Stability and Independence of the Shifts of a Multivariate Refinable Function

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Abstract. Due to their so-called time-frequency localization properties, wavelets have become a powerful tool in signal analysis and image processing. Typical constructions of wavelets depend on the stability of the shifts of an underlying refinable function $\phi \in L^2(\mathbb{R}^d)$. In this paper, we derive necessary and sufficient conditions for the stability of the shifts of certain compactly supported refinable functions. These conditions are in terms of the zeros of the refinement mask. We also provide a similar characterization of the (global) linear independence of the shifts.

§1 Introduction

In this paper we present a characterization of the stability and linear independence of the shifts of a compactly supported refinable function in terms of the refinement mask. Our results are applicable to a large class of multivariate functions which includes (but is not limited to) tensor products and box splines.

A function $\phi \in L^p(\mathbb{R}^d)$ is said to have *ℓ^p-stable shifts* if there exist positive constants C and D such that

$$C\|a\|_{\ell^p} \leq \| \sum_{\alpha \in \mathbf{Z}^d} a(\alpha)\phi(\cdot - \alpha)\|_{L^p} \leq D\|a\|_{\ell^p}$$

for all $a \in \ell^p(\mathbf{Z}^d, \mathbb{C})$ (it is often said that ϕ provides a *Riesz basis* in L^p in this case). A compactly supported function ϕ is said to have *linearly independent shifts* if the map

$$\phi *' : a \mapsto \sum_{\alpha \in \mathbf{Z}^d} a(\alpha)\phi(\cdot - \alpha)$$

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is one-to-one.

These two properties can be characterized in terms of $\widehat{\phi}$, the Fourier-Laplace transform of ϕ, which is an entire function defined on all of $\mathbb{C}^d$ for all compactly supported ϕ. In particular, the following results are stated in terms of the Fourier-Laplace transform.

Result 1. *Suppose $1 \leq p \leq \infty$ and $\phi \in L^p(\mathbb{R}^d)$ is compactly supported. Then ϕ has ℓ^p-stable shifts if and only if the set*

$$N_{\mathbb{R}}(\phi) := \left\{ \vartheta \in \mathbb{T}^d := [0 \mathinner{..} 2\pi)^d \ : \ \widehat{\phi}(\vartheta + 2\alpha\pi) = 0 \ \forall \alpha \in \mathbf{Z}^d \right\}$$

is empty.

Result 2. **[3]** *Suppose $\phi \in \mathcal{D}'(\mathbb{R}^d)$ is compactly supported. Then ϕ has linearly independent shifts if and only if the set*

$$N_{\mathbf{C}}(\phi) := \left\{ \vartheta \in \mathbb{T}^d + i\mathbb{R}^d \ : \ \widehat{\phi}(\vartheta + 2\alpha\pi) = 0 \ \forall \alpha \in \mathbf{Z}^d \right\}$$

is empty.

Notice that the condition that $N_{\mathbb{R}}(\phi)$ be empty in result 1 is independent of p. For this reason, we will say that a compactly supported function ϕ has *stable shifts* if $N_{\mathbb{R}}(\phi)$ is empty. So a function ϕ which has stable shifts has ℓ^p-stable shifts if and only if $\phi \in L^p$.

A compactly supported function ϕ is said to be *refinable* if (ϕ is not identically zero and) there exists a 2π-periodic function A satisfying

$$\widehat{\phi}(2\omega) = A(\omega)\widehat{\phi}(\omega) \text{ for all } \omega \in \mathbb{C}^d. \tag{1.1}$$

Equation (1.1) is called the *refinement equation* and A is called the *(refinement) mask.*

The characterizations given in results 1 and 2 are in terms of $\widehat{\phi}$. However, for refinable ϕ it is actually more desirable to characterize these properties in terms of the mask A. In the univariate case ($d = 1$), stability and linear independence of the shifts of a compactly supported function have been characterized in terms of the mask by Jia and Wang [2]. Their arguments relied on the following

Result 3. **[3]** *For a non-zero compactly supported $\phi \in \mathcal{D}'(\mathbb{R}^d)$, the set $N_{\mathbf{C}}(\phi)$ is finite.*

Unfortunately, this result is not true for multivariate functions To analyze the multivariate case, we consider functions ϕ whose Fourier-Laplace transform $\widehat{\phi}$ has the form:

$$\widehat{\phi} = \widehat{\phi}_{\Xi} := \prod_{\xi \in \Xi} \widehat{\phi}_{\xi}(\langle \cdot, \xi \rangle), \tag{1.2}$$

where Ξ is a finite subset of $\mathbf{Z}^d\backslash 0$ satisfying $\dim\operatorname{span}\Xi = d$ and, for each $\xi\in\Xi$, ϕ_ξ is a compactly supported member of $L^p(\mathbb{R}^d)$ for some $1\le p\le\infty$ (p may depend on ξ).

