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The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$

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Abstract: A simple characterization is given of finitely generated subspaces of $L_2(\mathbb{R}^d)$ which are invariant under translation by any (multi)integer, and used to give conditions under which such a space has a particularly nice generating set, namely a basis, and, more than that, a basis with desirable properties, such as stability, orthogonality, or linear independence. The last property makes sense only for 'local' spaces, i.e., shift-invariant spaces generated by finitely many compactly supported functions, and special attention is paid to such spaces.

As an application, we prove that the approximation order provided by a given local space is already provided by the shift-invariant space generated by just one function, with this function constructible as a finite linear combination of the finite generating set for the whole space, hence compactly supported. This settles a question of some 20 years' standing.

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1. Introduction; notation

We are interested in **shift-invariant subspaces** of $L_2(\mathbb{R}^d)$. By this we mean any closed linear subspace S of $L_2(\mathbb{R}^d)$ which is **closed under shifts**, i.e., which, for each $\phi \in S$, also contains $\phi(\cdot - \alpha)$ for every multi-integer $\alpha \in \mathbb{Z}^d$. Such a space contains, with each $\Phi \subset S$, the set

$$S_0(\Phi)$$

of all finite linear combinations of integer translates of all the $\phi \in \Phi$. We write

$$S(\Phi)$$

for the L_2-closure of $S_0(\Phi)$, and call it the shift-invariant space **generated by** Φ. In particular, we call S a **finitely generated shift-invariant**, or **FSI** space, if $S = S(\Phi)$ for some finite set Φ. We call the cardinality of a smallest generating set for S the **length** of S, and write

$$\operatorname{len} S := \min\{\#\Phi : S = S(\Phi)\}.$$

We note that the orthogonal projector P_S onto a closed shift-invariant space S commutes with any (integer) shift, i.e.,

$$P_S f(\cdot - \alpha) = (P_S f)(\cdot - \alpha), \quad \alpha \in \mathbb{Z}^d. \tag{1.1}$$

Shift-invariant spaces play a basic role in multivariate approximation theory, since such a space S, together with its **dilates**

$$S^h := \{f(\cdot/h) : f \in S\},$$

provides theoretically and practically convenient approximating families. This was recognized some time ago by those working with the Finite Element Method, but has been already exploited, in a univariate context, as long as there has been interpolation in a table. More recently, most work with so-called radial basis functions, and all the work on multiresolutions, hierarchical bases, and wavelets, involves **principal** shift-invariant, or **PSI** spaces, i.e., spaces $S = S(\{\phi\}) =: S(\phi)$ generated by one function.

In [BDR], we derived a simple characterization of a PSI space and of the orthogonal projection onto a PSI space. In the present paper, we derive corresponding results for FSI spaces and use them to explore in some detail the structure of FSI and PSI spaces. Of particular concern are the existence and construction of generating sets with desirable properties, such as stability, linear independence, or orthogonality. One of the fruits of this labor is a proof (cf. Theorem 4.1) that*, for every finite set $\Phi \subset L_2(\mathbb{R}^d)$ of compactly supported functions, and any compactly supported $g \in L_2(\mathbb{R}^d)$, there exists $\psi \in S_0(\Phi)$ so that, for any $f \in L_2(\mathbb{R}^d)$,

$$\operatorname{dist}(f, S(\Phi)) \le \operatorname{dist}(f, S(\psi)) \le \operatorname{dist}(f, S(\Phi)) + 2\operatorname{dist}(f, S(g)).$$

This settles, at least in the context of L_2, a problem of some twenty years' standing.

We next provide a more detailed overview of the paper's results and, in the process, introduce some notation.

It turns out that shift-invariant spaces are closely related to the doubly invariant spaces studied in operator theory. To bring out this useful connection, we recall the standard notation

$$\widehat{f}(x) := \int_{\mathbb{R}^d} e^{-ixy} f(y)\, dy$$

* the published paper fails to mention the function g.

for the Fourier transform of $f \in L_2(\mathbb{R}^d)$, and consider the map T, given by the rule

$$Tf : \mathbb{T}^d \to \ell_2(\mathbb{Z}^d) : x \mapsto \big(\widehat{f}(x + 2\pi\alpha)\big)_{\alpha \in \mathbb{Z}^d} =: \widehat{f}_{\|x}, \tag{1.2}$$

i.e., T associates with each $f \in L_2(\mathbb{R}^d)$ the element Tf of the Hilbert space $L_2(\mathbb{T}^d, \ell_2(\mathbb{Z}^d))$ of all $\ell_2(\mathbb{Z}^d)$-valued square-integrable functions on the d-dimensional torus

$$\mathbb{T}^d.$$

The identity

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{T}^d} \sum_{\alpha \in 2\pi\mathbb{Z}^d} f(\cdot + \alpha),$$

valid for any $f \in L_1(\mathbb{R}^d)$, shows that T is well-defined and is unitary, i.e., for every $f, g \in L_2(\mathbb{R}^d)$,

$$(2\pi)^d \langle f, g\rangle_{L_2(\mathbb{R}^d)} = \langle Tf, Tg\rangle := \int_{\mathbb{T}^d} [\widehat{f}, \widehat{g}]\, dx, \tag{1.3}$$

with

$$[f, g] : \mathbb{T}^d \to \mathbb{C} : x \mapsto \langle f_{\|x}, g_{\|x}\rangle_{\ell_2} = \sum_{\beta \in 2\pi\mathbb{Z}^d} f(x+\beta)\overline{g}(x+\beta) \tag{1.4}$$

the **bracket product** of f with g, i.e., the **periodization** of $f\overline{g}$. Note that $[f, g] \in L_1(\mathbb{T}^d)$ for any $f, g \in L_2(\mathbb{R}^d)$.

The map T is useful here because a subset S of $L_2(\mathbb{R}^d)$ is shift-invariant if and only if $T(S)$ is a **doubly-invariant** subset of $L_2(\mathbb{T}^d, \ell_2(\mathbb{Z}^d))$, i.e., invariant under pointwise multiplication by $e^{i\alpha\cdot}$ for any $\alpha \in \mathbb{Z}^d$. This is a well-known concept from operator theory that readily yields the characterization of FSI spaces given in Theorem 1.7 below and much used in this paper. Here is the derivation.

From the known characterization of doubly-invariant spaces (see [H; Theorem 8] for $d = 1$; the extension to the multivariate setting presents no difficulty), we conclude the following

Result 1.5. *The closed linear subspace S of $L_2(\mathbb{R}^d)$ is shift-invariant if and only if*

$$S = \{f \in L_2(\mathbb{R}^d) : \widehat{f}_{\|x} \in J(x), \text{ a.e. on } \mathbb{T}^d\},$$

with each $J(x)$, $x \in \mathbb{T}^d$, a closed linear subspace of $\ell_2(\mathbb{Z}^d)$, and such that the function $\mathbb{T}^d \to \mathbb{C} : x \mapsto \langle P_{J(x)} f, g\rangle_{\ell_2(\mathbb{Z}^d)}$ is measurable for every $f, g \in \ell_2(\mathbb{Z}^d)$. Here, $P_{J(x)}$ is the orthogonal projector from $\ell_2(\mathbb{Z}^d)$ onto $J(x)$.

Such a map J on $\mathbb{T}^d$ into the set of closed linear subspaces of $\ell_2(\mathbb{Z}^d)$ is called a **range function**. For a given closed shift-invariant subspace S of $L_2(\mathbb{R}^d)$, we denote by

$$J_S$$

the range function associated with S. Result 1.5 is useful for the study of S to the extent that we have information about the corresponding range function J_S. Fortunately, it is rather easy to prove (cf. Proposition 3.1) that, for a FSI space S with generating set Φ, necessarily

$$J_S(x) = \operatorname{span}\{\widehat{\phi}_{\|x} : \ \phi \in \Phi\} \quad a.e. \tag{1.6}$$

This formula for the range function provides the following very useful characterization of a FSI subspace of $L_2(\mathbb{R}^d)$ (the special case of a PSI space was treated in [BDR] without recourse to Result 1.5).

Theorem 1.7. *For any finite subset Φ of $L_2(\mathbb{R}^d)$ and any $f \in L_2(\mathbb{R}^d)$, $f \in S(\Phi)$ if and only if*

$$\widehat{f} = \sum_{\phi\in\Phi} \tau_\phi \widehat{\phi} \tag{1.8}$$

for some 2π-periodic functions τ_ϕ.

We call Φ a **basis** for $S(\Phi)$ if the periodic functions τ_ϕ in (1.8) are uniquely determined by f. Theorem 1.7 together with (1.6) implies that this happens if and only if $\widehat{\Phi}_{\|x}$ is a basis for $J_S(x)$ for a.e. x. Thus, a FSI space S cannot have a basis unless S is **regular** in the sense that $\dim J_S(x)$ is constant a.e. We prove (cf. Corollary 3.14) that, conversely, any regular FSI space has a basis. We also prove (cf. Corollary 3.11) that the τ_ϕ in (1.8) are necessarily measurable if Φ is a basis.

More generally, we call Φ a **quasi-basis** for $S(\Phi)$ if the periodic functions τ_ϕ in (1.8) are uniquely determined by f a.e. on

$$\sigma(S) := \{x \in \mathbb{T}^d : \dim J_S(x) > 0\},$$

the **spectrum** of S. Thus, a (nontrivial) FSI space S cannot have a quasi-basis unless it is **quasi-regular** in the sense that there is a unique $j > 0$ for which

$$\sigma_j(S) := \{x \in \mathbb{T}^d : \dim J_S(x) = j\}$$

has positive measure. We prove (cf. Corollary 3.14) that, conversely, every quasi-regular FSI space has a quasi-basis, and that any quasi-basis for S has cardinality $\operatorname{len} S$. We also show (cf. Theorem 3.5) that a FSI space S is the orthogonal sum of $\operatorname{len} S$ PSI spaces.

It is quite clear that not every FSI space contains a basis, but, since the sets $\sigma_j(S)$ provide a partition for $\mathbb{T}^d$, every FSI space is the orthogonal sum of quasi-regular spaces (see Theorem 3.2).

Assuming that Φ is a quasi-basis for $S(\Phi)$, we provide (in Theorem 3.9) the formula

$$\widehat{Pf} = \sum_{\phi\in\Phi} \frac{\det G_\phi(f)}{\det G(\Phi)}\, \widehat{\phi} \tag{1.9}$$

for the Fourier transform of the orthogonal projection Pf of $f \in L_2(\mathbb{R}^d)$ onto $S(\Phi)$. Here, $G(\Phi)$ is the **Gramian matrix**

$$G(\Phi) := \Big([\widehat{\phi}, \widehat{\psi}]\Big)_{\phi,\psi\in\Phi}$$

of the finite set $\Phi \subset L_2(\mathbb{R}^d)$, and the related matrix $G_\phi(f)$ is obtained from $G(\Phi)$ by changing its ϕth row to $([\widehat{f}, \widehat{\psi}])_{\psi\in\Phi}$.

The characterization (1.7) of the space $S(\Phi)$ and the formula (1.9) for the orthogonal projector onto $S(\Phi)$ provide a solid foundation for our main goal: the search for particularly suitable generators for a given FSI space S.

We call the generating set Φ for S **stable** if the map

$$\Phi *' : c \mapsto \sum_{\phi\in\Phi} \sum_{\alpha\in\mathbb{Z}^d} \phi(\cdot - \alpha) c(\phi, \alpha)$$

is a Hilbert-space isomorphism between $\ell_2(\Phi \times \mathbb{Z}^d)$ and $S(\Phi)$ (with the convergence above assumed to be unconditional in $L_2(\mathbb{R}^d)$). If every $\phi \in \Phi$ has compact support, hence $\Phi *' c$ makes sense (as a pointwise limit) for arbitrary $c \in \mathbb{C}^{\Phi\times\mathbb{Z}^d}$, then we call Φ **linearly independent** if $\Phi *'$ is 1-1 as a map on $\mathbb{C}^{\Phi\times\mathbb{Z}^d}$. Finally, whether or not the elements of Φ are compactly supported, we call Φ **orthogonal** if

$$\langle \phi(\cdot - \alpha), \psi(\cdot - \beta)\rangle \;=\; \delta_{\phi,\psi}\delta_{\alpha,\beta} \quad \text{for} \phi, \psi \in \Phi, \;\; \alpha, \beta \in \mathbb{Z}^d.$$

These properties are listed here in increasing order of difficulty of attainment.

We show (cf. Corollary 3.30) that a quasi-basis Φ is stable if and only if the eigenvalues of $G(\Phi)(x)$ are bounded and bounded away from zero uniformly for $x \in \mathbb{T}^d$ except, perhaps, for a set of measure zero. In particular, a quasi-basis which is not a basis cannot be stable. But, for this case, we introduce and characterize a suitable notion of **quasi-stability**, and show (cf. Theorem 3.20) that any FSI space with a quasi-basis has a quasi-stable basis. In particular (cf. Corollary 3.31), any FSI space with full spectrum has a stable basis and even has an orthogonal basis.

