On the pointwise limits of bivariate Lagrange projectors

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Abstract. A linear algebra proof is given of the fact that the nullspace of a finite-rank linear projector, on polynomials in two complex variables, is an ideal if and only if the projector is the bounded pointwise limit of Lagrange projectors, i.e., projectors whose nullspace is a radical ideal, i.e., the set of all polynomials that vanish on a certain given finite set. A characterization of such projectors is also given in the real case. More generally, a characterization is given of those finite-rank linear projectors, on polynomials in d complex variables, with nullspace an ideal that are the bounded pointwise limit of Lagrange projectors. The characterization is in terms of a certain sequence of d commuting linear maps and so focuses attention on the algebra generated by such sequences.

Keywords: ideal projector, ideal interpolation, multivariate, commuting matrices, minimal annihilating polynomial

1. Introduction

The purpose of this note is to provide a proof, self-contained and elementary in that it uses only tools from Linear Algebra, of the fact that every ideal projector of finite rank on the space of polynomials in *two* complex variables is the bounded pointwise limit of Lagrange projectors.

Here, to recall Birkhoff's definition [B], an **ideal projector** is an idempotent linear map on the space $k[\mathbf{x}]$, of polynomials in $\mathbf{x} = (x_1, \ldots, x_d)$ over the field k, whose kernel is an **ideal** (i.e., a linear subspace that is also closed under pointwise multiplication by any $g \in k[\mathbf{x}]$), while a **Lagrange projector** is an idempotent linear map on some space of functions whose kernel is the joint kernel of some linear functionals of point evaluation. Further, it is well-known (see, e.g., [M77] or the retelling in [dB05]) that a zero-dimensional ideal $\mathcal{I}$ in $\mathbb{C}[\mathbf{x}]$, such as the kernel of a finite-rank ideal projector, is the joint kernel of linear functionals of the form

$$g \mapsto q(D)g(v), \quad q \in Q_v,\ v \in \mathcal{V} := \mathcal{V}(\mathcal{I}),$$

with $\mathcal{V}$ the variety of the ideal and Q_v a D-invariant polynomial space depending on v and $\mathcal{I}$, all $v \in \mathcal{V}$. Hence, in general, an ideal projector involves not just matching of function values but also of certain derivatives, provided only that if a certain derivative is matched at a site, then also all 'lower-order' derivatives be matched at that site.

In the univariate case, this means that ideal interpolation is Hermite interpolation, and this is well-known to be the limit of Lagrange interpolation as some of the interpolation sites coalesce. This encouraged the first author to conjecture (in [dB05]) that, also in the multivariate setting, ideal interpolation is **Hermite interpolation** in the sense that it is the limit of Lagrange interpolation (thus deviating from [M77] where "Hermite interpolation" is used for what we are calling "ideal interpolation" here). However, this conjecture was disproved by the second author (in [S06]) when more than two variables are involved.

For the bivariate case, the second author proved the conjecture (in [S06]) albeit with the aid of tools from Algebraic Geometry. Given the basic nature at issue here (Lagrange interpolation and its limits as interpolation sites coalesce), it seems worthwhile to provide a proof that uses only linear algebra.

In the process, we prove that an ideal projector on $k[\mathbf{x}]$ can be approximated by Lagrange projectors if and only if a certain sequence of commuting matrices can be approximated by a sequence of diagonalizable matrices (with entries in k), and the known fact (see [MT] or [G]) that any pair of commuting matrices can be approximated by pairs of diagonalizable commuting complex matrices then supplies the proof in the bivariate case. However, the proof of this fact (in [MT] or [G]) uses Algebraic Geometry, so we felt obliged to supply a proof that only uses linear algebra. In addition, since [GS] also prove such approximation for certain commuting sequences of more than two matrices, we also obtain that certain low-rank multivariate ideal projectors are the limit of Lagrange projectors.

The note is organized as follows. In section 2, we bring a recipe for generating an ideal projector with a given range F in $k[\mathbf{x}]$ from a suitable sequence $A = (A_1, \ldots, A_d)$ of commuting linear maps on some linear space Y via the corresponding ring homomorphism $\Phi_A : p =: \sum_\alpha \widehat{p}(\alpha)\mathbf{x}^\alpha \mapsto \sum_\alpha \widehat{p}(\alpha)A^\alpha$, with the resulting projector $P_A := (\Phi_A|_F)^{-1}\Phi_A$ depending continuously on the sequence A, and relate this to the well-known multiplication maps associated with the quotient ring over a 0-dimensional ideal. In section 3, we prove that, for $k = \mathbb{C}$, the resulting ideal projector is a Lagrange projector if the linear maps A_i are diagonalizable. In section 4, we give a linear algebra proof that two commuting matrices can be approximated by diagonalizable commuting matrices and thereby finish the promised proof. Section 5 discusses the case $k=\mathbb{R}$, where we cannot hope to approximate by Lagrange projectors but only by certain projectors that are restrictions of Lagrange projectors on $\mathbb{C}[\mathbf{x}]$ to $\mathbb{R}[\mathbf{x}]$. In the final section, we urge further study of the relationship between the spectrum of a sequence A of commuting linear maps and the kernel of the corresponding ring homomorphism Φ_A, to match the very well understood relationship in the special case $d=1$.

2. Multiplication maps and ideal projectors

It is standard in algebraic geometry (see, e.g., the textbook [CLO: pp 51ff]) to consider, for a zero-dimensional ideal $\mathcal{I}$ in $k[\mathbf{x}]$, the map m, from $k[\mathbf{x}]$ into the linear maps on the quotient ring

$$k[\mathbf{x}]/\mathcal{I} = \{[g] := g + \mathcal{I} : g \in k[\mathbf{x}]\},$$

that carries $h \in k[\mathbf{x}]$ to the map

$$m_h : [g] \mapsto [hg].$$

In contrast, we are interested in *ideal projectors* (something not explicitly mentioned in the textbooks), i.e., linear projectors P on $k[\mathbf{x}]$ whose kernel is a zero-dimensional ideal, $\mathcal{I}$ say, hence their range is an algebraic complement, F say, of $\mathcal{I}$. To be sure, for such a projector, the factor map

$$P/\mathcal{I} : k[\mathbf{x}]/\mathcal{I} \to F : [f] \mapsto Pf$$

of P by its kernel is well defined, linear, and invertible, hence the map of interest to us, namely

$$M : k[\mathbf{x}] \to L(F) : g \mapsto M_g : F \to F : f \mapsto P(gf), \tag{2.1}$$

is similar to m in the sense that $[f] = [Pf]$, hence

$$[M_g f] = m_g[f], \quad g, f \in k[\mathbf{x}].$$

But since we are focusing on *all* ideal projectors with range a given F, we find it easier to deal directly with the map M, freely adapting the well-known arguments that establish the various corresponding properties of m. In particular, we will be constructing maps like M, from $k[\mathbf{x}]$ into the ring $L(F)$ of linear maps on F, before we even know a corresponding P or $\mathcal{I}$ in hopes of thereby obtaining P and $\mathcal{I}$ suitable for our needs, hence could not stick to the standard situation even if we wanted to.