Remark Each ϕ_ξ is univariate. However, (1.2) defines $\phi_{\{\xi\}}$ as a distribution on $\mathbb{R}^d$ (with support in the hyperplane $\mathbb{R}\xi$).

Remark If ϕ_ξ is refinable for every $\xi\in\Xi$, say with mask A_ξ, then ϕ_Ξ is also refinable with mask

$$A_\Xi := \prod_{\xi\in\Xi} A_\xi(\langle\cdot,\xi\rangle).$$

It is clear from (1.2) and result 1 (resp. result 2) that, if the shifts of ϕ_Ξ are stable (resp. linearly independent), then the set $N_{\mathbb{R}}(\phi_Y)$ (resp. $N_{\mathbf{C}}(\phi_Y)$) must be empty for every $Y\subset\Xi$. Now, suppose $\phi=\phi_\Xi$ is of the type (1.2) and suppose $Y\subset\Xi$ satisfies $d_Y:=\dim\operatorname{span}Y<d$. Then $\widehat{\phi}_Y$ is constant in directions orthogonal to Y. Therefore, if $N_{\mathbb{F}}(\phi_Y)$ (we use $\mathbb{F}$ to represent either of the fields $\mathbb{R}$ or $\mathbb{C}$) is non-empty, say $\vartheta\in N_{\mathbb{F}}(\phi_Y)$, then for any $\eta\in Y^\perp$, $\widehat{\phi}_Y(\vartheta+\eta+2\alpha\pi)$ is zero for all $\alpha\in\mathbf{Z}^d$; *i.e.*, the set $N_{\mathbb{F}}(\phi_Y)$ is infinite. And since $N_{\mathbb{F}}(\phi_Y)$ is a subset of $N_{\mathbb{F}}(\phi)$, $N_{\mathbb{F}}(\phi)$ is also infinite. One of the main results of this paper is the converse of this, namely:

Theorem 1. *If $\phi=\phi_\Xi$ is of the type (1.2), and if $N_{\mathbb{F}}(\phi)$ is infinite, then there is some $Y\subset\Xi$ with $d_Y=\dim\operatorname{span}Y<d$ so that $N_{\mathbb{F}}(\phi_Y)$ is already non-empty.*

Remark This theorem does not require that ϕ be refinable.

This theorem actually leads to a complete characterization of stability and linear independence in terms of the mask for refinable functions of the type (1.2). If the shifts of ϕ_Ξ are linearly dependent and $N_{\mathbf{C}}(\phi_\Xi)$ is infinite, for example, then any minimal $Y\subset\Xi$ with $N_{\mathbf{C}}(\phi_Y)\ne\{\}$ will satisfy $d_Y<d$ by theorem 1. In this case, ϕ_Y actually has its support in the subspace spanned by Y, and the map $\phi *'$ is not even one-to-one when restricted to $\mathbb{C}^{\mathbf{Z}^d\cap\operatorname{span}Y}$. We may then analyze those shifts of ϕ_Y with support in $\operatorname{span}Y$. Equivalently, we may analyze the set $N_{\mathbf{C}}(\phi_Y)\cap\operatorname{span}Y$ which, since Y is minimal, must be finite. This argument applies equally well to stability.

§2 Statement of main results

We have already stated one of our main results, namely theorem 1. Combining theorem 1 with theorems 2 and 3 leads to a complete characterization of stability and linear independence in terms of the mask for functions of the type (1.2).

We begin with a 2π-periodic function A defined on $\mathbb{C}^d$. We assume that there exists a non-trivial compactly supported function ϕ which satisfies the refinement equation (1.1). We will also assume that $A(0) = 1$, and that A is Lipschitz continuous at the origin.

Definition In the statement of results that follows, and throughout this paper, we will say that a function A has a

- *π-periodic zero in* $\mathbb{F}^d$ if there exists $z \in \mathbb{F}^d$ such that

$$A(z + \alpha\pi) = 0 \text{ for all } \alpha \in \mathbf{Z}^d;$$

- *contaminating zero in* $\mathbb{R}^d$ if there exists an integer $m \geq 2$ and $\mu \in \mathbf{Z}^d \backslash (2^m - 1)\mathbf{Z}^d$ such that

$$A(2^k \frac{2\mu\pi}{2^m - 1} + \nu\pi) = 0 \text{ for all } \nu \in \mathbf{Z}^d \backslash 2\mathbf{Z}^d,\ k \in \{0, 1, 2, \ldots\}.$$

Remark Example 2 is provided to generate some familiarity with contaminating zeros.