We call a shift-invariant space **local** if it is generated by a finite set Φ of *compactly supported* functions. For such Φ, the entries of $G(\Phi)$ are trigonometric polynomials and the spectrum of $S(\Phi)$ is full. We prove (cf. Proposition 3.42) that a *univariate* local space is the finite orthogonal sum of PSI spaces, each generated by some compactly supported function whose integer translates are linearly independent. In particular, the generating set for $S(\Phi)$ made up of these linearly independent generators is linearly independent.

Because of the present wide interest in PSI spaces, and despite the fact that most of our results on PSI spaces can be regarded as a specialization of their FSI counterparts, we found it important to devote a separate self-contained section to the analysis of PSI spaces. As it turns out, the analysis of that part can be handled with essentially no use of the general tools detailed above (such as the range function and the pointwise projection), although, of course, these notions are implicit in the discussion. As a matter of fact, we tried, whenever possible, to present our arguments via the generator(s) of the space rather than the range function, since, as a rule, in practical problems the generators constitute the explicitly known information.

Here is an outline of the paper. In Section 2, we determine under which circumstances a PSI space has a quasi-stable, respectively quasi-orthogonal generator. In case the PSI space is generated by a compactly supported function, we also look for the essentially unique linearly independent generator (if there is any), stressing the fact that the situation is rather different for $d = 1$ than for $d > 1$. Section 3 is devoted to the discussion of FSI spaces, providing proofs of the various results mentioned earlier. Finally, Section 4 considers the application of our structure results to Approximation Theory.

We finish this introduction with some observations concerning the bracket product (1.4), and some additional notations and conventions.

Given $f, g \in L_2(\mathbb{R}^d)$, we know that $[\widehat{f}, \widehat{g}] \in L_1(\mathbb{T}^d)$. This entitles us to consider the Fourier series of $[\widehat{f}, \widehat{g}]$. The following two lemmata follow from standard properties of the Fourier transform.

Lemma 1.10. *For $f, g \in L_2(\mathbb{R}^d)$, $[\widehat{f}, \widehat{g}]$ has the Fourier series $\sum_{\alpha\in\mathbb{Z}^d} \langle f, g(\cdot + \alpha)\rangle e_\alpha$. Consequently, f is orthogonal to the PSI space $S(g)$ if and only if $[\widehat{f}, \widehat{g}] = 0$.*

Here and below,

$$e_\theta : x \mapsto e^{i\theta x},\ \ \theta \in \mathbb{R}^d$$

is the complex exponential with **frequency** θ.

For $\phi \in L_2(\mathbb{R}^d)$, we define its **symbol**,

$$\widetilde{\phi} := [\widehat{\phi}, \widehat{\phi}]^{1/2}.$$

It is clear that $\widetilde{\phi}$ is a nonnegative 2π-periodic function. Also

Lemma 1.11. $\widetilde{\phi} \in L_2(\mathbb{T}^d)$, *and* $\|\widetilde{\phi}\|_{L_2(\mathbb{T}^d)} = \|\widehat{\phi}\| = (2\pi)^{d/2}\|\phi\|$.

Here and hereafter

$$\|\cdot\| := \|\cdot\|_{L_2(\mathbb{R}^d)}.$$

Note that the map $\phi \mapsto \widetilde{\phi}$ is not linear.

We reserve the notation $f * g$ for the standard convolution product (say, on $L_2(\mathbb{R}^d) \times L_2(\mathbb{R}^d)$), and, following a tradition in Approximation Theory, use the separate notation $*'$ for the semi-discrete convolution

$$f{*'}c := \sum_{\alpha\in\mathbb{Z}^d} f(\cdot - \alpha)c(\alpha),$$

which is well-defined under various assumptions on the function f and the sequence c.

All measurable subsets $\Omega \subset \mathbb{R}^d$ are defined in this paper modulo a null-set. For such Ω, we use the abbreviation

$$\Omega^\circ := \Omega + 2\pi\mathbb{Z}^d$$

for the 2π-periodic extension of the subset Ω of $\mathbb{T}^d$. Also, we identify any function τ on $\mathbb{T}^d$ with its 2π-periodic extension to $\mathbb{R}^d$.

If f is a measurable function, then its reciprocal is, offhand, only defined on the support of f. We find it convenient to extend $1/f$ to the whole domain of f by setting it to 0 off the support of f. Thus, if g is a measurable function with the same domain as f, then

$$g/f : x \mapsto \begin{cases} g(x)/f(x) & x \in \operatorname{supp} f \cap \operatorname{supp} g; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we use the (self-evident) notation

$$\widehat{F} := \{\widehat{f} :\ f \in F\}, \quad \text{for} F \subset L_2(\mathbb{R}^d)\}.$$

2. Principal shift-invariant spaces

2.1. Preliminaries

We study here the structure of the PSI space $S(\phi)$ and the action of certain operations (such as orthogonal projection or semi-discrete convolution) associated with this space. The analysis makes essential use of the following two results, proved in [BDR], which are special cases of (1.9) and Theorem 1.7, respectively.

Result 2.1. *Let $\widehat{P}$ be the orthogonal projector of $L_2(\mathbb{R}^d)$ onto $\widehat{S(\phi)}$. Then*

$$\widehat{P}(f) = \frac{[f, \widehat{\phi}]}{[\widehat{\phi}, \widehat{\phi}]}\widehat{\phi}. \tag{2.2}$$

Result 2.3. *For any $\phi \in L_2(\mathbb{R}^d)$,*

$$\widehat{S(\phi)} = \{\tau\widehat{\phi} \in L_2(\mathbb{R}^d) :\ \tau \text{ is } 2\pi\text{-periodic}\}.$$

Further, we shall make use of the following immediate corollaries of Result 2.3.

Corollary 2.4. *If $\psi \in S(\phi)$, then $S(\psi) = S(\phi)$ if and only if $\operatorname{supp}\widehat{\phi} \subset \operatorname{supp}\widehat{\psi}$.*

Note that, by symmetry, the corollary implies that $\operatorname{supp}\widehat{\phi} = \operatorname{supp}\widehat{\psi}$ for any two generators of a PSI space. Also, recall our convention that all measurable sets are determined up to a null-set.

Proof. The necessity follows directly from Result 2.3, hence we only discuss the sufficiency. Since $\psi \in S(\phi)$, we have $S(\psi) \subset S(\phi)$. To prove the reverse inclusion, we need to show that $\phi \in S(\psi)$. By Result 2.3, there exists a 2π-periodic τ such that $\widehat{\psi} = \tau\widehat{\phi}$. Since $\operatorname{supp}\widehat{\psi} = \operatorname{supp}\widehat{\phi}$, τ is nonzero a.e. on that set, and we conclude that $\widehat{\phi} = (1/\tau)\widehat{\psi}$. Since $\widehat{\phi} \in L_2(\mathbb{R}^d)$, Result 2.3 now provides the conclusion that $\phi \in S(\psi)$. □

The corollary implies in particular that we can always "undo finite differencing" in the following sense:

Corollary 2.5. *If $\psi \in S_0(\phi)\backslash 0$, then $S(\psi) = S(\phi)$.*

Proof. By assumption, $\widehat{\psi} = \tau\widehat{\phi}$ for some 2π-periodic trigonometric polynomial τ. Thus $\operatorname{supp}\widehat{\psi} = \operatorname{supp}\widehat{\phi}$. Now apply Corollary 2.4. $\square$

Since spaces that contain (nontrivial) compactly supported functions are very important in theory and applications, we will frequently restate our results for that special case, as we do now.

Corollary 2.6. *Let S be a PSI space. Then every compactly supported $\phi \in S\backslash 0$ generates S.*

Proof. Since ϕ is of compact support, $\widehat{\phi}$ is the restriction to $\mathbb{R}^d$ of an entire function, hence $\operatorname{supp}\widehat{\phi} = \mathbb{R}^d$. Therefore, if ψ is a generator for S, we have $\operatorname{supp}\widehat{\psi} \subset \operatorname{supp}\widehat{\phi}$. Now apply Corollary 2.4. $\square$

Here is a simple illustration of the usefulness of the last corollary.

Example 2.7. Let χ be the characteristic function of $[0\,\mathrm{d}1]$, and let $H := \chi - \chi(\cdot - 1)$ be the Haar function. It is easy to see that $S(\chi)$ is the space of all functions in $L_2(\mathbb{R})$ which are piecewise constant with breakpoints at the integers (this follows also from the fact that the shifts of χ are stable). It seems harder to understand the nature of $S(H)$; however, both Corollary 2.5 and Corollary 2.6 imply at once that $S(H) = S(\chi)$. $\square$

Definition 2.8. *Let S be a PSI space. The* **spectrum** *$\sigma(S)$ of S is $\operatorname{supp}\widetilde{\phi}$, where ϕ is some (any) generator for S. We say that S is* **regular** *if its spectrum is full, i.e., if $\sigma(S) = \mathbb{T}^d$.*

It is easy to check that the definition of spectrum given in the introduction coincides for a PSI space with the one given here. In particular, the definition is independent of the choice of the generator.

We remark in passing that the question whether $\psi \in S$ generates S can be settled in terms of $\sigma(S)$:

Proposition 2.9. *Let S be a PSI space, $\psi \in S$. Then ψ generates S if and only if $\operatorname{supp}\widetilde{\psi} = \sigma(S)$.*

Proof. The necessity follows from the previous discussion. Assume therefore that ψ does not generate S. We need to show that $\operatorname{supp}\widetilde{\psi} \neq \sigma(S)$. Let ϕ be a generator for S. By Result 2.3, $\widehat{\psi} = \tau\widehat{\phi}$ for some 2π-periodic τ. Since ψ does not generate S, τ must vanish on some set $\Omega \subset \sigma(S)$ of positive measure (otherwise, $\operatorname{supp}\widehat{\psi} = \operatorname{supp}\widehat{\phi}$ and Corollary 2.4 would imply that ψ generates S). It follows that $\widehat{\psi}_{\|\Omega} = 0$, and hence $\widetilde{\psi}$ vanishes on Ω. $\square$

2.2. Quasi-stable and quasi-orthogonal generators

A PSI space S is usually defined in terms of a generator ϕ. However, sometimes, one would like to find a better behaved generator for the space. For example, while Result 2.3 gives an explicit description of the Fourier transform of the PSI $S(\phi)$, it does not enable us to write every $f \in S(\phi)$ as a convergent series $\phi *' c$, for some sequence c. On the other hand, already the Preliminaries make clear that any (nontrivial) PSI space has many generators. Thus the following questions arise here in a natural way:

(1) Does there exist $\psi \in S(\phi)$ such that $S(\phi) = \psi *' \ell_2(\mathbb{Z}^d)$?

(2) Does there exist a generator $\psi \in S(\phi)$ such that each $f \in \widehat{S(\phi)}$ can be written in a *unique way* in the form $f = \tau\widehat{\psi}$?

(3) Does there exist a **stable** generator ψ for $S(\phi)$ in the sense that $\psi *'$ induces a Hilbert space isomorphism between $\ell_2(\mathbb{Z}^d)$ and $S(\phi)$?

(4) Does there exist an **orthogonal** generator for $S(\phi)$ in the sense that its shifts form an orthogonal system?

Clearly, (3) requires more than (2), and (4) requires more than (3). On the other hand, it is (essentially) known that (3) implies (4). As we will see below, question (1) is answered affirmatively for arbitrary PSI spaces, and the answer to question (2) is independent of the generator chosen. Remarkably, the affirmative answer to each of the questions (2-4) is equivalent to the regularity of the space S:

Theorem 2.10. *Let S be a PSI space. Then the following conditions are equivalent:*
(a) S is regular, i.e., $\sigma(S) = \mathbb{T}^d$.
(b) S contains a stable generator.
(c) S contains an orthogonal generator.
(d) For every (some) generator ϕ of S and every (some) $f \in S$, $\widehat{f}$ is uniquely *represented in the form $\tau\widehat{\phi}$ for some 2π-periodic τ.*
In particular, if S contains nontrivial compactly supported functions, then S satisfies (a) hence (b-d) as well.

We will prove this theorem later. Already at this point we emphasize that Theorem 2.10 implies, in case S contains a compactly supported function, the existence of a stable generator but not the existence of a *compactly supported* stable generator. As a matter of fact, we will sketch a counterexample to this hoped-for result. On the positive side, we mention that for *univariate* PSI spaces, the results of [R2] imply that we *do* have a compactly supported stable generator whenever S contains compactly supported functions. We elaborate on this last point in the next subsection.

While Theorem 2.10 provides valuable information on regular PSI spaces, it fails to quantify the extent to which a PSI space fails to be regular, nor does it suggest any alternatives to the notions of stability and orthogonality in the absence of regularity. In what follows, we introduce weaker versions of stability and orthogonality which we refer to as "quasi-stability" and "quasi-orthogonality". We show that any PSI space contains a quasi-stable and even a quasi-orthogonal generator, and then characterize the quasi-stability and quasi-orthogonality of a given generator ϕ for S in terms of $\widetilde{\phi}$. Theorem 2.10, as well as some related results, are eventually proved by specializing the aforementioned results to regular spaces.