Our main tool is the observation (which, for $d=2$ and $Y=\mathbb{C}^n$, can already be found in [N: p. 7, Theorem 1.9]) that any sequence $A=(A_1,\ldots,A_d)$ of pairwise commuting linear maps on a finite-dimensional linear space Y over the field k induces a map

$$\Phi_A : k[\mathbf{x}] \to L(Y) : g \mapsto g(A) := \sum_\alpha \widehat{g}(\alpha) A^\alpha, \tag{2.2}$$

with $\alpha \in \mathbb{Z}_+^d := \{\alpha \in \mathbb{Z}^d : \alpha(j) \ge 0, j=1{:}d\}$, $g =: \sum_\alpha \widehat{g}(\alpha)\mathbf{x}^\alpha$, and with

$$A^\alpha := \prod_j A_j^{\alpha(j)}$$

independent of the order in which this product is formed from its factors. This implies that the map Φ_A defined in (2.2) is a ring homomorphism, hence has an ideal as its kernel. Conversely, every ring homomorphism Φ on $k[\mathbf{x}]$ into $L(Y)$ is of the form Φ_A, with

$$A_j := \Phi x_j, \quad j = 1{:}d.$$

Now, since Y is finite-dimensional, $\ker\Phi_A$ has finite codimension (hence is a 0-dimensional ideal). Therefore, for any linear subspace F of $k[\mathbf{x}]$,

$$\Phi_A\big|_F : F \to \operatorname{ran}\Phi_A \ \text{ is invertible} \tag{2.3}$$

if and only if F is an algebraic complement of $\ker \Phi_A$ and, in that case,

$$P_A := (\Phi_A|_F)^{-1}\Phi_A : k[\mathbf{x}] \to k[\mathbf{x}] \tag{2.4}$$

is well-defined, linear, onto F, and the identity on F, and has $\ker \Phi_A$ as its kernel. In short, P_A is the ideal projector with $\operatorname{ran} P_A = F$ and $\ker P_A = \ker \Phi_A$.

(2.5) Example. Assume that the sequence $A = (A_1, \ldots, A_d)$ in $L(Y)$ satisfies

$$A_i A_j = 0, \quad i, j = 1{:}d.$$

Then, in particular, the A_i commute with each other, hence Φ_A is well-defined and maps every $\mathbf{x}^\alpha$ with $|\alpha| := \|\alpha\|_1 > 1$ to 0, therefore

$$\ker \Phi_A \supseteq k_{>1}[\mathbf{x}] := \operatorname{span}\{\mathbf{x}^\alpha : |\alpha| > 1\} \tag{2.6}$$

and

$$\Phi_A p = p(A) = \sum_{|\alpha|<2} \widehat{p}(\alpha) A^\alpha.$$

In particular,

$$\operatorname{ran} \Phi_A = \operatorname{ran}[\mathrm{id}_Y, A_1, \ldots, A_d].$$

Therefore, if $[\mathrm{id}_Y, A_1, \ldots, A_d]$ is 1-1 (i.e., if $(\mathrm{id}_Y, A_1, \ldots, A_d)$ is linearly independent), then Φ_A is 1-1 on $k_{<2}[\mathbf{x}] := \operatorname{span}\{\mathbf{x}^\alpha : |\alpha| < 2\}$, therefore, as $k_{<2}[\mathbf{x}]$ is an algebraic complement of $k_{>1}[\mathbf{x}]$, there is equality in (2.6), and

$$P_A = (\Phi_A|_{k_{<2}[\mathbf{x}]})^{-1}\Phi_A = T_{<2},$$

the **Taylor projector of order 2**, i.e., the linear projector that associates $p \in k[\mathbf{x}]$ with $\sum_{|\alpha|<2} \widehat{p}(\alpha)\mathbf{x}^\alpha$. □

How would one verify (2.3)? One way is to check that

$$F(A)y := \{f(A)y : f \in F\} \;=\; Y \quad \textit{for some } y \in Y. \tag{2.7}$$

This condition is offhand stronger than that A be **cyclic**, i.e., that $k[A]y = Y$ for some $y \in Y$, and readily implies that

$$F(A) = k[A] := \{g(A) : g \in k[\mathbf{x}]\} = \operatorname{ran} \Phi_A, \tag{2.8}$$

as follows: For any $g \in k[\mathbf{x}]$, $g(A)y = f(A)y$ for some $f \in F$ (since $g(A)y \in Y$), therefore, for every $p \in F$, $g(A)p(A)y = p(A)g(A)y = p(A)f(A)y = f(A)p(A)y$, i.e., $g(A) = f(A)$ on $F(A)y = Y$, hence $g(A) = f(A)$. In other words, (2.8) holds. Note that this argument even proves the stronger statement that

$$F(A) = \mathcal{C}(A) := \{B \in L(Y) : BA_j = A_j B, j = 1{:}d\} \tag{2.9}$$

since the only property of the linear map $g(A)$ used here is that it commutes with all the A_j.

In short, (2.7) implies that $\Phi_A|_F$ is onto $\operatorname{ran} \Phi_A$. It further implies that $\dim F \geq \dim \Phi_A(F) \geq \dim Y$, therefore, under the additional assumption $\dim Y \geq \dim F$, (2.7) also implies that $\Phi_A|_F$ is 1-1 (since $\dim F < \infty$).

Thus, with the assumption $\dim Y \geq \dim F$, (2.7) implies (2.3) (and $\dim Y = \dim F$). The converse does, in general, not hold, as Example 2.12 shows, but, as the discussion following that example shows, does hold when A is cyclic.