The following two theorems were proved for univariate ϕ in [2]. Since the authors dealt only with univariate functions, the set $N_{\mathbb{F}}(\phi)$ was guaranteed to be finite by result 3; and hence this was not part of their hypotheses.

Theorem 2. *Suppose the compactly supported function ϕ is refinable with mask A. Suppose further that $N_{\mathbb{R}}(\phi)$ is finite. Then the shifts of ϕ are stable if and only if*

(i) A has no π-periodic zeros in $\mathbb{R}^d$, and
(ii) A has no contaminating zeros in $\mathbb{R}^d$.

Theorem 3. *Suppose the compactly supported function ϕ is refinable with mask A. Suppose further that $N_{\mathbf{C}}(\phi)$ is finite. Then the shifts of ϕ are linearly independent if and only if*

(i) A has no π-periodic zeros in $\mathbb{C}^d$, and
(ii) A has no contaminating zeros in $\mathbb{R}^d$.

The assumption that $N_{\mathbb{F}}(\phi)$ be finite is only used to prove the sufficiency. We therefore have the following

Theorem 4. *Suppose the compactly supported function ϕ is refinable with mask A. If the shifts of ϕ are stable (resp. linearly independent), then*

(i) A has no π-periodic zeros in $\mathbb{R}^d$ (resp. $\mathbb{C}^d$) and
(ii) A has no contaminating zeros in $\mathbb{R}^d$.

Remark None of the theorems up to this point in this section required that ϕ be of the type (1.2).

Unfortunately, the assumption that $N_{\mathbb{F}}(\phi)$ be finite in theorems 2 and 3 cannot be easily verified in terms of the mask. The following two theorems show that it can be eliminated for functions of the type (1.2).

Theorem 5. *Suppose that $\phi = \phi_\Xi$, of the type (1.2), is refinable with mask $A := A_\Xi$. Then the shifts of ϕ are stable if and only if*
(i) A has no π-periodic zeros in $\mathbb{R}^d$,
(ii) A has no contaminating zeros in $\mathbb{R}^d$, and
(iii) $A_\xi(\pi) = 0$ for every $\xi \in \Xi$.

Theorem 6. *Suppose that $\phi = \phi_\Xi$, of the type (1.2), is refinable with mask $A := A_\Xi$. Then the shifts of ϕ are linearly independent if and only if*
(i) A has no π-periodic zeros in $\mathbb{C}^d$,
(ii) A has no contaminating zeros in $\mathbb{R}^d$, and
(iii) $A_\xi(\pi) = 0$ for every $\xi \in \Xi$.

These theorems may be best viewed in light of the following known necessary conditions for stability:

Result 4. *Suppose that $\phi = \phi_\Xi$ is of the type (1.2) and ϕ is refinable with mask $A := A_\Xi$. Suppose further that $A_\xi(\pi) = 0$ for every $\xi \in \Xi$. If the shifts of ϕ are stable, then $|\det B| = 1$ for every basis $B \subset \Xi$.*

However, the condition $|\det B| = 1$ for all bases is not sufficient. Example 1 illustrates a situation in which each ϕ_ξ has linearly independent shifts, each A_ξ has a zero at π, and $|\det B| = 1$ for all bases $B \subset \Xi$; yet the shifts of ϕ_Ξ are not even stable.

Other interesting results which follow from these theorems are

Corollary 1. *Suppose the compactly supported function ϕ is refinable with mask A. Suppose further that $N_{\mathbf{C}}(\phi)$ is finite. If the shifts of ϕ are stable but not linearly independent, then A has a (non-real) π-periodic zero.*

which follows immediately from theorems 2 and 3; and

Corollary 2. *Suppose $\phi = \phi_\Xi$ is of the type (1.2) and ϕ is refinable with mask $A = A_\Xi$. If the shifts of ϕ are stable but not linearly independent, then A_Ξ has a (non-real) π-periodic zero.*

which follows immediately from theorems 5 and 6.

