on the unit circle and other symmetric roots. Precisely, if we regard the existence of a symmetric zero in $a$ as the standard trace of linear dependence, then only *real* $2\pi$-periodic zeros of $\widehat{\phi}$ can escape without leaving that trace behind. Such a phenomenon deserves a closer look.

We let $a$, $a_0$, and $b$ be as in Theorem 4.1 with respect to $n = 2$. In particular,

$$a(z) = a_0(z)\frac{b(z^2)}{b(z)}. \tag{5.3}$$

Recall that $a_0$ and $b$ are known to be (Laurent) polynomials. But there is no reason to believe that $a$ should be a polynomial. *In fact, the assumption that $a$ is a polynomial excludes most of the compactly supported functions $\phi$ whose shifts are linearly dependent, among which many might still have stable shifts.* Off-hand, one might guess that the asymmetry between the real roots and the other roots of the mask (mentioned in the previous paragraph) is due to the fact that all the cases of rational masks were not covered in the [JW] study. This, however, is not true: if $a$ is properly rational, it must have a symmetric zero. The truth is straightforward: the lack of symmetric zeros leaves only very little room for linear dependence. Here are the details (compare with Lemma 1 of [JW]).

Suppose that we are given the linearly independent $\phi_0$ (of Corollary 3.4 and Theorem 4.1) and look for a trigonometric polynomial $B$ such that the symbol $a$ corresponding to the resulting $\phi$ (defined by $\widehat{\phi} = B\widehat{\phi}_0$) is a polynomial. Considering the relation (5.3) that $a$ satisfies, there are many ways to achieve such a goal, for example, we can choose $B$ such that $b$ divides the original $a_0$. However, almost all such choices leaves in $a$ a factor $z^2 - r$ of $b(z^2)$, which means that $a$ has a symmetric zero or equivalently that the mask $A$, hence $\widehat{\phi}$, has a $2\pi$-periodic zero. In such a case, the dependence assertion for the shifts of $\phi$ is evident from Result 3.1 (in this context see the discussions in [CW2], and in §5 of [BDR3]).

The only way to avoid symmetric zeros, is that during the cancellation of the various factors of $b(z)$, at least one factor from the two of each factor $z^2 - r$ of $b(z^2)$ is canceled. This implies that all factors of $b$ must be employed for such cancellation, and that no two of them are canceled against the same $z^2 - r$ factor (otherwise, there are not enough of them left to complete the job). This means that the polynomial $b$ must satisfy the following exceptional property: with $r_1, r_2, ..., r_k$ being the roots of $b$, there exists a permutation $\sigma$ of $1, ..., k$ such that $z - r_j$ divides $z^2 - r_{\sigma j}$, $\forall j$. Since we are assuming linear dependence, we know that $b$ must vanish at a point other than $0, 1$, which means that $\sigma$ cannot be the identity. Thus $\sigma$ has at least one cycle of length $m > 1$, which involves, without loss, the first $m$ roots, and in their previous order. In other words, the first $m$ roots are of the form

$$t, t^2, t^4, ..., t^{2^{m-1}}, \quad t^{2^m} = t.$$

Thus we immediately conclude that $t$ is a root of unity of order $2^m - 1$, and that $a$ must vanish at all the points

$$-t, -t^2, -t^4, ...., -t^{2^{m-1}},$$

which are roots of $b(z^2)$ that are not canceled by $b$.

The above observations can be combined to produce a proof of the sufficiency part of Theorem 5.1, with one minor change: instead of checking all primitive roots of unity of odd order, the condition we obtain requires the checking of all roots of unity of order $2^k - 1$, $k$ integer.

As a matter of fact, a similar argument allows us to provide extensions of Results 5.1 and 5.2 from $n = 2$ to a general $n$, and without assuming the polynomiality of the symbol. The statements and proofs of these results occupy the rest of this section.

**Theorem 5.4.** *Let $\phi$ be a compactly supported $n$-refinable $L_2(\mathbb{R})$-function, with a mask $A$ and symbol $a$. Then the shifts of $\phi$ are linearly independent if and only if the following three conditions hold:*

*(a)* $a(1) = 1$.

*(b)* $a(z)$ *is not divisible by polynomials of the form* $z^n - r$, $r \in \mathbb{C}\backslash 0$.

*(c) Given any positive integer $m$, and any root of unity $\xi \neq 1$ of order $n^m - 1$, there exists an $n$th root of unity $\zeta \neq 1$ and an integer $0 \leq \ell \leq m-1$ such that $a(\zeta\xi^{n^\ell}) \neq 0$.*

Note that we are not assuming $a$ to be a polynomial; still, it must be a rational polynomial. Correspondingly, any divisibility properties should be understood as referring to the numerator of the reduced form of $a$. Note also that, for $n = 2$, $\zeta$ is necessarily $-1$.