Our definitions are motivated by the following considerations: By Result 2.3, any $f \in S(\phi)$ is the inverse Fourier transform of $\tau\widehat{\phi}$ for some 2π-periodic function τ. Thus, *formally*, $f = \phi *' c$, with c the Fourier coefficients of τ (say, whenever $\tau \in L_1(\mathbb{T}^d)$). Further, since $\tau\widehat{\phi}$ is unchanged if we change τ off the spectrum of S, we expect $\phi *' c$ to be zero for any c whose Fourier series

$$\widehat{c} := \sum_{\alpha \in \mathbb{Z}^d} c(\alpha) e_\alpha$$

has its support in $\mathbb{T}^d \backslash \sigma(S)$. Defining

$$K_S := \{c \in \ell_2(\mathbb{Z}^d) : \operatorname{supp} \widehat{c} \subset (\mathbb{T}^d \backslash \sigma(S))\}, \tag{2.11}$$

we can only hope to find generators ϕ which behave nicely on the orthogonal complement of K_S, i.e., on the space

$$C_S := \{c \in \ell_2(\mathbb{Z}^d) : \operatorname{supp} \widehat{c} \subset \sigma(S)\}. \tag{2.12}$$

The discussion should be, of course, more careful, since the only requirement made of τ in Result 2.3 is that $\tau\widehat{\phi}$ be in L_2. In particular, there is offhand no reason to believe that $\tau = \widehat{c}$ for some $c \in \ell_2$.

Definitions 2.13. *Let $S = S(\phi)$. We say that ϕ is a* **quasi-stable** *(resp.,* **quasi-orthogonal***) generator if $\phi *' : \ell_2(\mathbb{Z}^d) \to L_2(\mathbb{R}^d)$ is well-defined (and bounded), vanishes on K_S, and provides an isomorphism (resp., an isometry) between C_S and S.*

We begin with a Lemma that makes the above discussion precise and provides the essential facts for our analysis of these properties.

Lemma 2.14. *Let $\phi \in L_2(\mathbb{R}^d)$.*

(a) If $\widetilde{\phi}$ is bounded, then $\phi*'$ is a bounded linear map from $\ell_2(\mathbb{Z}^d)$ into $S(\phi) \subset L_2(\mathbb{R}^d)$. Further,

$$\|1/\widetilde{\phi}\|^{-1}_{L_\infty(\sigma(S))}\|c\|_{\ell_2(\mathbb{Z}^d)} \;\le\; \|\phi*'c\|_{L_2(\mathbb{R}^d)} \;\le\; \|\widetilde{\phi}\|_{L_\infty(\sigma(S))}\|c\|_{\ell_2(\mathbb{Z}^d)}, \qquad \text{all } c \in C_S,$$

and these inequalities are sharp.

(b) If $\widetilde{\phi}$ is not bounded, then $\phi*'$ fails to be a bounded map on C_S.

Proof. (a): For any finitely supported sequence c, $(\phi*'c)\widehat{\ } = \widehat{c}\widehat{\phi}$, therefore

$$(2.15) \qquad (2\pi)^{d/2}\|\phi*'c\|_{L_2(\mathbb{R}^d)} = \|\widehat{c}\widehat{\phi}\|_{L_2(\mathbb{R}^d)} = \|\widehat{c}\widetilde{\phi}\|_{L_2(\mathbb{T}^d)} = \|\widehat{c}\widetilde{\phi}\|_{L_2(\sigma(S))}$$

for such a c, with $S := S(\phi)$.

Thus, if $\widetilde{\phi}$ is bounded, then, for arbitrary $c \in \ell_2(\mathbb{Z}^d)$, the series $\phi*'c$ converges unconditionally in $L_2(\mathbb{R}^d)$, necessarily to an element of $S(\phi)$, and so $\phi*'$ is a bounded linear map from $\ell_2(\mathbb{Z}^d)$ into $S(\phi) \subset L_2(\mathbb{R}^d)$. Moreover, (2.15) then holds for arbitrary $c \in \ell_2(\mathbb{Z}^d)$, and the inequalities of (a) as well as their sharpness now follow from (2.15) and from the fact that $\|\widehat{c}\|_{L_2(\mathbb{T}^d)} = \|c\|_{\ell_2(\mathbb{Z}^d)}(2\pi)^{d/2}$.

(b): if $\widetilde{\phi}$ is unbounded, then we can find, for each $n \in \mathbb{N}$, a set $\Omega_n \subset \sigma(S)$ of positive measure such that $\widetilde{\phi} \ge n+1$ on Ω_n. Approximating the characteristic function of this set by a sequence of trigonometric polynomials $\{\tau_k\}_k$, we conclude that for large enough k

$$(2\pi)^{d/2}\|\phi*'(\tau_k^\vee)\|_{L_2(\mathbb{R}^d)} = \|\tau_k\widehat{\phi}\|_{L_2(\mathbb{R}^d)} = \|\tau_k\widetilde{\phi}\|_{L_2(\mathbb{T}^d)} \ge n\|\tau_k\|_{L_2(\mathbb{T}^d)} = (2\pi)^{d/2}n\|\tau_k^\vee\|_{\ell_2(\mathbb{Z}^d)}.$$

Consequently, $\phi*'$ does not admit a continuous extension from the finitely supported sequences to all of $\ell_2(\mathbb{Z}^d)$, hence fails to be a bounded map on $\ell_2(\mathbb{Z}^d)$. □

The next two theorems provide characterizations of quasi-stability and quasi-orthogonality, along the lines of the above motivation.

Theorem 2.16. *Let S be a PSI space.*

(a) $\phi \in S$ is a quasi-stable generator for S if and only if both $\widetilde{\phi}$ and $1/\widetilde{\phi}$ are essentially bounded on $\sigma(S)$.

(b) Let ϕ be a quasi-stable generator for S, and $\psi \in S$. Then ψ is also a quasi-stable generator for S if and only if $\widehat{\psi} = \tau\widehat{\phi}$ for some 2π-periodic τ with $|\tau|$ and $1/|\tau|$ essentially bounded on $\sigma(S)$.

Proof. The proof is a direct consequence of Lemma 2.14.

(a): By Lemma 2.14, $\phi*'$ is a bounded map from $\ell_2(\mathbb{Z}^d)$ to $L_2(\mathbb{R}^d)$ if and only if $\widetilde{\phi}$ is bounded, and thus, we may assume the boundedness of $\widetilde{\phi}$ (on $\sigma(S)$, hence everywhere), and need only prove the equivalence between the quasi-stability of ϕ and the boundedness on $\sigma(S)$ of $1/\widetilde{\phi}$. Now, since $\widetilde{\phi}$ is bounded, $\phi*'$ is well-defined (and bounded) on K_S and C_S, and this map then clearly vanishes on K_S, hence $\phi*'\ell_2(\mathbb{Z}^d) = \phi*'C_S$. Thus, $S_0(\phi)$ is contained in $\phi*'C_S$, and since $S_0(\phi)$ is dense in $S(\phi)$, $\phi*'C_S$ is a dense subspace of $S(\phi)$. This finishes the proof, since then, by Lemma 2.14, $\phi*'$ is bounded below on C_S if and only if $1/\widetilde{\phi}$ is bounded on $\sigma(S)$, and, in such a case, $\phi*'C_S$ is closed, hence coincides with all of $S(\phi)$.

For (b), conclude from Proposition 2.9 that ψ generates $S(\phi)$ if and only if $\widehat{\psi} = \tau\widehat{\phi}$ for some 2π-periodic τ with $\operatorname{supp}\tau \supset \sigma(S)$. Since then $\widetilde{\psi} = |\tau|\widetilde{\phi}$, (b) follows from (a). □

The last result shows that quasi-stability competes with the *localizing power* of the generator. If $\sigma(S) \neq \mathbb{T}^d$, and ϕ is quasi-stable, then $\widetilde{\phi}$ cannot be continuous, since otherwise $1/\widetilde{\phi}$ must be unbounded near the boundary of $\sigma(S)$. This does *not* imply that quasi-stable generators do not exist. It only means that their associated $\widetilde{\phi}$ *must* be discontinuous at the boundary of the spectrum. Substituting $f = g = \phi$ in Lemma 1.10, this discontinuity translates into the fact that *a quasi-stable generator of a nonregular space does not decay fast at* ∞.

On the other hand, if S is regular, we might even have a compactly supported stable generator. In any event, if the generator ϕ decays fast enough at ∞ to make $\widetilde{\phi}$ continuous, then Theorem 2.16 implies the following.

Corollary 2.17. *If $S(\phi)$ is regular and $\widetilde{\phi}$ is continuous, then ϕ is a stable generator if and only if $\widetilde{\phi}$ vanishes nowhere.*

This last result is essentially known. For a compactly supported ϕ, it was stated in [SF] (see the proof in [DM]). Extensions from compactly supported functions to functions which have some decay at ∞ can be found in [JM2]. In all these references, the authors considered the shift-invariant space $\phi *' \ell_2(\mathbb{Z}^d)$. However, when $\widetilde{\phi}$ is continuous, hence bounded, $\|\tau\widehat{\phi}\| = \|\tau\widetilde{\phi}\|_{L_2(\mathbb{T}^d)} \leq \|\widetilde{\phi}\|_{L_\infty(\mathbb{T}^d)}\|\tau\|_{L_2(\mathbb{T}^d)}$, and therefore $\phi *' \ell_2(\mathbb{Z}^d)$ is then a (trivially dense) subspace of $S(\phi)$ (again, by Result 2.3).

We turn now our attention to quasi-orthogonality. First, we give several characterizations of quasi-orthogonality, and subsequently prove that *every PSI space contains a quasi-orthogonal generator* (hence has quasi-stable generators).

Theorem 2.18. *Let $\phi \in L_2(\mathbb{R}^d)$ and $S = S(\phi)$. Then the following conditions are equivalent:*

(a) ϕ is a quasi-orthogonal generator for $S(\phi)$.

(b) $\widetilde{\phi} = \chi_{\sigma(S)}$.

(c) The orthogonal projector onto $\widehat{S(\phi)}$ is given by

$$\widehat{P} : f \mapsto [f, \widehat{\phi}]\widehat{\phi}.$$

Proof. Assuming (a), we may transform the quasi-orthogonality into the Fourier transform domain to obtain that we have

$$\|\tau\|_{L_2(\sigma(S))} = \|\tau\widehat{\phi}\| = \|\tau\widetilde{\phi}\|_{L_2(\mathbb{T}^d)}$$

for any $\tau \in L_2(\mathbb{T}^d)$ with support in $\sigma(S)$. By Lemma 2.14(a), this can happen if and only if $\widetilde{\phi}$ and $1/\widetilde{\phi}$ have ∞-norm 1 on $\sigma(S)$, i.e., if and only if $\widetilde{\phi} = 1$ a.e. on $\sigma(S)$, and we obtain (b). Conversely, assuming (b), Theorem 2.16 implies that ϕ is quasi-stable, and the argument above can then be reversed to imply that ϕ is also quasi-orthogonal.

The implication (b)$\Longrightarrow$(c) follows from Result 2.1.

Finally, we show that (c) implies (b). Since $\widehat{P}\widehat{\phi} = \widehat{\phi}$, we obtain from (c) that

$$\widehat{\phi} = [\widehat{\phi}, \widehat{\phi}]\widehat{\phi} = \widetilde{\phi}^2\widehat{\phi}.$$

By periodization, this implies that $\widetilde{\phi}^2 = \widetilde{\phi}^4$, hence, since $\widetilde{\phi}$ is non-negative, it must be the characteristic function of its support. This support is, by definition, the spectrum $\sigma(S)$. $\square$

Corollary 2.19. *Assume that ϕ is a quasi-orthogonal generator for S. Then:*
(a) The orthogonal projector P onto S is given by

$$P : f \mapsto \sum_{\alpha\in\mathbb{Z}^d} \langle f, \phi(\cdot+\alpha)\rangle \phi(\cdot+\alpha). \tag{2.20}$$

(b)

$$\sum_{\alpha\in\mathbb{Z}^d\backslash 0} |\langle \phi, \phi(\cdot+\alpha)\rangle|^2 = \|\phi\|^2\ (2\pi)^{-d} \operatorname{meas}(\mathbb{T}^d\backslash\sigma(S)).$$

(c) $\psi \in S$ is a quasi-orthogonal generator for S if and only if $\widehat{\psi} = \tau\widehat{\phi}$ for some 2π-periodic τ that satisfies $|\tau| = 1$ on $\sigma(S)$.