§3 Examples

Example 1. This example will show that the sufficient conditions provided in result 4 are not necessary in general. We define the univariate mask

$$\begin{aligned} A_\theta &:= \frac{e^{-3i\cdot} + (1 - 2\cos\theta)e^{-2i\cdot} + (1 - 2\cos\theta)e^{-i\cdot} + 1}{4 - 4\cos\theta} \\ &= \frac{\left(e^{-i\cdot} + 1\right)\left(e^{-i\cdot} - e^{-i\theta}\right)\left(e^{-i\cdot} - e^{i\theta}\right)}{4 - 4\cos\theta} \end{aligned}$$

for $\pi/3 < \theta < \pi$. Then A_θ is a trigonometric polynomial with real coefficients which satisfies $A_\theta(0) = 1$. This is enough to imply the existence of a real-valued compactly supported refinable distribution with mask A_θ. There is a unique such distribution, ϕ_θ, if we insist further that $\widehat{\phi}_\theta(0) = 1$. In fact, for $\pi/3 < \theta < \pi$, ϕ_θ is a continuous function with $\operatorname{supp} \phi_\theta = [0\,..\,3]$. The functions ϕ_θ are plotted for $\theta = \frac{15\pi}{32}$ and $\theta = \frac{17\pi}{32}$ in figure 1.

The zeros of the mask A_θ are

$$\mathcal{Z}(A_\theta) = \{\pi, \theta, 2\pi - \theta\} + 2\mathbf{Z}\pi.$$

From this we can see that the shifts of ϕ_θ are linearly independent for all $\theta \neq \frac{\pi}{2}$. We also see that $A_\theta(\pi) = 0$.

In this example, we consider the bivariate function ϕ_Ξ of type (1.2) given by

$$\Xi = \{\xi, \eta, \zeta\} := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$
$$\phi_\xi = \phi_{15\pi/32}, \quad \phi_\eta = \phi_{17\pi/32}, \quad \text{and} \quad \phi_\zeta = \chi_{[0..1)},$$
$$\widehat{\phi}_\Xi = \widehat{\phi}_\xi(\langle\cdot,\xi\rangle)\widehat{\phi}_\eta(\langle\cdot,\eta\rangle)\widehat{\phi}_\zeta(\langle\cdot,\zeta\rangle)$$

which has mask

$$A_\Xi(\omega_1, \omega_2) = A_{\frac{15\pi}{32}}(\omega_1) A_{\frac{17\pi}{32}}(\omega_2) \left(\frac{e^{-i(\omega_1+\omega_2)} + 1}{2} \right).$$

This defines a function $\phi := \phi_\Xi \in C^1(\mathbb{R}^2)$.

Each of the univariate functions has linearly independent shifts. Moreover, convolving any two also results in a function with linearly independent shifts. Also note that every basis $B \subset \Xi$ satisfies $|\det B| = 1$. However, the shifts of ϕ_Ξ are not even stable, as you can see by observing that the zero set of A_Ξ contains all points $(x_1, x_2) \in \mathbb{R}^2$ for which

$$x_1 \in \left\{ \frac{15\pi}{32}, \frac{49\pi}{32} \right\} + 2\mathbf{Z}\pi \text{ or } x_2 \in \left\{ \frac{17\pi}{32}, \frac{47\pi}{32} \right\} + 2\mathbf{Z}\pi \text{ or } x_1 + x_2 \in \pi + 2\mathbf{Z}\pi.$$

Thus $\mathcal{Z}(A_\Xi)$ contains $(\frac{15\pi}{32}, \frac{17\pi}{32}) + \mathbf{Z}\pi$; *i.e.*, the shifts of ϕ are not stable.

Example 2. We provide an example of a box spline whose mask has a contaminating zero to generate some familiarity with contaminating zeros.

As far as this paper is concerned, it will be sufficient to define box splines in terms of their refinement mask. We can define a box spline M_Ξ associated with $\Xi \subset \mathbf{Z}^d \backslash 0$ by the refinement equation

$$\widehat{M}_\Xi(2\cdot) = A_\Xi \widehat{M}_\Xi \text{ where } A_\Xi := \prod_{\xi \in \Xi} A_\xi(\langle\cdot,\xi\rangle) \text{ and } A_\xi := \left(\frac{1 + e^{-i\cdot}}{2} \right)^{n_\xi}$$

(along with $\widehat{M_\Xi}(0) = 1$). Here, $n := (n_\xi)_{\xi\in\Xi} \in \mathbb{N}^\Xi$ is the *multiplicity* of the *direction set* Ξ.

Remark The necessary conditions stated in result 4 are also sufficient for box splines.