Conversely, *for any ideal projector P with $\operatorname{ran} P = F$, the map M defined in (2.1) is a ring homomorphism into $L(F)$ with kernel* $\ker P$, as follows at once from the identity

$$P(fPg) = P(fg), \quad f, g \in k[\mathbf{x}] \tag{2.10}$$

which characterizes ideal projectors among all linear maps on $k[\mathbf{x}]$; see, e.g., [dB05]: Indeed, (2.10) implies that

$$M_f M_g - M_{fg} : h \mapsto P(fP(gh)) - P((fg)h) = 0, \quad f, g, h \in k[\mathbf{x}],$$

showing that M is a ring homomorphism, hence $M = \Phi_A$ with

$$A_j := M_{x_j} : F \to F : f \mapsto P(x_j f), \quad j = 1{:}d.$$

Further, $\ker M = \ker P$ since $Pf = 0$ implies that $M_f g = P(fg) = P(gPf) = P(0) = 0$, hence $\ker P \subset \ker M$, while, conversely, $M_f = 0$ implies that, in particular, $Pf = P(f\mathbf{x}^0) = M_f \mathbf{x}^0 = 0$, hence, also $\ker M \subset \ker P$.

Therefore, (2.3) holds, and $(M|_F)^{-1} M$ is defined and equal to P. More than that, for all $f \in F$, $f = Pf = P(f\mathbf{x}^0) = P(fP\mathbf{x}^0) = M_f(P\mathbf{x}^0)$, hence even (2.7) holds for this A (with $Y = F$ and $y = P\mathbf{x}^0$).

The following theorem summarizes the fruits of this discussion.

(2.11) Theorem. *Let F be a finite-dimensional linear subspace of the linear space $k[\mathbf{x}]$ of polynomials in $\mathbf{x} := (x_1, \dots, x_d)$ with coefficients in the field k.*

(i) Every sequence $A = (A_1, \dots, A_d)$ of commuting linear maps on the finite-dimensional linear space Y (over k) whose corresponding map Φ_A as defined in (2.2) satisfies (2.3) gives rise to an ideal projector with range F, namely the ideal projector $P_A := (\Phi_A|_F)^{-1}\Phi_A$ whose kernel is $\ker \Phi_A$.

(ii) Every ideal projector P with range F gives rise to a sequence $A = (A_1, \dots, A_d)$ of commuting linear maps on $Y = F$ (namely the linear maps $A_j = M_{x_j} : f \mapsto P(x_j f)$) for which $P = P_A = (\Phi_A|_F)^{-1}\Phi_A$ and, in particular, (2.7) with $y = P\mathbf{x}^0$, hence also (2.3), is satisfied.

Here is an example to show how tenuous might be the relationship between the sequence A and the ideal $\ker \Phi_A$.

(2.12) Example. Choose $Y = k^3$ and (with $\mathbf{i}_j$ the jth coordinate vector)

$$A_i : \mathbf{i}_j \mapsto \begin{cases} \mathbf{i}_1, & j = i+1; \\ 0 & \text{otherwise}, \end{cases} \quad j = 1{:}3, \quad i = 1, 2.$$

Then

$$A_i A_j = 0, \quad i, j \in \{1, 2\},$$

while $[\mathrm{id}_Y, A_1, A_2]$ is 1-1, therefore, by Example 2.5, $\ker \Phi_A = k_{>1}[\mathbf{x}]$ and, with $F = k_{<2}[\mathbf{x}]$, we get $P_A = T_{<2}$. In particular, (2.3) holds in this example. On the other hand,

$$A_i w \in \mathrm{span}(\mathbf{i}_1), \quad w \in k^3, \quad i = 1, 2,$$

therefore $\mathrm{span}(w, A_1 w, A_2 w) \subset \mathrm{span}(w, \mathbf{i}_1)$. Hence, for any $y \in Y$, $(y, A_1 y, A_2 y)$ is linearly dependent, therefore (2.7) does not hold in this example.

On the other hand,

$$\ker \Phi_A = \ker \Phi_B,$$

with $B = (B_i := A_i^T : i = 1, 2)$, since $p(A^T) = p(A)^T$ for $p \in k[\mathbf{x}]$, while

$$B_i = A_i^T : \mathbf{i}_j \mapsto \begin{cases} \mathbf{i}_{i+1}, & j = 1; \\ 0 & \text{otherwise}, \end{cases} \quad j = 1{:}3, \quad i = 1, 2,$$

hence, in contrast to A, B satisfies (2.7) since, e.g., $[\mathbf{i}_1, B_1\mathbf{i}_1, B_2\mathbf{i}_1] = \mathrm{id}_3$ is evidently 1-1. □

The example also illustrates the fact that, while **similarity** of A and B (in the sense that, for some invertible linear map S, $B_i = S^{-1}A_iS$ for all i) implies that $\ker \Phi_A = \ker \Phi_B$, the converse does not hold. This, however, changes if we know, in addition, that both A and B are cyclic. For, if $k[A]y = Y$ for some $y \in Y$, then the map

$$\Phi_{A,y} : k[\mathbf{x}] \to Y : g \mapsto g(A)y$$

is linear and onto, and $\ker \Phi_{A,y} \supseteq \ker \Phi_A$ but, also, for any $g \in \ker \Phi_{A,y}$ and any $p \in k[\mathbf{x}]$, $g(A)p(A)y = p(A)g(A)y = p(A)0 = 0$, hence $g \in \ker \Phi_A$. In short,

$$\ker \Phi_{A,y} = \ker \Phi_A,$$

hence $\Phi_{A,y}$ maps any algebraic complement F of $\ker \Phi_A$ 1-1 onto Y, thus providing the linear invertible map

$$S : F \to Y : f \mapsto f(A)y.$$

In particular, (2.7) holds. Now consider the linear maps

$$C_i := S^{-1}A_iS : F \to F, \quad i = 1{:}d.$$

With P the linear projector with range F and nullspace $\ker \Phi_A$, we compute, for arbitrary $f \in F$,

$$A_i f(A)v = (x_i f)(A)v = (P(x_i f))(A)v,$$

hence $C_i = M_{x_i}$, all i, with M the ring homomorphism defined in (2.1). In short, *if A is cyclic, then it is similar to $(M_{x_i} : i = 1{:}d)$, with M as defined in (2.1) hence depends only on the ideal* $\ker \Phi_A$ (and the choice of the algebraic complement F to $\ker \Phi_A$).

Remark. This is far from the first paper to consider polynomial ideals in terms of ring homomorphisms whose kernel they are. A recent, quite pertinent example is [R] which considers ring homomorphisms from $k[\mathbf{x}]$ into the ring of linear maps on arbitrary finite-dimensional vector-spaces, as a means for constructing ideal bases for their kernel. In particular, the claim in the preceding paragraph is proved there.