**Proof:** By Theorem 2.3, the condition $a(1) = 1$ is necessary for linear independence, hence can be assumed without loss.

Further, the condition "$a(z)$ is divisible by $z^n - r$, $r \in \mathbb{C}\backslash 0$" implies that $A$ vanishes on $\theta + 2\pi\mathbb{Z}$, with $e^{i\theta} = r$, and hence, by Result 3.1, implies also the linear dependence of the shifts of $\phi$. Thus, this condition is necessary, too, and we can assume that it holds.

It remains to show that, assuming (a) and (b), condition (c) is equivalent to the linear independence. Suppose that condition (c) does not hold, and let $\xi$ be the number that violates it. Defining

$$b(z) := \prod_{j=1}^{m}(z - \xi^{n^j}),$$

one checks that the *polynomial*

$$\frac{b(z^n)}{b(z)}$$

divides $a$, since the roots of this polynomial are exactly the numbers of the form $\zeta\xi^{n^\ell}$ mentioned in condition (c). This means that we had found a factorization $a(z) = a'(z)b(z^n)/b(z)$ with $a'$ and $b$ being Laurent polynomials. Invoking Theorem 2.3, we conclude that the shifts of $\phi$ are linearly dependent.

Now assume that the shifts of $\phi$ are linearly dependent and let $a(z) = a_0(z)b(z^n)/b(z)$ be the factorization provided by Theorem 2.3. Let $r_1, r_2, ..., r_k$ be the roots of $b$ ordered in any fashion. Since factors of the form $z^n - r$ do not exist in $a(z)$, $b(z^n)$ must contain at least $k$ linear factors which appear also in $b$, one per each factor of the form $z^n - r$. This means that $a$ is a polynomial and (as in the case $n = 2$) that there is a permutation $\sigma$ of the numbers $1, 2, ..., k$ such that $r_j^n = r_{\sigma j}$. Again, we choose a cycle of this permutation which does not consist of 0 or 1 alone (such a cycle exists since $b$ vanishes at points other than $0, 1$), and assume without loss that this cycle consists of the first $m$ roots, and in their present order. We denote $\xi := r_1$. Since $\xi^{n^m} = \xi$, $\xi$ is a root of unity of order $n^m - 1$, and certainly $\xi \neq 1$. Now, we know that the polynomial

$$\frac{\prod_{l=0}^{m-1}(z^n - \xi^{n^l})}{\prod_{l=0}^{m-1}(z - \xi^{n^l})}$$

divides $a$, and hence $a$ vanishes at all the roots of this polynomial. But the roots of this polynomial are exactly all the numbers of the form $\zeta\xi^{n^l}$ specified in condition (c) with respect to the present $\xi$ and $m$. Hence, condition (c) is violated. $\square$

**Theorem 5.5.** *Let $\phi$ be a compactly supported $n$-refinable $L_2(\mathbb{R})$-function, with mask $A$ and symbol $a$. Then the shifts of $\phi$ are stable if and only if the following three conditions hold:*

*(a)* $a(1) = 1$.

*(b)* *$a(z)$ is not divisible by polynomials of the form $z^n - r$, $r \in \mathbb{T}\backslash 1$.*

*(c) The same as (c) of Theorem 5.4.*

**Proof:** One possible proof can be obtained by modifying the arguments in the proof of the previous theorem. Instead, we give a proof that *employs* Theorem 5.4.

Assume first that the shifts of $\phi$ are stable. The necessity of $a(1) = 1$ was proved before. Let $\phi_0$ and $B$ be as in Corollary 3.4 (for $k = 1$) has no real zeros, hence, with $b$ as in Theorem 4.1, $b$ has no zeros on $\mathbb{T}$, and the same thus holds for $b(z^n)$. Further, by the same theorem,

$$a(z) = a_0(z)\frac{b(z^n)}{b(z)},$$

with $a_0$ being the symbol of $\phi_0$. We conclude that the $\mathbb{T}$-zeros of $a$ and $a_0$ are the same. Since $a_0$ is the symbol of the linearly independent $\phi_0$, Theorem 5.4 implies that $a_0$ satisfies the conditions (b) and (c) there, and the fact that our $a$ has the same zeros on $\mathbb{T}$ as $a_0$ than implies that $a$ satisfies conditions (b) and (c) of the present theorem.