Proof. Since ϕ is quasi-orthogonal, hence quasi-stable, we know that each $f \in \widehat{S(\phi)}$ can be written in the form $\tau\widehat{\phi}$, with $\tau \in L_2(\mathbb{T}^d)$. Thus, from the equivalence of (a) and (c) in Theorem 2.18 we know that the orthogonal projection of f into $S(\phi)$ is given by $\phi*'c$, with the sequence $c \in \ell_2(\mathbb{Z}^d)$ the Fourier coefficients for $[\widehat{f}, \widehat{\phi}]$. Invoking Lemma 1.10 (with $g = \phi$), we obtain (a).

To prove (b), we use Lemma 1.10 once more, now with $f = g = \phi$, which together with the equivalence of (b) and (a) in Theorem 2.18 shows that the Fourier coefficients of the function $\chi_{\sigma(S(\phi))} = [\widehat{\phi}, \widehat{\phi}]$ are $\{\langle \phi, \phi(\cdot+\alpha)\rangle\}_{\alpha\in\mathbb{Z}^d}$. Thus Parseval's identity implies that

$$(\operatorname{meas}\sigma(S))/(2\pi)^d = \|\chi_{\sigma(S(\phi))}\|^2/(2\pi)^d = \langle\phi,\phi\rangle^2 + \sum_{\alpha\in\mathbb{Z}^d\backslash 0} |\langle \phi, \phi(\cdot+\alpha)\rangle|^2.$$

This proves (b), since $\langle\phi,\phi\rangle = \|\phi\|^2 = \|\widetilde{\phi}\|^2_{L_2(\mathbb{T}^d)}/(2\pi)^d = \operatorname{meas}\sigma(S)/(2\pi)^d$, and $\operatorname{meas}(\mathbb{T}^d\backslash\sigma(S)) = (2\pi)^d - \operatorname{meas}\sigma(S)$.

To prove (c), let $\psi \in S(\phi)$. By Result 2.3, $\widehat{\psi} = \tau\widehat{\phi}$ for some 2π-periodic τ, and hence $\widetilde{\psi} = |\tau|\widetilde{\phi} = |\tau|\chi_{\sigma(S)}$, the last equation by the equivalence of (a) and (b) in Theorem 2.18. Furthermore, this equivalence implies that ψ is quasi-orthogonal if and only if $\widetilde{\psi} = \chi_{\sigma(S)}$, and we see that indeed this is equivalent to $|\tau| = 1$ on $\sigma(S)$. □

Remark. We emphasize that the formula (2.20) just proved for the orthogonal projection Pf of f into $S(\phi)$ in case ϕ is quasi-orthogonal is one of many formulas for Pf in case the spectrum of S differs from $\mathbb{T}^d$. For, in that case, $\phi*'$ fails to be 1-1 on $\ell_2(\mathbb{Z}^d)$, hence $Pf = \phi*'c$ for many different ℓ_2-sequences c. However, (2.20) is optimal in the sense that the coefficient sequence $(\langle f, \phi(\cdot+\alpha)\rangle)_{\alpha\in\mathbb{Z}^d}$ is of smallest possible ℓ_2-norm: Indeed, if $Pf = \phi*'c$ for some $c \in \ell_2$, then $[\widehat{f}, \widehat{\phi}]\widehat{\phi} = \widehat{Pf} = \widehat{c}\widehat{\phi}$, therefore, by periodizing, $[\widehat{f}, \widehat{\phi}] = \widehat{c}$ on $\sigma(S)$. Since $[\widehat{f}, \widehat{\phi}] = 0$ off $\sigma(S)$, it follows that $\|\widehat{c}\|_{L_2(\mathbb{T}^d)} \geq \|\chi_{\sigma(S)}\widehat{c}\|_{L_2(\mathbb{T}^d)} = \|[\widehat{f}, \widehat{\phi}]\|_{L_2(\mathbb{T}^d)}$. □

One obtains a quasi-orthogonal generator by quasi-orthogonalizing (or, normalizing) *any* given generator.

Theorem 2.21. *Let $\phi \in L_2(\mathbb{R}^d)$ and $S := S(\phi)$. Then the function q_ϕ, defined via its Fourier transform by*

$$\widehat{q_\phi} := \widehat{\phi}/\widetilde{\phi},$$

is a quasi-orthogonal generator for S.

Proof. Since $\widetilde{q_\phi} = (1/\widetilde{\phi})\widetilde{\phi} = \chi_{\sigma(S)} \in L_2(\sigma(S))$, we conclude that $(1/\widetilde{\phi})\widehat{\phi} = \widehat{q_\phi} \in L_2(\mathbb{R}^d)$. Since $1/\widetilde{\phi}$ is 2π-periodic, this implies with Result 2.3 that $q_\phi \in S$, hence with Theorem 2.18 that q_ϕ is a quasi-orthogonal generator for $S(q_\phi) \subset S$. But, since $\operatorname{supp}\widetilde{q_\phi} = \sigma(S)$, we conclude from Proposition 2.9 that, in fact, $S(q_\phi) = S$. □

We are now ready to prove Theorem 2.10.

Proof of Theorem 2.10. By Theorem 2.21, S contains a quasi-orthogonal generator. Assuming (a), we conclude that this generator is an orthogonal generator (say by Corollary 2.19(b)). This shows that (a)$\Longrightarrow$(c), while certainly (c)$\Longrightarrow$(b). On the other hand, if $\sigma(S) \neq \mathbb{T}^d$, then K_S is not trivial, then, the moment $\phi*'$ is well-defined on $\ell_2(\mathbb{Z}^d)$ (as required for stability), it must vanish on K_S. Therefore no generator for S can be stable. This shows that (b)$\Longrightarrow$(a).

It remains to shows that (a) and (d) are equivalent. If (a) is violated, then, with τ the 2π-periodic extension of $\chi_{\mathbb{T}^d\backslash\sigma(S)}$, we obtain that $\tau\widehat{\phi} = 0$ for every generator ϕ for S, hence no representation is unique. Conversely, if $\tau\widehat{\phi} = 0$ for some 2π-periodic τ and some generator ϕ, then also $|\tau|\widetilde{\phi} = 0$. This implies that $\tau = 0$ a.e., in case $\sigma(S) = \mathbb{T}^d = \operatorname{supp}\widetilde{\phi}$. □

In light of Theorem 2.10, the characterization of quasi-stability (Theorem 2.16) and quasi-orthogonality (Theorem 2.18) readily give:

Corollary 2.22. *The generator ϕ for $S(\phi)$ is stable, respectively orthogonal, if and only if $\widetilde{\phi}$ is bounded away from 0 and infinity (a.e.), respectively $\widetilde{\phi} = 1$ (a.e.).*

2.3. Compactly supported generators for PSI spaces

Let S be a PSI space containing a compactly supported function ϕ. Since $\widehat{\phi}$ is entire, $\widetilde{\phi}$ only vanishes on a set of measure zero, i.e., $S(\phi)$ is regular, and therefore $S = S(\phi)$ (see Corollary 2.6). Hence, by Theorem 2.10, there are stable generators and even orthogonal generators for S. Our interest is then in finding a *compactly supported stable/orthogonal* generator for S. But in the context of compactly supported generators, one studies as well another significant property of the generator, *the linear independence of its shifts.* Note that if $\phi \in L_2(\mathbb{R}^d)$ is compactly supported, then the series $\phi*'c$ converges uniformly on compact sets, for *any* sequence c (of arbitrary growth). Correspondingly, we define

$$\ker \phi*' := \{c : \mathbb{Z}^d \to \mathbb{C} : \phi*'c = 0\}$$

when ϕ is compactly supported.

Definition 2.23. *We say that the shifts of the compactly supported $\phi \in L_2(\mathbb{R}^d)$ are* **linearly independent** *if $\ker \phi*' = \{0\}$. With a slight abuse of language, we call ϕ a* **linearly independent generator** *for $S(\phi)$, in case ϕ is compactly supported and its shifts are linearly independent.*

The following characterization of linear independence was obtained in [R1]. Recall that for a compactly supported ϕ, $\widehat{\phi}$ is the restriction of an entire function. We use the same notation, $\widehat{\phi}$, for its analytic extension to $\mathbb{C}^d$.

Result 2.24. *The shifts of the compactly supported $\phi \in L_2(\mathbb{R}^d)$ are linearly dependent if and only if $\widehat{\phi}_{\|x} = \big(\widehat{\phi}(x+\beta)\big)_{\beta\in 2\pi\mathbb{Z}^d} = 0$ for some $x \in \mathbb{C}^d$.*

In view of this result, and the characterization of stability (see (2.22)), we can "rank" the properties of stability, linear independence, and orthogonality, as follows:

Proposition 2.25. *Let ϕ be a compactly supported $L_2(\mathbb{R}^d)$-function. Consider the following properties:*
(a) ϕ is a stable generator for $S(\phi)$;
(b) ϕ is a linearly independent generator for $S(\phi)$;

(c) ϕ is an orthogonal generator for $S(\phi)$.

Then (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a).

Proof. (c)$\Longrightarrow$(b): For any $c \in \ker \phi *'$,

$$0 = \langle \phi, \phi *' c \rangle = \sum_{\operatorname{supp} \phi \cap \operatorname{supp} \phi(\cdot - \alpha) \neq \emptyset} \langle \phi, \phi(\cdot - \alpha) \rangle c(\alpha),$$

the sum being finite since ϕ is compactly supported. With this, (c) implies that $c(0) = 0$. Since $\ker \phi *'$ is shift-invariant, this can only happen if $\ker \phi *' = \{0\}$.

(b)$\Longrightarrow$(a): Since ϕ is compactly supported, $\widetilde{\phi}^2$ is a trigonometric polynomial (by Lemma 1.10), hence continuous, therefore bounded on $\mathbb{T}^d$. Further, by Result 2.24, $\widehat{\phi}_{\|x} \neq 0$ for every x, hence $\widetilde{\phi}(x) > 0$ for every x, therefore, by continuity, also $1/\widetilde{\phi}$ is bounded. $\square$

In the context of compactly supported generators, it seems more interesting to study linear independence rather than stability. For example, if ϕ is a stable univariate generator for $S(\phi)$, then we can replace ϕ by $\psi := \phi - a\phi(\cdot - 1)$ and still get a stable generator, provided only that $|a| \neq 1$. This means that we can construct stable generators of arbitrarily large support. By contrast, a given PSI space can contain at most one linearly independent generator (up to shifts and multiplication by a scalar), as we show in a moment.

Thus, given a compactly supported function ϕ, we can ask whether $S(\phi)$ contains a linearly independent (or even orthogonal) compactly supported generator and whether such a generator is unique (up to shifts and scalar multiples). In this regard, we first recall the following result from [R2].

Result 2.26. *Let ϕ be a* univariate *compactly supported function. Then there exists a compactly supported ψ such that*

(a) The shifts of ψ are linearly independent;

(b) $\phi \in S_0(\psi)$.

It follows that $\operatorname{diam} \operatorname{supp} \psi \leq \operatorname{diam} \operatorname{supp} \phi$ with equality if and only if the shifts of ϕ itself are linearly independent. In the light of Corollary 2.5, this implies the following.

Corollary 2.27. *Let $\phi \in L_2(\mathbb{R})$ be compactly supported. Then $S(\phi)$ contains a linearly independent generator.*

We will see that already in two dimensions not every local PSI space contains a linearly independent generator. However, the *uniqueness* property of such a generator holds in arbitrary dimensions.

Theorem 2.28. *Let ϕ be a linearly independent generator for S. Then:*

(a) $\psi \in S(\phi)$ has compact support if and only if $\psi \in S_0(\phi)$.

(b) Up to shifts and scalar multiples, ϕ is the unique linearly independent generator for S.

Proof. Since ϕ is compactly supported by assumption, so is any element of $S_0(\phi)$. On the other hand, since ϕ is linearly independent, it is stable (by Proposition 2.25), hence every $\psi \in S(\phi)$ has a unique representation $\psi = \phi *' c_\psi$ with $c_\psi \in \ell_2(\mathbb{Z}^d)$. By [BR], the resulting linear functional $\psi \mapsto c_\psi(0)$ is *local*, i.e., there exists a ball B_R of some (finite) radius R (depending on ϕ) so that $c_\psi(0) = 0$ whenever $\operatorname{supp} \psi \cap B_R = \emptyset$. Since $(\phi *' c)(\cdot - \alpha) = \phi *' c(\cdot - \alpha)$ for any $\alpha \in \mathbb{Z}^d$, this proves (a).