3. Lagrange projectors and diagonalizable commuting matrices

In this section, we choose the underlying field to be $\mathbb{C}$ but any algebraically closed field would do since we only use the Nullstellensatz in the proof of the following proposition.

(3.1) Proposition. *Let Φ be a ring homomorphism from $\mathbb{C}[\mathbf{x}]$ (with $\mathbf{x} = (x_1, \ldots, x_d)$) into the ring $L(Y)$ of linear maps on the finite-dimensional linear space Y over $\mathbb{C}$, and, for $j = 1{:}d$, let B_j be the matrix representing the linear map Φx_j on Y with respect to some fixed basis $V : \mathbb{C}^n \to Y$ for Y, and assume that $n := \dim Y = \operatorname{codim} \ker \Phi$.*
If $B := (B_1, \ldots, B_d)$ is approximable by diagonalizable commuting sequences in $\mathbb{C}^{n \times n}$, then every linear projector on $k[\mathbf{x}]$ with nullspace $\ker \Phi$ is an **Hermite** *projector, i.e., the (pointwise) limit of Lagrange projectors.*
If B is cyclic, then also the converse holds, i.e., the fact that some linear projector with nullspace $\ker \Phi$ is Hermite implies that B is approximable by diagonalizable commuting sequences.

For the proof, we need the following variant of [CLO: (4.5)Theorem, on p. 54], for which we also provide a simple proof, for completeness.

(3.2) Lemma. *Let $A = (A_1, \ldots, A_d)$ be a sequence of commuting linear maps on the finite-dimensional vector space Y over $\mathbb{C}$, and let $\mathcal{I} := \ker \Phi_A$ be the corresponding ideal, necessarily zero-dimensional.*
Then, for every $g \in \mathbb{C}[\mathbf{x}]$,

$$\operatorname{spect}(g(A)) = g(\mathcal{V}),$$

with

$$\mathcal{V} := \mathcal{V}_{\mathcal{I}} := \{z \in \mathbb{C}^d : g(z) = 0,\ g \in \mathbb{C}[\mathbf{x}]\} \tag{3.3}$$

the (necessarily finite) **variety** *of the ideal* $\mathcal{I}$.

Proof: We need the well-known fact that

$$\mathbb{C}[\mathbf{x}] \to \mathbb{C}^{\mathcal{V}} : g \mapsto g|_{\mathcal{V}} \text{ is onto.} \tag{3.4}$$

Perhaps the fastest proof is the following: For each $v \in \mathcal{V}$, define

$$\ell_v := \prod_{w \in \mathcal{V} \backslash v} \langle \cdot - w, v - w \rangle,$$

with $\langle v, w \rangle := \sum_j v(j)\overline{w(j)}$ the standard scalar product in $\mathbb{C}^n$. Evidently, ℓ_v is a polynomial (of degree $< \#\mathcal{V}$), and vanishes on all of $\mathcal{V} \backslash v$ but not at v, hence the columns of $[\ell_v|_{\mathcal{V}} : v \in \mathcal{V}]$ form a basis for $\mathbb{C}^{\mathcal{V}}$.

Now, take $g \in \mathbb{C}[\mathbf{x}]$, $\mu \in \mathbb{C}$, and consider

$$g - \mu =: h.$$

If $\mu = g(v)$ for some $v \in \mathcal{V}$, then $r := h\ell_v$ vanishes on $\mathcal{V}$, hence, by the Nullstellensatz, $(h\ell_v)^j \in \mathcal{I} = \ker \Phi_A$ for some j, therefore $h(A)^j \ell_v(A)^j = 0$, yet $\ell_v(A)^j \neq 0$ since $\ell_v(v) \neq 0$ and therefore $\ell_v^j \notin \mathcal{I}$ for any j, therefore, finally, $h(A)$ is not invertible, hence $g(v) \in \operatorname{spect}(g(A))$. In short, $g(\mathcal{V}) \subseteq \operatorname{spect}(g(A))$.

If $\mu \notin g(\mathcal{V})$, then h does not vanish on $\mathcal{V}$, therefore, by (3.4), for some polynomial r, $1 - hr$ vanishes on $\mathcal{V}$, hence, by the Nullstellensatz, some power of it, say the jth, lies in $\mathcal{I} = \ker \Phi_A$. This says that

$$0 = (1 - hr)^j(A) = (A^0 - h(A)r(A))^j = \mathrm{id} - h(A)C$$

for some $C \in L(Y)$, showing $h(A) = g(A) - \mu\mathrm{id}$ to be invertible. In short, $\operatorname{spect}(g(A)) \subseteq g(\mathcal{V})$. □

Proof of Proposition 3.1: Since $B_j = V^{-1}(\Phi x_j)V$, all j, with $V : \mathbb{C}^n \to Y$ linear and invertible, the matrices B_j commute with each other, and the corresponding ring homomorphism, Φ_B, has the same kernel as Φ.

Let P be a linear projector on $\mathbb{C}[\mathbf{x}]$ with nullspace $\ker \Phi = \ker \Phi_B$ and let $F := \operatorname{ran} P$. Then P is an ideal projector, and Φ_B maps F 1-1 onto $\operatorname{ran} \Phi_B$, hence

$$P = (\Phi_B|_F)^{-1} \Phi_B$$

and, in particular, $\Phi_B(F)P\mathbf{x}^0 = F$. By assumption, we can find, for each $\varepsilon > 0$, commuting sequences $A = (A_1, \ldots, A_d)$ consisting of diagonalizable matrices of order n for which $\|B_j - A_j\| < \varepsilon$ (in whatever norm we choose to use on $\mathbb{C}^n$).