Now, we assume that the shifts of $\phi$ are unstable, and retain the meaning of $\phi_0$, $B$, $b$ and $a_0$ as above. We factor $B = B_1 B_2$ such that $B_1$ has only real zeros and $B_2$ has no real zeros. By Corollary 3.6, $B_1$ has a positive degree, i.e., vanishes somewhere. The factorization of $B$ gives rise to an analogous factorization $b = b_1 b_2$, hence to the representation

$$a(z) = a_0(z) \frac{b_1(z^n)}{b_1(z)} \frac{b_2(z^n)}{b_2(z)}.$$

By Theorem 4.1, the rational polynomial $a_1(z) := a_0(z) b_1(z^n)/b_1(z)$ is the refinement symbol of the function $\phi_1$ defined by

$$\widehat{\phi}_1 = B_1 \widehat{\phi}_0.$$

The shifts of $\phi_1$ are linearly dependent, since $B_1$ has zeros (these shifts are even unstable). Thus, by Theorem 5.4, $a_1$ must violate one of the conditions (a-c) there. However, $a_1$ cannot be divisible by anything of the form $z^n - r$, $r \notin \mathbb{T}$, since $a_0$ was not divisible by such a factor (being associated with the linearly independent $\phi_0$), and $b_1$ has zeros only in $\mathbb{T}$. We conclude therefore that $a_1$ must violate one of conditions (a-c) of the present theorem. Since $a$ is obtained from $a_1$ by multiplying by the expression $b_2(z^n)/b_2(z)$, and that function takes the value 1 at 1 and has no poles on $\mathbb{T}$, $a$ must violate the same condition $a_1$ does. □

## 6. An algorithm

In this final section we sketch an algorithm which does the following: given the symbol $a$ of the mask $A$ of the compactly supported function $\phi$, the algorithm produces the symbol $a_0$ of the linearly independent $\phi_0$ that appears in Corollary 3.4. It is a finite algorithm if we assume that all roots and poles of $a$ are known.

The input of the algorithm is a list $N$ of the roots of the numerator of $a$ and list $D$ of the roots of the denominator of $a$. $N$ is assumed to be disjoint from $D$. The output is a list $A$ of the roots of $a_0$.

**Step 1**: For each $d \in D$, we add to $N$ and $D$ any root of $z^n - d$ which is not yet in $N$. The procedure must terminate after finitely many steps, although $D$ is being continuously changed during that process.

**Step 2**: For each $d \in D$, we remove from $N$ all roots of $z^n - d$.

**Step 3**: We check whether $N$ contains a set $R_r$ of all the $n$th order roots of some $r \in \mathbb{C} \backslash 0$. If it does, we replace $R_r$ by $r$.

**Step 4**: We compute the product $\prod_{r\in N}(1-r)$. It must be a power of $n$, say, $n^k$. We remove from $N$ $k$ appearances of each $n$th root of unity other than 1 (each such root must appear with multiplicity at least $k$).

**Step 5**: We move to $A$ every $r \in N$, unless $r$ is an $m$th root of unity, with $m$ relatively prime to $n$.

**Step 6**: We take $\xi \in N$, and check whether $\xi$ generates a cycle: first, we multiply $\xi$ by all $n$th roots of unity, to obtain a collection $M$ of $n$ products, and define a new set $T$ by $T := M$. If $N$ misses at least two numbers from $M$, we move $T \cap N$ to $A$, and pick a new $\xi$ in the (smaller) $N$. If $N$ misses only one number, we replace $\xi$ by $\xi^n$ and repeat the process, which means that, at any intermediate iteration, we generate an $n$-set $M$, update $T = T \cup M$, and check: if $N$ misses two (or more) of the numbers in the present $M$, we move $N \cap T$ into $A$. Otherwise, we proceed. After finitely many iterations we must obtain $\xi$ again. In such a case, we remove $N \cap T$ from $N$. We repeat this step until $N$ becomes empty.

**Discussion**: We want to find the factorization

$$a(z) = a_0(z)b(z^n)/b(z).$$

Since $a$ and $b$ are polynomials, if $z - d$ appears in the present denominator, it must be a factor of $b$. Thus $z^n - d$ must divide the numerator. If it does not, this can be only as a result of previous cancellations with other factors of $b$. Step 1 undoes these cancellations. At the end of Step 1, every factor $z - d$ in the denominator has its $z^n - d$ counterpart in the numerator. The self-understood Step 2 leaves us then with no denominator.

Step 3 checks for a possible division of $a$ (which now is a polynomial) by an expression $z^n - r$. If such divisor exists, then we divide $a$ by $(z^n - r)/(z - r)$. Step 4 takes care of possible occurrences of the root 1 in $b$. The proof of Theorem 2.3 provides the necessary background.

After the completion of Step 4, the only parts of $b(z^n)/b(z)$ which can still remain in $a$ are the "cycles" that violate condition (c) of Theorem 2.3. Since all the corresponding roots of such cycles are roots of unity of order $n^m - 1$, for some $m$, we can already move to $a_0$ all factors which do not vanish at such unity roots. This is Step 5.

Finally, Step 6 is a careful check of condition (c) of Theorem 2.3. It removes factors identified as a "cycle" and moves to $a_0$ all the rest.

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