Assume now that ψ is also a linearly independent generator for $S(\phi)$. Then, by (a), we also must have $\phi = \psi *' c_\phi$ for some unique finitely supported c_ϕ, therefore $\phi = \phi *' (c_\phi * c_\psi)$, hence, by linear independence, $c_\phi * c_\psi$ must be the delta-sequence, which can happen only if both c_ϕ and c_ψ have one-point support. This proves (b). $\square$

As an immediate consequence, we find that many of the spline spaces now in the literature do not have a compactly supported orthogonal generator:

Theorem 2.29. *Let ϕ be a linearly independent generator for S. If ϕ is not an orthogonal generator, then S has no compactly supported orthogonal generator.*

Proof. By Proposition 2.25, any compactly supported orthogonal generator is also a linearly independent generator, hence must be a multiple of a shift of ϕ, by Theorem 2.28, therefore ϕ itself must be orthogonal. □

For the univariate case, Theorem 2.28 together with Corollary 2.27 provides the following result:

Theorem 2.30. *Let S be a* univariate *PSI space, and let $\phi \in S$ be compactly supported. Then the following conditions are equivalent:*
(a) ϕ is a linearly independent generator for S.
(b) $S_0(\phi)$ contains all compactly supported elements of S.
(c) ϕ is of **minimal support**, *i.e., for every $\psi \in S$, $\operatorname{diam}\operatorname{supp}\phi \le \operatorname{diam}\operatorname{supp}\psi$.*
Furthermore, up to shifts and multiplication by constants, there exists a unique function $\psi \in S$ that satisfies any (hence all) of these conditions.

Proof. The implication (a)$\Longrightarrow$(b) follows from Theorem 2.28. The implication (b)$\Longrightarrow$(c) is trivial. To prove that also (c)$\Longrightarrow$(a), we invoke Corollary 2.27 to conclude that there exists a compactly supported linearly independent generator $\psi \in S$ such that $\phi \in S_0(\psi)$. Since we are assuming that ϕ is of minimal support, this can only happen if ψ is a constant multiple of a shift of ϕ, hence ϕ is also a linearly independent generator.

The existence of a linearly independent generator follows from Corollary 2.27, while the uniqueness assertion has been proved in Theorem 2.28. □

Since a compactly supported orthogonal generator is, in particular, linearly independent, we have the following result.

Corollary 2.31. *Let S be a local PSI space, and let ϕ be a compactly supported orthogonal generator of S. Then ϕ is of minimal support among all functions in S, $S_0(\phi)$ contains all compactly supported functions in S, and ϕ is, up to shifts, the unique compactly supported orthogonal generator of S.*

Theorem 2.32. *There exist local (multivariate) PSI spaces that contain no compactly supported stable generator (hence, in particular, no linearly independent generator).*

Proof. Perhaps the simplest example is the PSI space generated by the characteristic function χ of the L_1 unit ball in the plane, i.e., the diamond with vertices $(\pm 1, 0)$, $(0, \pm 1)$.

Suppose that ψ is a compactly supported function in $S(\chi)$. Then, by Result 2.1,

$$\widehat{\psi} = \frac{[\widehat{\psi}, \widehat{\chi}]\widehat{\chi}}{[\widehat{\chi}, \widehat{\chi}]}.$$

Further, by Lemma 1.10, $[\widehat{\psi}, \widehat{\chi}]$ is a trigonometric polynomial, since χ and ψ are compactly supported. It is easy to see that

$$\widehat{\chi} = 2\operatorname{sinc}(\xi x)\operatorname{sinc}(\zeta x)$$

with $\xi := (1, 1)$, $\zeta := (-1, 1)$, and with $\operatorname{sinc}(t) := \sin(t/2)/(t/2)$. From Lemma 1.10, one concludes that

$$[\widehat{\chi}, \widehat{\chi}](x_1, x_2) = 2 + \cos x_1 + \cos x_2.$$

Thus, $(\pi,\pi)+2\pi\mathbb{Z}^2$ are the only zeros of $[\widehat{\chi},\widehat{\chi}]$ on $\mathbb{R}^2$ and for $x\in\mathbb{R}^2\backslash 0$, $(D_x)^2[\widehat{\chi},\widehat{\chi}](\pi,\pi)\neq 0$. Now, let $\beta\in(\pi,\pi)+2\pi\mathbb{Z}^2$. Then $\widehat{\chi}$, hence $[\widehat{\psi},\widehat{\chi}]\widehat{\chi}$, is an entire function that vanishes on whichever line of the following two

$$\{\beta+t\xi:\ t\in\mathbb{R}\},\quad \{\beta+t\zeta:\ t\in\mathbb{R}\}$$

does not go through 0. Without loss, we may assume that it is the first line. By L'Hôpital rule,

$$\widehat{\psi}(\beta)=\frac{D_\xi^2([\widehat{\psi},\widehat{\chi}]\widehat{\chi})(\beta)}{D_\xi^2([\widehat{\chi},\widehat{\chi}])(\beta)}=0.$$

We conclude that $\widehat{\psi}$ vanishes on $(\pi,\pi)+2\pi\mathbb{Z}^2$, and hence, by Corollary 2.17, ψ is not a stable generator of $S(\chi)$. $\square$

3. Finitely generated shift-invariant spaces

3.1. General

Our analysis of finitely generated closed shift-invariant (or, FSI) spaces is based on the results concerning closed doubly-invariant subspaces of $L_2(\mathbb{T}^d,\ell_2(\mathbb{Z}^d))$ quoted in the introduction. Still, as already stressed before, most of the questions that we are interested in focus on the nature of the generating set Φ of the space rather than on the content of the "fibers" $J_S(x)$, $x\in\mathbb{T}^d$, and this explains our efforts to avoid the application of these tools, whenever such efforts do not interfere with the efficiency of the analysis. We divide our discussion into two parts: in the first subsection, we discuss various basic properties of general FSI spaces. The rest of the section is devoted to quasi-regular spaces (as defined in the introduction) and to problems concerning the properties of their possible generating sets.

3.2. General FSI spaces

Given a closed shift-invariant space S, we can always describe it in terms of its range function J_S, but, if we know a countable generating set Φ, then there is a simple description of the range function in terms of Φ, as follows.

Proposition 3.1. *If $S=S(\Phi)$ for some (at most) countable Φ, then*

$$J_S(x)=\operatorname{Span}\widehat{\Phi}_{\|x}=:\widehat{S}_{\|x},\quad \text{for a.e. } x\in\mathbb{T}^d,$$

with Span *denoting the* closed *linear span.*

Proof. If Φ is a countable generating set for S and $f\in S$, then $\widehat{f}$ is the limit of sums of the form $\widehat{f}_k:=\sum_{\phi\in\Phi}\tau_{k,\phi}\widehat{\phi}$, where each $\tau_{k,\phi}$ is a trigonometric polynomial and all but finitely many of these polynomials are zero. For each k and almost all $x\in\mathbb{T}^d$, $\widehat{f}_{k\|x}\in\operatorname{span}\widehat{\Phi}_{\|x}$. Since $\widehat{f}_k$ converges to $\widehat{f}$, we may assume, after going to a subsequence, that Tf_k converges pointwise a.e. to Tf. This says that, for a.e. $x\in\mathbb{T}^d$, the sequence $\{\widehat{f}_{k\|x}\}_k$ converges in ℓ_2 to $\widehat{f}_{\|x}$, showing that $\widehat{f}_{\|x}\in\operatorname{Span}\widehat{\Phi}_{\|x}$ a.e. Since f was arbitrary, this shows that $J_S(x)\subset\operatorname{Span}\widehat{\Phi}_{\|x}$ for a.e. $x\in\mathbb{T}^d$, while the converse inequality follows from Result 1.5. $\square$

Theorem 1.7 is nothing but a convenient rewrite of the above proposition for an FSI space.

We remark in passing that *every closed shift-invariant subspace of $L_2(\mathbb{R}^d)$ is countably generated.* Indeed, let S be a closed shift-invariant space and P_S its corresponding orthogonal projector. For $\beta \in 2\pi\mathbb{Z}^d$, let C_β be the cube $\beta + [-\pi \,\mathrm{d}\pi]^d$ and S_β the translation-invariant space with spectrum C_β, i.e.,

$$S_\beta := \{f \in L_2(\mathbb{R}^d) : \ \operatorname{supp} \widehat{f} \subset C_\beta\}.$$

By Result 2.3, S_β is the PSI space generated by $\psi_\beta := (\chi_{C_\beta})^\vee$, and hence it follows that $L_2(\mathbb{R}^d)$ is the (orthogonal) sum of the PSI spaces $\{S_\beta\}_{\beta\in 2\pi\mathbb{Z}^d}$. Since P_S commutes with shifts (cf. (1.1)), it follows that S is generated by $\{\phi_\beta := P_S\psi_\beta\}_\beta$.

Actually, these arguments provide the first step in the proof of Result 1.5 (cf. [H] for more details).

The following theorem shows that each FSI space can be written as the orthogonal sum of quasi-regular spaces with pairwise disjoint spectra. It provides a useful tool for extending results (such as the explicit formula for the orthogonal projection) from quasi-regular spaces to arbitrary FSI spaces.

In this theorem, we use, for $\Omega \subset \mathbb{R}^d$, the notation P_Ω for the orthogonal projector onto the translation-invariant space with spectrum $\Omega^\circ := \Omega + 2\pi\mathbb{Z}^d$, i.e.,

$$\widehat{P_\Omega f} = \chi_{\Omega^\circ}\widehat{f}.$$

Since $\operatorname{ran} P_\Omega$ is translation-invariant, P_Ω commutes with translations, and in particular with shifts, and hence maps any shift-invariant space S onto a (possibly different) shift-invariant space. Moreover, since χ_{Ω° is 2π-periodic, Theorem 1.7 (or, more generally, Result 1.5, if S is not FSI), shows that P_Ω maps S into *itself.* If now $\{\Omega_j\}_{j=0}^\infty$ is a partition of $\mathbb{T}^d$ (into measurable sets), it is clear that $\sum_{j=0}^\infty P_{\Omega_j}(L_2(\mathbb{R}^d)) = L_2(\mathbb{R}^d)$ and this sum is orthogonal because $\Omega_j^\circ \cap \Omega_k^\circ = \emptyset$ for $j \neq k$.

Theorem 3.2. *Let $S = S(\Phi)$ for some finite Φ, and set*

$$\sigma_j := \sigma_j(S) = \{x \in \mathbb{T}^d : \dim\operatorname{span} \widehat{\Phi}_{\|x} = j\}.$$

Then,

$$S = \sum_{j=1}^{\#\Phi} S(P_{\sigma_j}\Phi) \tag{3.3}$$

is an orthogonal decomposition of S into quasi-regular spaces

$$S_j := P_{\sigma_j}S = S(P_{\sigma_j}\Phi),$$

with $\sigma(S_j) = \sigma_j$ (up to nullsets), all j.

Proof. Since σ_j cannot have positive measure for $j > \#\Phi$, it follows that σ_j, $j = 0, \ldots, \#\Phi$ provides a finite partition of $\mathbb{T}^d$. Since $P_{\sigma_0}S = \{0\}$, the remarks preceding this theorem prove that, indeed, (3.3) is an orthogonal decomposition of S and S_j is generated by $P_{\sigma_j}\Phi$. It remains, thus, to show that each S_j is quasi-regular.

This last fact is also straightforward: it is clear that $\sigma(P_{\sigma_j}S) \subset \sigma_j$. On the other hand, for $x \in \sigma_j$, $\widehat{\Phi}_{\|x} = \chi_{\sigma_j^\circ}\widehat{\Phi}_{\|x}$, showing thus that $\sigma_k(P_{\sigma_j}S)$ is a null-set unless $k \in \{0, j\}$. In other words, $P_{\sigma_j}S$ is quasi-regular. $\square$

The theorem is valid with respect to a countable Φ, too, only that in this case one needs to verify that each S_j is a FSI space before concluding that it is quasi-regular. Furthermore, since at each $x \in \mathbb{T}^d$, $\dim \widehat{S}_{j\,\|x} \le j$, it seems plausible that only j functions will be required to generate S_j. This is true, indeed, and is proved subsequently. We first require the following fact, which shows that the orthogonal complement of a shift-invariant space in another shift-invariant space can be well-understood in terms of the corresponding range functions.

Corollary 3.4. *Let S' be a closed shift-invariant subspace of the closed shift-invariant subspace S of $L_2(\mathbb{R}^d)$, and let S'' be the orthogonal complement of S' in S. Then S'' is a closed shift-invariant space and, a.e. on $\mathbb{T}^d$, $\widehat{S}_{\|x}$ is the orthogonal sum of $\widehat{S'}_{\|x}$ and $\widehat{S''}_{\|x}$.*

Proof. Since $P_{S'}$ commutes with any integer shift (cf. (1.1)), S'' is shift-invariant. Let Φ (Ψ) be an at most countable set of generators for S' (S''). Since $S' \perp S''$, Lemma 1.10 implies that $[\widehat{\phi}, \widehat{\psi}] = 0$ for every $\phi \in \Phi$ and $\psi \in \Psi$, or, in other words, that $\widehat{\phi}_{\|x} \perp \widehat{\psi}_{\|x}$ for every $\phi \in \Phi$, $\psi \in \Psi$ and almost every $x \in \mathbb{T}^d$. Thus, a.e., $\widehat{S'}_{\|x} \perp \widehat{S''}_{\|x}$, and these two spaces sum up to $\widehat{S}_{\|x}$ since $\Phi \cup \Psi$ generates S. □

Theorem 3.5. *Let S be a closed shift-invariant space. Then S is a FSI space if and only if*

$$d_S := \operatorname{ess\,sup} \dim\{\widehat{S}_{\|x} : \; x \in \mathbb{T}^d\}$$

is finite. In such a case, $d_S = \operatorname{len} S$, and S can be written as the orthogonal sum of $d_S = \operatorname{len} S$ PSI spaces.