The corresponding ring homomorphism $\Phi_A : g \mapsto g(A)$ converges boundedly pointwise to Φ_B as $A_j \to B_j$, all j, hence the fact that $\Phi_B(F)P\mathbf{x}^0 = F$ implies that, for all $\varepsilon > 0$ small enough, also $F(A)P\mathbf{x}^0 = F$, therefore $\Phi_A|_F$ is invertible and

$$P_A := (\Phi_A|_F)^{-1} \Phi_A$$

is an ideal projector with range F and kernel the ideal

$$\mathcal{I} := \ker \Phi_A.$$

We claim that P_A *is a Lagrange projector*, i.e., that the ideal $\mathcal{I}$ is **radical** or, what is the same thing, that the variety $\mathcal{V} := \mathcal{V}_{\mathcal{I}}$ (see (3.3)) has cardinality $\operatorname{codim} \mathcal{I} = \dim F$ (hence all the points in the variety are simple). For the proof, recall, e.g., from [HJ: 1.3.19 Theorem] that any such finite sequence of commuting *and* diagonalizable matrices has a common eigenbasis, i.e., we can so choose the basis

$$V : \mathbb{C}^n \to Y$$

that, for all $g = x_j$, hence for all $g \in \mathbb{C}[\mathbf{x}]$, $V^{-1}g(A)V$ is a diagonal matrix. Since the map $g \mapsto V^{-1}g(A)V$ is linear, this implies the existence of linear functionals $\lambda_1, \ldots, \lambda_n$ so that

$$V^{-1}g(A)V = \operatorname{diag}[\lambda_i g : i = 1{:}n], \quad g \in \mathbb{C}[\mathbf{x}],$$

therefore, since the map $g \mapsto V^{-1}g(A)V$ has $\ker \Phi_A$ as its kernel,

$$\ker \Phi_A = \cap_i \ker \lambda_i.$$

Since $\ker \Phi_A$ has codimension n, this implies that the linear map

$$\mathbb{C}[\mathbf{x}] \to \mathbb{C}^n : g \mapsto (\lambda_i g : i = 1{:}n)$$

is onto. This implies that $\#\mathcal{V}(\mathcal{I}) = n = \dim Y$, hence that P_A is a Lagrange projector, since, with Lemma 3.2,

$$\{\lambda_i g : i = 1{:}n\} = \operatorname{spect}(g(A)) = g(\mathcal{V}),$$

while always $\#\mathcal{V} \le n$.

For the proof of the converse, let P be an Hermite projector with $\ker P = \ker \Phi$ and let $F := \operatorname{ran} P$. Then we can find a bounded sequence (P_k) of Lagrange projectors that converges pointwise to P. In particular (see, e.g., [dB06: section 1]), for all sufficiently large k, $F = \operatorname{ran} P$ is an algebraic complement of $\ker P_k$, hence there is the linear projector R_k with range F and nullspace $\ker P_k$, and P is also the pointwise limit of the resulting sequence (R_k). In particular, on the finite-dimensional linear space $F + \sum_{i=1}^d x_i F$, R_k converges pointwise to P. But this implies that, for each i, the linear maps $M_i^{[k]} : F \to F : f \mapsto R_k(x_i f)$ converge as $k \to \infty$ to the corresponding linear map $M_i : F \to F : f \mapsto P(x_i f)$. Also, by Lemma 3.2, for any such k and any $g \in \mathbb{C}[\mathbf{x}]$, $\operatorname{spect}(g(M^{[k]})) = g(\mathcal{V}^{[k]})$, with $\mathcal{V}^{[k]}$ the variety for the ideal $\ker R_k = \ker P_k$, and that ideal is radical since P_k is a Lagrange projector, i.e., $\#\mathcal{V}^{[k]} = n$, therefore, with (3.4), for some $g \in \mathbb{C}[\mathbf{x}]$, the linear map $g(M^{[k]})$ has n distinct eigenvalues, hence is, in particular, non-derogatory. But this implies (see, e.g., the proof of Fact 4.1 for details) that all linear maps commuting with it are diagonalizable, and this is, in particular, true of the $M_i^{[k]}$. If now B is cyclic, then we know that B is similar to $M := (M_1, \ldots, M_d)$ via the linear map $V^{-1}S$ with $S : F \to Y : f \mapsto \Phi(f)y$ and y a cyclic vector for Y. But then

$$B^{[k]} := (V^{-1} S M_i^{[k]} S^{-1} V : i = 1{:}d)$$

is a sequence of diagonalizable commuting matrices, and it converges as $k \to \infty$ to the sequence B. □

(3.5) Example. It seems worthwhile to point out, by way of an example, that the approximability of the commuting sequence B in Proposition 3.1 by sequences of diagonalizable commuting matrices is not necessary for the ideal $\ker \Phi$ to be approximable by radical ideals, which further stresses the tenuous relationship between an ideal and a sequence A of commuting linear maps for which $\ker \Phi_A$ is that ideal. To be sure, in light of the last part of Proposition 3.1, any example like the following must fail to be cyclic.

For this, we use, once more, Example 2.5, this time with $d = 16$, hence $n = \dim Y = d + 1 = 17$, assured that the resulting $\ker \Phi_A$ is $k_{>1}[\mathbf{x}]$ which is well-known to be approximable by radical ideals.

Specifically, we take

$$Y := \mathbb{C}^{17} = Y_0 \oplus Y_1 \oplus \cdots \oplus Y_8,$$

with

$$Y_0 := \mathbb{C}, \quad Y_j := \mathbb{C}^2, \quad j = 1{:}8,$$

and, for

$$\gamma \in \Gamma := \{1{:}4\} \times \{5{:}8\},$$

consider the linear map A_γ on Y that carries $y =: (y_k : k = 0{:}8)$ to

$$(\underbrace{0, \ldots, 0}_{\gamma(2) \text{ terms}}, y_{\gamma(1)}, 0, \ldots).$$

Then

$$A_\gamma A_\beta = 0, \quad \gamma, \beta \in \Gamma,$$

hence, from Example 2.5, $\ker \Phi_A = k_{>1}[\mathbf{x}]$, as hoped for.

Now assume that $A := (A_\gamma : \gamma \in \Gamma)$ were approximable by sequences of pairwise commuting diagonalizable matrices. Being commuting and diagonalizable would mean that they would be simultaneously diagonalizable, therefore

$$A_\gamma = \lim_{n\to\infty} S_n \operatorname{diag}[d_{\gamma,n}, D_{\gamma,n}] S_n^{-1}, \quad \gamma \in \Gamma, \tag{3.6}$$

with S_n invertible matrices, $d_{\gamma,n}$ scalars, and $D_{\gamma,n}$ diagonal matrices of order 16. More than that, since $A_\gamma^2 = 0$, hence $\operatorname{spect}(A_\gamma) = \{0\}$, we would know that

$$\lim_n d_{\gamma,n} = 0, \quad \lim_n D_{\gamma,n} = 0, \quad \gamma \in \Gamma. \tag{3.7}$$