We have suppressed the dependence on the multiplicity n because the results of this paper are in terms of $\mathcal{Z}(A_\Xi)$, the zeros of the mask. It is clear that this set is independent of n. In fact, we see that $\mathcal{Z}(A_\xi) = (\mathbf{Z}\backslash 2\mathbf{Z})\pi$ and hence that

$$\mathcal{Z}(A_\Xi) = \bigcup_{\xi\in\Xi} \Big\{\, x \in \mathbb{R}^d \;:\; \langle x, \xi\rangle \in (\mathbf{Z}\backslash 2\mathbf{Z})\pi \,\Big\}.$$

We should point out that, although our definition of box splines requires $\Xi \subset \mathbf{Z}^d$, the standard definition of box splines allows for arbitrary $\Xi \subset \mathbb{R}^d\backslash 0$. However, box splines with non-integer directions are, in general, not refinable. Hence we are not concerned with them here.

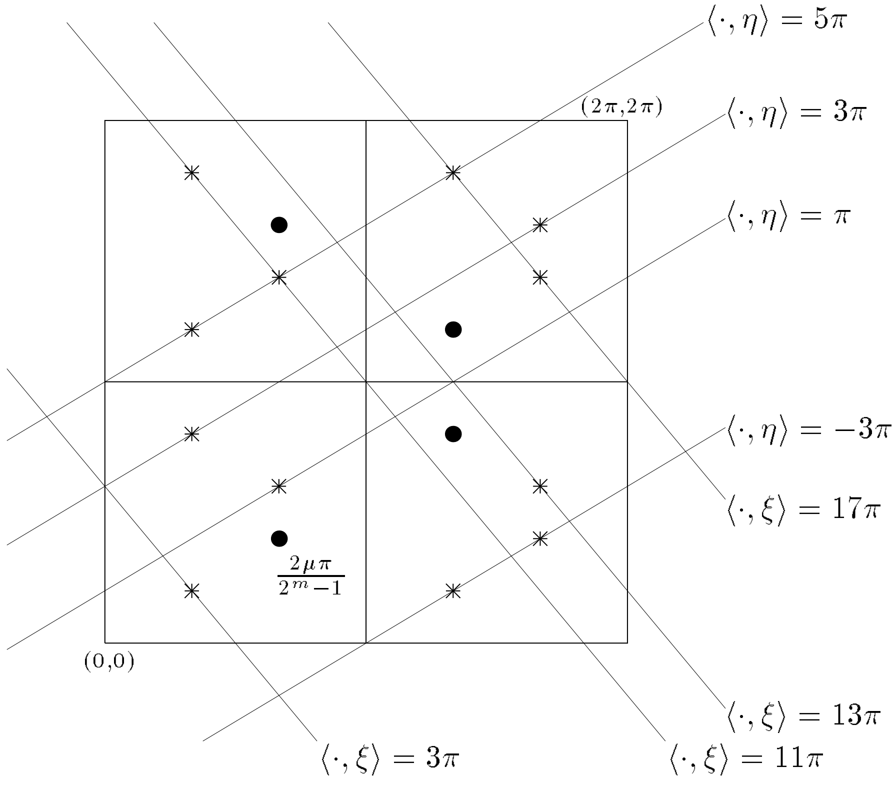

Figure 1. Contaminating zero set with $\frac{2\mu\pi}{2^m-1} = \left(\frac{2\pi}{3}, \frac{2\pi}{5}\right)$.

In this example, we let $d = 2$ and we consider $\phi_\Xi := M_\Xi$, where

$$\Xi = \{\xi, \eta\} := \left\{ \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right\}.$$

Recall that $\mathcal{Z}(A_\xi) = \mathcal{Z}(A_\eta)$ consist of all odd multiples of π. Since $|\det[\xi \quad \eta]| = 45 \neq 1$, result 4 implies that the shifts of ϕ_Ξ are not stable. Indeed, ϕ_Ξ has a contaminating zero with $m = 4$ and $\mu = (5,3)$. In figure 1 we have denoted the points $2^k \frac{2\mu\pi}{2^m - 1} \in \mathbb{T}^d$ by bullets($\bullet$). The contaminating zero set is marked by asterisks($*$). We have also displayed particular curves $\langle \cdot, \xi \rangle \in \pi\mathbf{Z}\backslash 2\mathbf{Z}$ and $\langle \cdot, \eta \rangle \in \pi\mathbf{Z}\backslash 2\mathbf{Z}$ which cover this contaminating zero set.

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