Proof. Assume that S is a FSI space, and let Φ be a generating set for S of cardinality $\operatorname{len} S$. Then $\widehat{S}_{\|x} = \operatorname{span} \widehat{\Phi}_{\|x}$ a.e., and hence $d_S \le \operatorname{len} S$, and in particular, d_S is finite.

The proof of the rest of the theorem is based on the following lemma:

Lemma 3.6. *Let S be a closed shift-invariant space. Then S contains a PSI subspace with spectrum $\sigma(S)$.*

We first show how, based on the lemma, the proof of the theorem can be completed.

Assume that d_S is finite. If $d_S = 0$, then $S = 0$ and there is nothing to prove. Otherwise, let $S' \subset S$ be the PSI space of the lemma, and S'' its orthogonal complement in S. Since $\sigma(S') = \sigma(S)$, $\dim \widehat{S'}_{\|x} = 1$ on $\sigma(S)$, and therefore, by Corollary 3.4,

$$\dim \widehat{S''}_{\|x} = \dim \widehat{S}_{\|x} - \dim \widehat{S'}_{\|x} \le d_S - 1,$$

a.e. on $\sigma(S)$ (and trivially on $\mathbb{T}^d \backslash \sigma(S)$). Thus, $d_{S''} \le d_S - 1$. By induction on d_S, S'' can be written as the orthogonal sum of $d_S - 1$ PSI spaces, and hence S can be written as the orthogonal sum of d_S PSI spaces. The generators of these spaces certainly generate S and hence $d_S \ge \operatorname{len} S$. This completes the proof of the theorem.

It remains to prove the lemma. Let $\Phi := (\phi_j)_{j=1}^{\infty}$ be a generating set for S. Since $x \in \sigma(S)$ iff $\widehat{\Phi}_{\|x} \ne 0$, and $\widehat{\phi}_{\|x} = 0$ iff $\widetilde{\phi}(x) = 0$, we conclude that

$$\sigma(S) = \cup_j \operatorname{supp} \widetilde{\phi}_j.$$

Therefore the sets

$$\Omega_j := \operatorname{supp} \widetilde{\phi}_j \backslash (\cup_{k=1}^{j-1} \operatorname{supp} \widetilde{\phi}_k), \quad j = 1, 2, ...$$

form a (measurable) partition of $\sigma(S)$. We define now g via its Fourier transform by

$$\widehat{g} := \sum_j \chi_{\Omega_j^\circ} \widehat{\phi}_j / 2^j.$$

Since $\chi_{\Omega_j^\circ}$ is 2π-periodic and bounded, each $\chi_{\Omega_j^\circ} \widehat{\phi}_j$ is in $\widehat{S(\phi_j)}$, hence in $\widehat{S}$. Since the series converges and $\widehat{S}$ is closed, the limit $\widehat{g}$ is in $\widehat{S}$, too, and therefore $g \in S$. Because $\{\Omega_j\}_j$ are pairwise disjoint, $\operatorname{supp} \widetilde{g} = \cup_j \Omega_j = \sigma(S)$. $S(g)$ is then the required space. □

3.3. Basic facts about quasi-regular spaces

We shall next describe the projector $\widehat{P_S}$ onto $\widehat{S}$ for a quasi-regular space. We begin with the following characterization of $\widehat{P_S}$ whose proof may be found in [H; p.58] and on which the proof of Result 1.5 is based.

Result 3.7. *Let S be a shift-invariant space. Then $\widehat{P_S}$, the projector onto $\widehat{S}$, is pointwise in the sense that for each $f \in L_2(\mathbb{R}^d)$ and each $x \in \mathbb{T}^d$, $\widehat{P_S} f_{\|x}$ is the orthogonal projection of $f_{\|x}$ into the space $\widehat{S}_{\|x}$.*

If, now, S is a *quasi-regular* space and Φ is a quasi-basis for S, then the orthogonal projection at x, $\widehat{P_S} f_{\|x}$, onto $\widehat{S}_{\|x} = \operatorname{span} \widehat{\Phi}_{\|x}$ can be computed by solving (pointwise on $\sigma(S)$) the normal equations

$$\sum_{\phi\in\Phi} [\widehat{\phi}, \widehat{\psi}](x)\tau_\phi(x) = [\widehat{f}, \widehat{\psi}](x), \quad \psi \in \Phi,$$

where the sought-for 2π-periodic functions $\{\tau_\phi\}_\phi$ are the functions in the representation

$$\widehat{P_S f}(x) = \sum_{\phi\in\Phi} \tau_\phi(x)\widehat{\phi}(x),$$

(cf. Theorem 1.7), which can be defined *a priori* to be 0 on $\mathbb{T}^d \backslash \sigma(S)$. Since Φ is a quasi-basis, $\widehat{\Phi}_{\|x}$ is linearly independent for a.e. $x \in \sigma(S)$. Since

$$\det G(\Phi)(x) = 0 \quad \Longleftrightarrow \quad \widehat{\Phi}_{\|x} \text{ is linearly dependent,} \tag{3.8}$$

with

$$G(\Phi) := ([\widehat{\phi}, \widehat{\psi}])_{\phi,\psi\in\Phi}$$

the Gramian of Φ, the fact that Φ is a quasi-basis implies that $\det G(\Phi) \neq 0$ a.e. on $\sigma(S)$. Thus, Cramer's rule provides the standard formula

$$\widehat{P_S f}(x) = \sum_{\phi\in\Phi} \frac{\det G_\phi(f)(x)}{\det G(\Phi)(x)}\, \widehat{\phi}(x),$$

with

$$G_\phi(f)$$

the "modified Gramian" obtained from $G(\Phi)$ by changing its ϕth row to $([\widehat{f}, \widehat{\psi}])_{\psi\in\Phi}$.

This proves

Theorem 3.9. *Let Φ be a quasi-basis for the FSI space S and let $f \in L_2(\mathbb{R}^d)$. Then the orthogonal projection $\widehat{P}\widehat{f}$ of $\widehat{f}$ into $\widehat{S}$ is given by*

$$\widehat{P}\widehat{f} = \sum_{\phi\in\Phi} \frac{\det G_\phi(f)}{\det G(\Phi)}\, \widehat{\phi}. \tag{3.10}$$

Corollary 3.11. *If S is a FSI space, with quasi-basis Φ, then the 2π-periodic functions τ_ϕ in the representation $\widehat{f} = \sum_{\phi\in\Phi} \tau_\phi \widehat{\phi}$ for $f \in S$ (cf. (1.8)), uniquely determined on $\sigma(S)$ (up to null-sets), are measurable there.*

Proof. Since $\widehat{P}\widehat{f} = \widehat{f}$ for every $f \in S$, we conclude from (3.10) that, on $\sigma(S)$, $\tau_\phi = \det G_\phi(f)/\det G(\Phi)$, and these are measurable. □

Since the orthogonal projector $\widehat{P_S}$ acts pointwise and each of the "fiber" spaces $\widehat{S}_{\|x}$ is finite-dimensional, many of the basic linear algebra facts give rise to analogous results for FSI spaces. The following theorem collects some of them.

Theorem 3.12. *Let Ψ be a finite subset of the FSI space S.*

(a) *If $\#\Psi > j$, then $\det G(\Psi)$ vanishes a.e. on $\sigma_j(S)$. In particular, $\det G(\Psi)$ vanishes a.e., in case $\#\Psi > \operatorname{len} S$.*

(b) *The following are equivalent:*
   (1) *Ψ is a quasi-basis for S.*
   (2) *$\det G(\Psi)$ is nonzero a.e. on $\sigma(S)$, and $\#\Psi \geq \operatorname{len} S$.*
   (3) *S is quasi-regular, Ψ generates S, and and $\#\Psi \leq \operatorname{len} S$.*

(c) *Let Φ be a quasi-basis for S. Then Ψ is a quasi-basis for S if and only if $\widehat{\Psi} = \mathrm{T}\widehat{\Phi}$ for some square matrix T whose entries are 2π-periodic and which is nonsingular a.e. on $\sigma(S)$.*

Proof. (a): $\widehat{\Psi}_{\|x}$ is a subset of the space $\widehat{S}_{\|x}$, the latter has dimension j on $\sigma_j(S)$. Thus, if $\#\Psi > j$, $\widehat{\Psi}_{\|x}$ must be linearly dependent a.e on $\sigma_j(S)$, and, by (3.8), $\det G(\Psi)(x) = 0$ a.e.

(b): For $x \in \sigma(S)$, $\widehat{\Psi}_{\|x}$ is a subset of $\widehat{S}_{\|x}$ and $\dim \widehat{S}_{\|x} \leq \operatorname{len} S$. Now, (1) says that $\widehat{\Psi}_{\|x}$ is a basis for $\widehat{S}_{\|x}$, (2) says that $\dim \widehat{S}_{\|x} \leq \operatorname{len} S \leq \#\Psi$ and that (by 3.8)) $\widehat{\Psi}_{\|x}$ is linearly independent, and (3) says that $\dim \widehat{S}_{\|x} = \operatorname{len} S \geq \#\Psi$ and that $\widehat{\Psi}_{\|x}$ spans $\widehat{S}_{\|x}$. Thus, these three conditions are, indeed, equivalent for any fixed x, and (b) easily follows from that.

(c): If Φ is a quasi-basis, it is, in particular, a generating set for S, and the fact that $\widehat{\Psi} = \mathrm{T}\widehat{\Phi}$ follows from Theorem 1.7. Further, if this matrix is not square, $\#\Psi > \#\Phi$, hence Ψ is not a quasi-basis (say, by (a) here). Now, if T is non-singular a.e. on $\sigma(S)$, then we can write $\widehat{\Phi} = \mathrm{T}^{-1}\widehat{\Psi}$, which implies (by Theorem 1.7 when applied to $S(\Psi)$) that $\Phi \subset S(\Psi)$, hence $S = S(\Phi) \subset S(\Psi)$, and equality must hold since the reverse inclusion in assumed. Conversely, if Ψ is a quasi-basis, $\widehat{\Psi}_{\|x}$ is linearly independent a.e. on $\sigma(S)$, which implies that $\mathrm{T}(x)$ is non-singular a.e., since $\widehat{\Psi}_{\|x} = \mathrm{T}(x)\widehat{\Phi}_{\|x}$ a.e. □

Some of the results of the previous section admit improved versions for quasi-regular spaces. For example, we have the following consequence of Corollary 3.4 and Theorem 3.5:

Theorem 3.13. *Let S' be a quasi-regular subspace of the quasi-regular space S, assume that $\sigma(S) = \sigma(S')$, and let S'' be the orthogonal complement of S' in S. Then, S'' is quasi-regular, too, $\sigma(S'') = \sigma(S)$, and $\operatorname{len} S'' = \operatorname{len} S - \operatorname{len} S'$.*

Proof. Since, both, S and S' are quasi-regular with spectrum $\sigma(S)$, we know that $\dim \widehat{S}_{\|x}$ as well as $\dim \widehat{S'}_{\|x}$ are constants a.e. on $\sigma(S)$. By Corollary 3.4,

$$\dim \widehat{S''}_{\|x} = \dim \widehat{S}_{\|x} - \dim \widehat{S''}_{\|x},$$

and hence $\dim \widehat{S''}_{\|x}$ is also constant on $\sigma(S)$ (and vanishes on $\mathbb{T}^d \backslash \sigma(S)$). Therefore, S'' is quasi-regular with spectrum $\sigma(S)$, and, further, $d_{S''}$ (defined as in Theorem 3.5) is $d_S - d_{S'}$. By Theorem 3.5, for any shift-invariant S, $d_S = \operatorname{len} S$, and we have thus proved that $\operatorname{len} S'' = \operatorname{len} S - \operatorname{len} S'$. □

As an easy consequence of the above results, we obtain that every quasi-regular space has a quasi-basis:

Corollary 3.14. *A FSI space S is quasi-regular if and only if it is the orthogonal sum of $\operatorname{len} S$ PSI spaces, each with spectrum $\sigma(S)$. By selecting a generator for each of these PSI spaces, one obtains a quasi-basis for S (of cardinality $\operatorname{len} S$). Further, any quasi-basis for S has cardinality $\operatorname{len} S$. In particular, every regular FSI space has a basis.*

Proof. Let S be a quasi-regular space. By Lemma 3.6, S contains a PSI subspace S' with spectrum $\sigma(S)$. By Theorem 3.13, the orthogonal complement, S'', of S' in S is quasi-regular of length $\operatorname{len} S - 1$. Employing induction, we obtain that S is the orthogonal sum of $\operatorname{len} S$ PSI spaces, each with spectrum $\sigma(S)$. The rest of the claims of this Corollary are straightforward. $\square$

In general, though, the generating set Φ for S, which was constructed inductively in the proof of Theorem 3.5, may fail to be a quasi-basis for the simple reason that a FSI space need not be quasi-regular. For example, the univariate space generated by $\phi := \chi_{[0,2\pi]}{}^{\vee}$ and $\psi := \chi_{[0,3\pi]}{}^{\vee}$ is not quasi-regular.