Now write all matrices in block form corresponding to the blocking of the diagonal matrices. Specifically,

$$S_n =: \begin{bmatrix} s_{0,n} & s_{01,n} \\ s_{10,n} & s_{1,n} \end{bmatrix}, \quad S_n^{-1} =: \begin{bmatrix} t_{0,n} & t_{01,n} \\ t_{10,n} & t_{1,n} \end{bmatrix}, \quad A_\gamma =: \operatorname{diag}[0, a_\gamma], \quad \gamma \in \Gamma.$$

Then, in particular,

$$s_{10,n}t_{01,n} + s_{1,n}t_{1,n} = \mathrm{id}_{16}, \tag{3.8}$$

and so

$$\begin{aligned} a_\gamma &= \lim_n (d_{\gamma,n} s_{10,n} t_{01,n} + s_{1,n} D_{\gamma,n} t_{1,n}) \\ &= \lim_n (d_{\gamma,n}(\mathrm{id}_{16} - s_{1,n}t_{1,n}) + s_{1,n} D_{\gamma,n} t_{1,n}) \\ &= \lim_n s_{1,n}(D_{\gamma,n} - d_{\gamma,n}\mathrm{id}_{16}) t_{1,n}, \end{aligned}$$

the last equality by (3.7).

This implies that, for any choices of scalars x_γ,

$$\sum_\gamma x_\gamma a_\gamma = \lim_n s_{1,n} \left(\sum_\gamma x_\gamma (D_{\gamma,n} - d_{\gamma,n}\mathrm{id}_{16}) \right) t_{1,n},$$

hence suggests that we choose, for each n, scalars $x_{0,n}$ and $x_{\gamma,n}$, $\gamma \in \Gamma$, so that

$$x_{0,n}\mathrm{id}_{16} = \sum_\gamma x_{\gamma,n}(D_{\gamma,n} - d_{\gamma,n}\mathrm{id}_{16}). \tag{3.9}$$

This is a homogeneous linear system of 16 equations in 17 unknowns, hence has nontrivial solutions. In particular, we choose a solution with

$$|x_{0,n}| + \sum_\gamma |x_{\gamma,n}| = 1, \quad n = 1, 2, \ldots,$$

hence, after going to a subsequence, may assume that

$$x_0 := \lim_n x_{0,n}, \quad x_\gamma := \lim_n x_{\gamma,n}, \quad \gamma \in \Gamma,$$

exist and satisfy $|x_0| + \sum_\gamma |x_\gamma| = 1$.

But then,

$$\begin{aligned} \sum_\gamma x_\gamma a_\gamma &= \lim_n \sum x_{\gamma,n} a_\gamma \\ &= \lim_n \sum_n s_{1,n} \left(\sum_\gamma x_{\gamma,n}(D_{\gamma,n} - d_{\gamma,n}\mathrm{id}_{16}) \right) t_{1,n} \\ &= \lim_n x_{0,n} s_{1,n} t_{1,n} \\ &= \lim_n x_{0,n}(\mathrm{id}_{16} - s_{10,n}t_{01,n}) \end{aligned}$$

while $\lim_n x_{0,n} = 0$ by (3.9) and (3.7). Therefore, finally,

$$\sum_\gamma x_\gamma a_\gamma = -\lim_n x_{0,n} s_{10,n} t_{01,n},$$

with the left side a matrix of rank ≥ 2 (since $\sum_\gamma |x_\gamma| = 1$ while the a_γ have disjoint support and each is of rank 2) while all the terms in the sequence on the right side are of rank 1, which is impossible. $\square$

4. Hermite projectors and commuting matrices

(4.1) Fact. *([MT], [G]) Any two matrices $A, B \in \mathbb{C}^{n\times n}$ that commute can be approximated by two diagonalizable matrices that commute.*

Proof: This result follows from stronger statements (namely the irreducibility of the variety of all pairs of commuting matrices of a given order with entries in an algebraically closed field; see [MT], [G]), but can be proved directly, by purely linear algebra arguments, as follows.

Recall that $A \in \mathbb{C}^{n\times n}$ is called **non-derogatory** if all its eigenvalues have geometric multiplicity 1, hence, equivalently, each eigenvalue of A is associated with only one Jordan block, hence, equivalently, its characteristic polynomial is its minimal annihilating polynomial, hence, equivalently, if A has a **cyclic vector**, i.e., if, for some $v \in \mathbb{C}^n$, $[v, Av, A^2v, \ldots, A^{n-1}v]$ is a basis for $\mathbb{C}^n$, for which reason we will use here the shorter, but non-standard, term **cyclic** for such A (as we did already earlier in the more general situation of d commuting matrices). This notion is important here since, *if A is cyclic, then the set*

$$\mathcal{C}(A)$$

of matrices commuting with A equals $\mathbb{C}[A] = \{g(A) : g \in \mathbb{C}[\mathbf{x}]\}$ (a special case of the implication (2.7) $\Longrightarrow$ (2.9) proved earlier).

The crux of the argument for (4.1) is Guralnick's observation that *the cyclic matrices are dense in* $\mathcal{C}(A)$. His proof: *(i) $\mathcal{C}(A)$ contains cyclic matrices*; e.g., assuming without loss that A is in Jordan form, $A = \mathrm{diag}[J_i : i = 1{:}r]$ say, with μ_i the eigenvalue of J_i, then any matrix $R = \mathrm{diag}[(\nu_i - \mu_i) + J_i : i = 1{:}r]$ is in $\mathcal{C}(A)$ and is cyclic whenever the ν_i are pairwise distinct. Therefore, *(ii) for every $B \in \mathcal{C}(A)$ and any cyclic matrix $R \in \mathcal{C}(A)$ and every $z \in \mathbb{C}$, $B + zR$ is in $\mathcal{C}(A)$ and is cyclic with at most $n(n-1)/2$ exceptions* since, with v a cyclic vector for such R and $V(z) := [(B + zR)^j v : j = 0{:}n{-}1]$, $\det V(z)$ is a polynomial in z of degree $\le n(n-1)/2$ that is nonzero for large $|z|$, hence can be zero only for at most $n(n-1)/2$ values of z, and must be nonzero otherwise, i.e., $V(z)$ is a basis for $\mathbb{C}^n$ otherwise and, in particular, $B + zR$ is cyclic for all nonzero z close to 0.