Given a minimal generating set for a FSI space S', this set cannot always be extended to a minimal generating set for a FSI space $S \supset S'$. However, this is true for quasi-regular spaces:

Theorem 3.15. *Let S' be a quasi-regular proper subspace of the quasi-regular space S, and assume that $\sigma(S) = \sigma(S')$. Then any quasi-basis for S' can be completed to a quasi-basis for S. Furthermore, the completion can be chosen independently of the given quasi-basis for S'.*

Proof. By Theorem 3.13, the orthogonal complement, S'', of S' in S is quasi-regular. One easily verifies that, given any quasi-basis Φ for S' and quasi-basis Ψ for S'', the union $\Phi \cup \Psi$ is a quasi-basis for S. $\square$

3.4. Quasi-stability in quasi-regular spaces

We earlier characterized quasi-stable PSI spaces in term of the behavior of $\widetilde{\phi}$ on $\sigma(S)$. The analogous results in the finitely generated case are obtained along the same lines, with $\det G(\Phi)$ replacing $\widetilde{\phi}$. First, we define the notion of quasi-stability for a FSI space S. As before, we define

$$C_S := \{c \in \ell_2(\mathbb{Z}^d) : \operatorname{supp} \widehat{c} \subset \sigma(S)\},$$

and denote by K_S the orthogonal complement of C_S in $\ell_2(\mathbb{Z}^d)$.

Definition 3.16. *Let Φ be a finite set of $L_2(\mathbb{R}^d)$-functions. We say that Φ is* **a quasi-stable generating set** *for $S(\Phi)$ if the map*

$$\Phi *' : C_S^\Phi \to S(\Phi) : c \mapsto \Phi *' c := \sum_{\phi \in \Phi} \phi *' c_\phi \tag{3.17}$$

is a Hilbert-space isomorphism. If, in addition, $\sigma(S(\Phi)) = \mathbb{T}^d$, then we call Φ **stable**.

It should be understood that the search for quasi-stable generators can succeed only in quasi-regular spaces:

Proposition 3.18. *Let Φ be a quasi-stable generating set for the space S. Then Φ is a quasi-basis for S (hence S is quasi-regular).*

Proof. Assume that Φ is not a quasi-basis for S. Then there exists $\Omega \subset \sigma(S)$ of positive measure and a proper subset $\Psi \subset \Phi$ such that, for almost all $x \in \Omega$, $\widehat{\Psi}_{\|x}$ is linearly independent, but $\{\widehat{\Psi}_{\|x}, \widehat{\phi}_{\|x}\}$ is linearly dependent for some $\phi \in \Phi \backslash \Psi$. This implies the existence of bounded 2π-periodic functions $\{\tau_\psi\}_{\psi \in \Psi}$, τ_ϕ with support in Ω so that $\tau_\phi \widehat{\phi} + \sum_{\psi \in \Psi} \tau_\psi \widehat{\psi} = 0$, and τ_ϕ vanishes nowhere on Ω. By going to a proper subset of Ω (still of positive measure) if necessary, we can assume that τ_ϕ is bounded away from zero on Ω, hence $\widehat{\phi} = \sum_{\psi \in \Psi} \tau'_\psi \widehat{\psi}$ on Ω for certain bounded functions τ'_ψ with support in Ω. Since $P_\Omega \Psi$ is a quasi-basis (for the space it generates), τ'_ψ are measurable (Corollary 3.11), hence are in $L_\infty(\mathbb{T}^d)$. This implies that each τ'_ψ is the Fourier transform of some $c_\psi \in C_S$. We conclude that $\Phi *'$ cannot be defined in a 1-1 manner on C_S^Φ, hence Φ is not a quasi-stable generating set. $\square$

In view of the above proposition, we assume for the remainder of the discussion that S is quasi-regular, and further, that the generating set in question is a quasi-basis for S. Our main result concerning quasi-stability is the following:

Theorem 3.19. *Let Φ be a quasi-basis for the FSI space S.*
(a) If Φ is a quasi-stable quasi-basis, then $\det G(\Phi)$ is essentially bounded below and above on $\sigma(S)$ by positive constants.
(b) If $[\widehat{\phi}, \widehat{\psi}]$ is continuous on $\sigma(S)$ for every $\phi, \psi \in \Phi$, then Φ is a quasi-stable quasi-basis if and only if $\det G(\Phi)$ vanishes nowhere on the closure of $\sigma(S)$.
In particular:
(c) If S is regular and each $[\widehat{\phi}, \widehat{\psi}]$ is continuous, then Φ is a stable basis if and only if $\det G(\Phi)$ vanishes nowhere.

We note that part (c) of the theorem is essentially due to Jia and Micchelli, [JM1,2]. See Corollary 3.30 below for a characterization of quasi-stability even in case the entries of the Gramian $G(\Phi)$ are not continuous.

Another important question is the *existence* of a quasi-stable quasi-basis for a given FSI space. The following theorem provides a complete answer to this question.

Theorem 3.20. *Let S be a FSI space. Then:*
(a) S contains a quasi-stable quasi-basis if and only if S is quasi-regular.
In particular:
(b) S contains a stable basis if and only if S is regular.

In the next subsection (cf. Theorem 3.35), we prove that every space that is generated by finitely many *compactly supported* functions is regular. Hence:

Corollary 3.21. *If S is generated by finitely many compactly supported functions, then S contains a stable basis.*

We prepare for the proofs of Theorems 3.19 and 3.20 with the following observations. Let Φ be a quasi-basis for S and let $f = \sum_{\phi\in\Phi} \phi *' c_\phi$, with each c_ϕ in C_S, i.e., $c_\phi \in \ell_2(\mathbb{Z}^d)$ and the corresponding 2π-periodic function $\tau_\phi := \widehat{c_\phi}$ having its support in $\sigma(S)$. Then $\widehat{f} = \sum_{\phi\in\Phi} \tau_\phi \widehat{\phi}$, hence

$$\|\widehat{f}\|^2_{L_2(\mathbb{R}^d)} = \int_{\mathbb{T}^d} [\widehat{f}, \widehat{f}] = \int_{\sigma(S)} \tau(x)^H G(x) \tau(x)\, dx, \tag{3.22}$$

with $G := G(\Phi)$, the Gramian of Φ, and with

$$\tau : \mathbb{T}^d \mapsto \ell_2(\Phi) : x \mapsto (\tau_\phi(x))_{\phi\in\Phi} = (\widehat{c_\phi}(x))_{\phi\in\Phi}.$$

Further, since $G(x)$ is a Hermitian positive semidefinite matrix, it satisfies the *sharp* inequalities

$$\lambda_\Phi(x)|v|_2^2 \le v^H G(x) v \le \Lambda_\Phi(x)|v|_2^2, \quad v \in \ell_2(\Phi), \tag{3.23}$$

with

$$|v|_2^2 := v^H v = \sum_{\phi\in\Phi} |v_\phi|_2^2, \qquad v \in \ell_2(\Phi),$$

and

$$\Lambda_\Phi(x) := \sup_{v\in\ell_2(\Phi)} v^H G(x) v/|v|_2^2 = \max_{\lambda\in\mathrm{spect}(G(x))} \lambda = \max_{v\in\ell_2(\Phi)} |G(x)v|_2/|v|_2 = \|G(x)\|,$$

and, correspondingly,

$$\lambda_\Phi(x) := \inf_{v\in\ell_2(\Phi)} v^H G(x)v/|v|_2^2 = \min_{\lambda\in\mathrm{spect}(G(x))} \lambda = \min_{v\in\ell_2(\Phi)} |G(x)v|_2/|v|_2 = 1/\|G(x)^{-1}\|.$$

In particular,

$$\lambda_\Phi^{\#\Phi} \le \det G \le \Lambda_\Phi^{\#\Phi}. \tag{3.24}$$

Lemma 3.25. *The functions λ_Φ and Λ_Φ are measurable and nonnegative and have their support in $\sigma(S)$.*

Proof. Only the measurability of these functions requires verification. Since each entry of G is in $L_1(\mathbb{T}^d)$, it can be approximated in $L_1(\mathbb{T}^d)$ by simple functions. Upon passing to a subsequence, we obtain a sequence $\{G_n\}_{n=1}^\infty$, such that each entry of each matrix G_n is a simple function in $L_1(\mathbb{T}^d)$, and such that $G_n(x)$ converges to $G(x)$ for almost every $x \in \mathbb{T}^d$. Let Λ_n and λ_n be the maximal and minimal eigenvalue functions associated with G_n. Because each of the entries of G_n is simple and measurable, it is clear that Λ_n and λ_n are simple and measurable. On the other hand, $\Lambda_n(x) \to \Lambda_\Phi(x)$, whenever $G_n(x) \to G(x)$. Consequently, $\{\Lambda_n\}_n$ converges pointwise a.e. on $\mathbb{T}^d$ to Λ_Φ. Hence Λ_Φ is measurable, and, by an analogous argument, λ_Φ is measurable, too. □

Also, $\lambda_\Phi(x) = 0$ if and only if $\det G(\Phi)(x) = 0$, and therefore, since Φ is a quasi-basis for S, λ_Φ is nonzero a.e. on $\sigma(S)$. In other words, $1/\lambda_\Phi$ is finite a.e. on $\sigma(S)$.

On combining (3.22) with (3.23), we find that

$$1/\|1/\lambda_\Phi\|_{L_\infty(\sigma(S))} \;\le\; \|\widehat{f}\|^2_{L_2(\mathbb{R}^d)}/\int_{\sigma(S)} |\tau(x)|_2^2\,dx \;\le\; \|\Lambda_\Phi\|_{L_\infty(\sigma(S))}. \tag{3.26}$$

Here, $1/\|1/\lambda_\Phi\|_{L_\infty(\sigma(S))}$ equals the essential infimum of λ_Φ on its support.

We next show that (3.26) is sharp:

Lemma 3.27. *The constant $\|\Lambda_\Phi\|_{L_\infty(\sigma(S))}$ (respectively $1/\|1/\lambda_\Phi\|_{L_\infty(\sigma(S))}$) in (3.26) cannot be replaced by any smaller (respectively, larger) constant.*

Proof. If, e.g., $1/\|1/\lambda_\Phi\|_{L_\infty(\sigma(S))} < \varepsilon$, then, for some $\varepsilon_1 < \varepsilon$, the set $\Omega := \{x \in \sigma(S) : \lambda_\Phi(x) < \varepsilon_1/2\}$ has positive measure. The natural way to proceed is by choosing the function $\tau = \{\tau_\phi\}_\phi$ to consist, for each $x \in \Omega$, of an eigenvector of $G(x)$ corresponding to $\lambda_\Phi(x)$. However, such an approach leaves questionable the measurability of τ, hence we use instead a simple function approximation.

As in the previous lemma, we let $\{G_n\}$ be a sequence of matrices which converges pointwise a.e. to G, and whose entries are simple 2π-periodic functions. By Egorov's Theorem, G_n converges *uniformly* (i.e., each entry converges uniformly) on a subset of Ω of positive measure, which we may assume, without loss, to be the original Ω. With λ_n the minimal eigenvalue function associated with G_n, we can find sufficiently large n for which $1/\|1/\lambda_n\|_{L_\infty(\Omega)} \le \varepsilon_1$, and such that $\|(G_n - G)(x)\|_2 \le \delta$, for some small δ and uniformly for $x \in \Omega$. Since the entries of G_n are simple, we can replace Ω by a subset of it, still of positive measure (which we still denote by Ω), on which G_n, hence λ_n, are constant. Let $v = \{v_\phi\}$ be a (constant) eigenvector of $G_n(x)$, $x \in \Omega$, corresponding to the (constant) eigenvalue $\lambda_n(x)$, normalized to have $|v|_2 = 1$. We define $\tau_\phi \in L_\infty(\mathbb{T}^d)$, $\phi \in \Phi$, by

$$\tau_\phi := v_\phi \chi_\Omega.$$

Then $\tau^H G_n \tau = \lambda_n \le \varepsilon_1 < \varepsilon$ on Ω, and $\tau^H G_n \tau = 0$ elsewhere. With an appropriate choice of δ, we can ensure that the same holds with G replacing G_n and with ε replacing ε_1 (and with τ unchanged). Defining

$$\widehat{f} := \sum_{\phi\in\Phi} \tau_\phi \widehat{\phi},$$

we see that $\widehat{f} \in L_2(\mathbb{R}^d)$, and hence, by Corollary 3.11, $\widehat{f} \in \widehat{S}$. Therefore, from (3.22),

$$\|\widehat{f}\|^2_{L_2(\mathbb{R}^d)} = \int_\Omega \tau(x)^H G(x)\tau(x)\,dx \le \int_\Omega \varepsilon = \varepsilon \int_{\sigma(S)} |\tau|_2^2,$$

showing that $\int_{\sigma(S)} |\tau|_2^2 \ne 0$ and $\|\widehat{f}\|^2_{L_2(\mathbb{R}^d)} / \int_{\sigma(S)} |\tau|_2^2 \le \varepsilon$. □

We thus arrive at the following result:

Proposition 3.28. *Let Φ be a basis for the FSI space S. For $c \in C_S^\Phi$, let $|c|_2$ be its norm, i.e., $|c|_2^2 = \sum_{\phi\in\Phi} \|c_\phi\|^2_{\ell_2(\mathbb{Z}^d)}$. Then the inequalities*

$$1/\|1/\lambda_\Phi\|_{L_\infty(\sigma(S))} \le \|\Phi{*}'c\|^2_{L_2(\mathbb{R}^d)}/|c|_2^2 \le \|\Lambda_\Phi\|_{L_\infty(\sigma(S))} \tag{3.29}$$

are valid and sharp.