With that, if $AB = BA$, then there are diagonalizable B' close to B and cyclic $B'' \in \mathcal{C}(A)$ close to B, therefore $A = g(B'')$ for some $g \in k[\mathbf{x}]$, and then $A' := g(B')$ is close to A, diagonalizable, and commutes with B'. □

(4.2) Theorem. *([S06]) Any ideal projector on the bivariate polynomials with complex coefficients is an Hermite projector, i.e., the bounded pointwise limit of Lagrange projectors.*

Proof: Combine Theorem 2.11.(ii), Proposition 3.1, and Fact 4.1. □

Along the same lines, the fact (proved in [GS: Theorem 8]) that any triplet of commuting matrices of order 4 can be approximated by a triplet of commuting diagonalizable matrices implies that any ideal projector of rank ≤ 4 on trivariate polynomials is an Hermite projector. In particular, this holds for any ideal projector onto the space $k_{<2}[\mathbf{x}]$ of trivariate linear polynomials. By now (see [H], [Si]), it is known that, for $n \le 8$, every commuting triplet of (complex) matrices of order n is approximable by diagonalizable commuting triplets and that this is not true for $n \ge 30$.

5. The real case

Theorem 4.2 relies on Proposition 3.1 whose proof does not apply to the real case since it uses the Nullstellensatz. At the same time, there is, of course, no hope of approximating every real matrix by real diagonalizable matrices and, correspondingly, we should not expect to approximate all ideal projectors by Lagrange projectors in the real case. However, as is made clear in [S0y], one can do the next best thing which is to approximate ideal projectors by ideal projectors whose kernel has $\mathbb{R}_{<n}[x_1]$ as an algebraic complement. The next proposition makes clear why that is such a good thing.

(5.1) Proposition. *If P is an ideal projector on $k[\mathbf{x}]$ with range $F := k_{<1}[x_1]$, where k is some field with algebraic closure $\overline{k}$, then P can be approximated by restrictions to $k[\mathbf{x}]$ of Lagrange projectors on $\overline{k}[\mathbf{x}]$.*

Proof: Consider the univariate polynomial $r := x_1^n - Px_1^n$. As an element of $\overline{k}[x_1]$, it has n roots, counting multiplicities, and, after an arbitrarily small perturbation, we may assume these roots τ_j, $j = 1{:}n$, to be distinct. Correspondingly, let

$$z_j := ((Px_i)(\tau_j) : i = 1{:}d) \in \overline{k}^d, \quad j = 1{:}n.$$

Since $z_j(1) = (Px_1)(\tau_j) = \tau_j$, all j, any $f \in F$ vanishing on all the τ_j is necessarily zero. Since there are $n = \dim F$ distinct z_j, there is therefore a Lagrange projector R on $\overline{k}[\mathbf{x}]$ corresponding to interpolation from F at the n sites z_j, $j = 1{:}n$. Moreover, *on $k[\mathbf{x}]$, R agrees with P*. Indeed, $R = P$ on F. Also, $Rx_1^n = x_1^n - r = Px_1^n$, hence, for any $f \in F$, $P(x_1 f) = R(x_1 f)$. Therefore, as P and R are ideal projectors even when restricted to $k[x_1]$ (recall the pointwise characterization (2.10) of an ideal projector), this implies, by (ii) of Theorem 2.11, that $R = P$ on $k[x_1]$. Further, $Rx_i = Px_i$ for $i = 2{:}d$ since

$$(Rx_i)(z_j) = x_i(z_j) = z_j(i) = (Px_i)(\tau_j) = (Px_i)(z_j), \quad j = 1{:}n.$$

Therefore, finally, for $j = 0{:}n-1$ and all i,

$$R(x_1^j x_i) = R(x_1^j Rx_i) = P(x_1^j Px_i) = P(x_1^j x_i),$$

the middle equality since $R = P$ on $k[x_1] \ni Rx_i = Px_i$, and the outer equalities since both R and P are ideal projectors. With that, we know that $R(x_i f) = P(x_i f)$ for $i = 1{:}d$ and all $f \in F$, hence, by (ii) of Theorem 2.11, that $R = P$ on all of $k[\mathbf{x}]$. $\square$

This suggests the following theorem as a proper analogue of Theorem 4.2 for $k = \mathbb{R}$.

(5.2) Theorem. *Every ideal projector of finite rank n on the space $\mathbb{R}[x_1, x_2]$ of bivariate real polynomials is the (pointwise) limit of ideal projectors whose kernel has $\mathbb{R}_{<n}[x_1]$ as an algebraic complement.*

In particular, this settles (in the negative) the question, raised in Remark 3.2 of [S0y], whether there might be "bad" ideals in $\mathbb{R}[x_1, x_2]$, i.e., ideals of colength n having nontrivial intersection with $\mathbb{R}_{<n}[x_1]$.

Before proving this theorem, we discuss some ancillary results. In this discussion, we call, for simplicity, a real matrix **cyclic** if it is cyclic on $\mathbb{C}^n$, i.e., is non-derogatory over $\mathbb{C}$.

(5.3) Proposition. *Let $A = (A_1, \ldots, A_d)$ be a sequence of commuting real matrices of order n. If A_1 is cyclic, then the real ideal*

$$\mathcal{I}_A^{(\mathbb{R})} := \{p \in \mathbb{R}[\mathbf{x}] : p(A) = 0\}$$

is an algebraic complement of $\mathbb{R}_{<n}[x_1] = \operatorname{ran}[1, x_1, \ldots, x_1^{n-1}]$ in $\mathbb{R}[\mathbf{x}]$.

Proof: Let y be a cyclic vector for A_1. Then

$$\mathbb{C}^n = \{g(A_1)y : \deg g < n\},$$

hence (2.7) holds with $k = \mathbb{C}$, $Y = \mathbb{C}^n$ and $F = \mathbb{C}_{<n}[x_1]$, thus with $\dim Y \geq \dim F$, and therefore, as shown earlier for an arbitrary such k, Y, and F, also (2.3) holds in this case. In particular, $\mathbb{C}_{<n}[x_1]$ is an algebraic complement of $\ker \Phi_A$ in $\mathbb{C}[\mathbf{x}]$, and, since the A_j are real matrices, this implies that $\mathbb{R}_{<n}[x_1]$ is an algebraic complement of $= \mathcal{I}_A^{(\mathbb{R})}$ in $\mathbb{R}[\mathbf{x}]$. $\square$

(5.4) Fact. *For every real matrix of order n, there are cyclic real matrices that commute with them.*

(5.5) Corollary. *Any pair (A_1, A_2) of real commuting matrices can be approximated by pairs (B_1, B_2) of real commuting matrices with B_1 cyclic.*

Proof: The earlier proof that cyclic matrices are dense in $\mathcal{C}(A)$ goes through verbatim after $\mathbb{C}$ is replaced by $\mathbb{R}$ except, perhaps, for the claim that there are real cyclic matrices in $\mathcal{C}(A)$. But this is taken care of by Fact 5.4. $\square$

Proof of Theorem 5.2: Let P be an ideal projector of rank n on $\mathbb{R}[x_1,x_2]$, set $F := \operatorname{ran} P$, and, for $j=1,2$, let A_j be the matrix representation of the linear map $F \to F : f \mapsto P(x_j f)$. By Corollary 5.5, we can approximate the pair $A := (A_1, A_2)$ with commuting pairs $B := (B_1, B_2)$ of real matrices with B_1 cyclic and, for all such B close enough to A, the linear map $\Phi_B : p \mapsto p(B)$ on $\mathbb{R}[x_1,x_2]$ also carries F 1-1 onto $\operatorname{ran}\Phi_B$ (since Φ_A does), hence the map

$$P_B := (\Phi_B|_F)^{-1}\Phi_B$$

is well-defined, a linear projector with range F and nullspace $\ker P_B = \ker \Phi_B$, and close to P, while, by Proposition 5.3, $\ker \Phi_B$ has $\mathbb{R}_{<n}[x_1]$ as an algebraic complement. □

6. Minimal annihilating polynomials

In this final section, we briefly touch on questions raised by our use of the algebra

$$\operatorname{ran}\Phi_A = \{p(A) : p \in k[\mathbf{x}]\}$$

generated by a sequence $A = (A_1, \ldots, A_d)$ of commuting linear maps on some finite-dimensional linear space Y over some field k.

We did not find much discussion of it in the Linear Algebra literature, – except, of course, for the case $d = 1$, in which the ideal $\ker \Phi_A$ is principal and its generator is the minimal annihilating polynomial for the sole matrix involved, and for the case of arbitrary d, in which the A_i are simultaneously upper triangularizable (over $\mathbb{C}$).

Because of the major role played by minimal annihilating polynomials in basic Linear Algebra, we had expected to find the analogous discussion for $d > 1$, with the role of minimal annihilating polynomial played by some suitable basis for the ideal $\ker \Phi_A$.

In view of the fact that [R] proposes to obtain such an ideal basis by a version of the Möller-Buchberger algorithm [MB], it might be worthwhile to point out that straightforward Gauss elimination suffices for this task, as described in [dB07], applied to the column map

$$[A^\alpha : \alpha \in \mathbb{Z}_+^d] : k_0^{\mathbb{Z}_+^d} \to L(Y) : a \mapsto \sum_\alpha a(\alpha) A^\alpha,$$

with the columns so ordered that the corresponding ordering

$$\prec$$

of $\mathbb{Z}_+^d$ is **monomial** in the sense that (i) every subset has a first element, and (ii) $\alpha \prec \beta$ implies that $\alpha + \gamma \prec \beta + \gamma$ for all $\alpha, \beta, \gamma \in \mathbb{Z}_+^d$.

Recall that Gauss elimination classifies the columns of a matrix, hence, more generally, the columns of a column map, into *free* and *bound*, with a column **free** if it is a linear combination of the columns to the left of it, and **bound** otherwise. The particular ordering of the columns of $[A^\alpha : \alpha \in \mathbb{Z}_+^d]$ guarantees that if column α is free, then so is column $\alpha + \gamma$ for every $\gamma \in \mathbb{Z}_+^d$. Thus we should look for the **minimally free** columns, i.e., those columns α for which all columns $\alpha - \gamma$ with $0 \neq \gamma \in \mathbb{Z}_+^d$ are bound. Each such column can be written as a weighted sum of *bound* columns to the left of it, thus supplying an element of $\ker \Phi_A$ of the form

$$p_\alpha = \mathbf{x}^\alpha - \sum_{\beta \prec \alpha} a(\beta)\mathbf{x}^\beta,$$

with $a(\beta) = 0$ for any free column β. The resulting set $\{p_\alpha\}$ is not only a reduced Groebner basis for $\ker \Phi_A$, it is a reduced H-basis.

It remains to discuss the actual determination of the free columns. This can be done for any column map $W := [w_1, w_2, \ldots]$ into a linear space X by choosing a linear map Q on X into some k^s that carries $\operatorname{ran} W$ onto k^s and then applying Gauss elimination to the matrix $QW = [Qw_1, Qw_2, \ldots]$. In our specific case, the columns are k-valued matrices of some order n, hence a natural choice for Q would associate a matrix with the 'vector' of its entries. Further, we are not really interested in finding all free columns but only the minimally free ones. Hence, as soon as we find a free column, α say, we immediately remove all columns $\alpha + \gamma$ with $\gamma \in \mathbb{Z}_+^d \backslash 0$ from further consideration. This guarantees that the next free column found is minimally free, too, and the Hilbert Basis theorem guarantees that we will run out of columns to look at after finitely many steps.

For the univariate case, i.e., $d = 1$, we think of the minimal annihilating polynomial as giving us much information about the spectrum of the sole matrix in question. For arbitrary d, recall from Lemma 3.2 that

$$\operatorname{spect}(g(A)) = g(\mathcal{V}), \quad g \in \mathbb{C}[\mathbf{x}]. \tag{6.1}$$

Also, the (univariate) minimal annihilating polynomial $h_i \in \mathbb{C}[x_i]$ of the linear map A_i is, as a polynomial in $\mathbf{x}$, in $\mathcal{I}$, hence vanishes at the ith coordinates of the points in $\mathcal{V}$. This connection between the variety of the ideal $\mathcal{I}$ and the spectrum of the A_i has been put to good use, e.g., in [AS], [St] and [M93], to determine the former from the latter in the special case that the A_i are the linear maps m_{x_i} that carry $[g] = g + \mathcal{I}$ to $[x_i g]$, all i.

In this special case, A is cyclic, hence any cyclic B with $\ker \Phi_B = \ker \Phi_A$ is similar to A. However, (6.1) holds for any A, cyclic or not, as long as the underlying field is $\mathbb{C}$ (or, more generally, algebraically closed). This raises the question of how much Φ_A will tell us about the spectral structure of the A_i or, more generally, of $g(A)$ when A is not cyclic. In that case, as Example 2.12 shows, there may not even exist $y \in Y$ with $\ker \Phi_{A,y} = \ker \Phi_A$.

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