Proof. It should be understood that the middle quantity of (3.29) is defined to equal ∞ whenever $\Phi{*}'$ does not extend continuously from finitely supported sequences to all of $\ell_2(\mathbb{Z}^d \times \Phi)$. Thus, we show first that in such a case Λ_Φ is unbounded (on $\sigma(S)$): Since Φ is a basis, $\Phi{*}'$ does not extend to $\ell_2(\mathbb{Z}^d \times \Phi)$ exactly when, for some $\phi \in \Phi$, $\phi{*}'$ does not extend to $\ell_2(\mathbb{Z}^d)$, and the latter is equivalent (cf. Lemma 2.14) to $\widetilde{\phi}$ being unbounded on its support $\operatorname{supp}\widetilde{\phi} \subset \sigma(S)$. Yet, $\widetilde{\phi}^2 = [\widehat{\phi}, \widehat{\phi}]$ is one of the (diagonal) entries of the Gramian G, and we conclude that G has an entry unbounded on $\sigma(S)$, hence Λ_Φ must be unbounded on that set, too.

On the other hand, if $\Phi{*}'$ is well-defined, then, the claim of the proposition follows from the preceding analysis and the following equality, valid for every $c \in C_S^\Phi$,

$$\|\Phi{*}'c\|^2_{L_2(\mathbb{R}^d)}/|c|_2^2 = \|\sum_{\phi\in\Phi} \tau_\phi \widehat{\phi}\|_{L_2(\mathbb{R}^d)} / \int_{\sigma(S)} |\tau|_2^2,$$

where $\tau := \{\tau_\phi := \widehat{c_\phi}\}$. □

Corollary 3.30. *Let Φ be a quasi-basis for the (quasi-regular) FSI space S. Then Φ is a quasi-stable basis if and only if both Λ_Φ and $1/\lambda_\Phi$ are in $L_\infty(\sigma(S))$. In particular, if S is regular, then Φ is a stable basis for S if and only if Λ_Φ and $1/\lambda_\Phi$ are bounded a.e. on $\mathbb{T}^d$.*

Note that Λ_Φ vanishes on $\mathbb{T}^d \backslash \sigma(S)$, hence its boundedness on $\sigma(S)$ is equivalent to its boundedness everywhere.

We prove now the two theorems stated earlier in this subsection:

Proof of Theorem 3.19. (a): If Φ is a quasi-stable basis, then by Corollary 3.30, Λ_ϕ and $1/\lambda_\Phi$ are bounded on $\sigma(S)$. Since $\lambda_\Phi^{\#\Phi} \le \det G(\Phi) \le \Lambda_\Phi^{\#\Phi}$, we conclude that $\det G(\Phi)$ must be bounded below and above by some positive constants a.e. on $\sigma(S)$.

(b): If all the entries of $G(\Phi)$ are continuous, so are Λ_Φ and λ_Φ. It follows that Λ_Φ is necessarily bounded on $\mathbb{T}^d$. As to $1/\lambda_\Phi$, it is bounded on $\sigma(S)$ if and only if λ_Φ does not vanish on the closure of $\sigma(S)$. The latter is equivalent to $\det G(\Phi)$ being nonzero on the closure of $\sigma(S)$.

(c) is a special case of (b). □

Proof of Theorem 3.20. Statement (b) is a special case of (a). By Proposition 3.18, if Φ is a quasi-stable generating set for S, then S must be quasi-regular, so that we only need to prove that a quasi-regular space contains a quasi-stable quasi-basis. By Corollary 3.14, S is the orthogonal sum of $n := \operatorname{len} S$ PSI spaces $\{S_j\}_j$, each of which with spectrum $\sigma(S)$. By Theorem 2.21, each S_j contains a quasi-orthogonal generator ϕ_j. By the orthogonality of the S_j spaces, $[\widehat{\phi}_j, \widehat{\phi}_k] = 0$ for $j \ne k$, and by the quasi-orthogonality of each ϕ_j, we have $\widetilde{\phi}_j = \chi_{\sigma(S_j)} = \chi_{\sigma(S)}$. Therefore, we conclude that $G(\{\phi_j\}_j)(x)$ is the identity matrix for each $x \in \sigma(S)$. In particular, by Corollary 3.30, $\{\phi_j\}_j$ is a quasi-stable generating set. □

In particular, the construction of the last proof implies the following:

Corollary 3.31. *Let S be a regular FSI space. Then S contains a basis Φ which is orthogonal in the sense that $\langle \phi(\cdot - \alpha), \psi(\cdot - \beta)\rangle = \delta_{\phi,\psi}\delta_{\alpha,\beta}$, for all $\phi, \psi \in \Phi$, $\alpha, \beta \in \mathbb{Z}^d$.*

We conclude this section with the following result concerning the connection between two quasi-stable quasi-bases for S:

Corollary 3.32. *Let Φ be a quasi-stable quasi-basis for the FSI space S. Let $\Psi \subset S$. Then Ψ is also a quasi-stable quasi-basis for S if and only if there exists a square matrix T with 2π-periodic entries such that $\widehat{\Psi} = \mathrm{T}\widehat{\Phi}$ and $\|\mathrm{T}\|$, $\|\mathrm{T}^{-1}\|$ are bounded (a.e.) on $\sigma(S)$.*

Proof. Because of Theorem 3.12(c), we may assume that Ψ is a quasi-basis, that $\widehat{\Psi} = \mathrm{T}\widehat{\Phi}$, with T a square matrix which is non-singular on $\sigma(S)$, and prove that the quasi-stability of Ψ is equivalent to the boundedness of $\|\mathrm{T}\|$ and $\|\mathrm{T}^{-1}\|$ on $\sigma(S)$.

For this, recall that, for any matrix A, $\|A\|^2 = \|A^H\|^2 = \max_c \frac{c^H AA^H c}{c^H c} = \|AA^H\|$ and $1/\|A^{-1}\|^2 = \min_c \frac{c^H AA^H c}{c^H c} = 1/\|(AA^H)^{-1}\|$, while $\lambda_\Psi = \min_c \frac{c^H G(\Psi) c}{c^H c}$ and $\Lambda_\Psi = \max_c \frac{c^H G(\Psi)c}{c^H c}$. Since $G(\Psi) = \mathrm{T}G(\Phi)\mathrm{T}^H$, we have

$$\frac{c^H G(\Psi)c}{c^H c} = \frac{(\mathrm{T}^H c)^H G(\Phi)(\mathrm{T}^H c)}{(\mathrm{T}^H c)^H(\mathrm{T}^H c)} \; \frac{c^H \mathrm{T}\mathrm{T}^H c}{c^H c},$$

from which we conclude that, for a.e. $x \in \sigma(S)$,

$$\lambda_\Phi(x)\|\mathrm{T}(x)\|^2 \le \Lambda_\Psi(x) \le \Lambda_\Phi(x)\|\mathrm{T}(x)\|^2 \quad \text{and} \quad \lambda_\Phi(x)/\|\mathrm{T}(x)^{-1}\|^2 \le \lambda_\Psi(x) \le \Lambda_\Phi(x)/\|\mathrm{T}(x)^{-1}\|^2.$$

Thus, since Φ is quasi-stable by assumption, Ψ is also quasi-stable if and only if $\|\mathrm{T}\|$ and $\|\mathrm{T}^{-1}\|$ are both bounded functions on $\sigma(S)$. □

3.5. Local spaces

We call a space **local** if it is a FSI space with *compactly supported* generators. Such spaces are locally finite-dimensional, hence particularly attractive in applications. The recent work by Jia and Micchelli (cf. [JM2] and the surveys [M1], [M2]) on FSI spaces deals largely with local spaces. For a local space, our earlier theorems concerning FSI spaces can be sharpened because of the following observation.

Lemma 3.33. *Let Φ be a finite set of compactly supported functions. Then $\det G(\Phi)$ is a trigonometric polynomial.*

Proof. By Lemma 1.10, $[\widehat{\phi}, \widehat{\psi}]$ is a trigonometric polynomial for any compactly supported ϕ and ψ. This implies that all entries of $G(\Phi)$ are trigonometric polynomials, hence so is $\det G(\Phi)$. □

Since we have already used the term 'linearly independent' in connection with the linear independence of the integer shifts of a given ϕ, we now use the related term **free** to denote the fact that Φ is a finite set of compactly supported functions for which $G(\Phi)$ is invertible a.e. That is, Φ is free iff it forms a basis for the space it generates. It follows that a compactly supported set Φ is either free, or else $\det G(\Phi)$ vanishes a.e.

Corollary 3.34. *Let Φ and Ψ be finite sets of compactly supported functions. If Ψ is free and in $S := S(\Phi)$, then Ψ can be completed to a basis for S by elements from Φ.*

Note the difference between this result and Theorem 3.15: the latter merely says that Φ can be completed to a basis. The claim here shows in particular that the completion can be made by compactly supported functions.

Proof. Let Ψ' be a maximal free subset of $\Phi \cup \Psi$ containing Ψ. Since for any compactly supported ϕ, the singleton set $\{\phi\}$ is free, it follows that Ψ' is not empty (even if Ψ is). But then, by the maximality of Ψ', $\det G(\Psi \cup \{\phi\})$ must vanish a.e. for every $\phi \in \Phi \backslash \Psi'$. By (3.8), this implies that $\widehat{\Psi}_{\|x} \cup \{\widehat{\phi}_{\|x}\}$ is linearly dependent, while Ψ is free by construction, hence $\widehat{\Psi}_{\|x}$ is linearly independent a.e. Therefore, $\widehat{\phi}_{\|x} \in \operatorname{span} \widehat{\Psi}'_{\|x}$, a.e., and hence (say, by Theorem 1.7) $\phi \in S(\Psi')$. Since $\phi \in \Phi$ was arbitrary, we obtain that $\Phi \subset S(\Psi')$, while also $\Psi' \subset S(\Phi)$, hence $S(\Psi') = S(\Phi)$.

Since $\det G(\Psi')$ vanishes almost nowhere, this implies that Ψ' is a basis for $S(\Phi)$. □

Application of Corollary 3.34 with $\Psi = \emptyset$ gives the following.

Theorem 3.35. *Let Φ be a finite set of compactly supported functions. Then Φ contains a basis for $S(\Phi)$, hence $S(\Phi)$ is regular.*

Corollary 3.36. *Let S be a FSI space, and assume that the space S_c of all compactly supported functions in S is dense in S. Then S is local.*

Proof. Let $\Phi \subset S_c$ be a maximal set with respect to the property $\det G(\Phi) \neq 0$ a.e. Since S is finitely generated, Theorem 3.12 (together with Lemma 3.33) implies that Φ is finite. Clearly, Φ is a basis for the regular space $S(\Phi)$, hence, by (3.8) and the maximality of Φ, $S_c \subset S(\Phi)$, and, as S_c is dense in S, this shows that $S = S(\Phi)$. □

We can also interpret the two last results directly in terms of Gramians (which in any case were involved in the arguments) as follows:

Corollary 3.37. *Let S be a local FSI, and Ψ a finite set of compactly supported functions in S.*
(a) If $\det G(\Psi) \neq 0$, then Ψ can be completed to a basis for S.
(b) If $\det G(\Psi) = 0$, then Ψ contains a basis for $S(\Psi)$.

The following is the local version of Theorem 3.13.

Theorem 3.38. *Let S be a local space. Let $\Psi \subset S$ be a finite collection of compactly supported functions. Then the orthogonal complement S' of $S(\Psi)$ in S is local as well, and we have*

$$\operatorname{len} S' + \operatorname{len} S(\Psi) = \operatorname{len} S. \tag{3.39}$$

Moreover, if Ψ is a basis for $S(\Psi)$, then a compactly supported basis Φ' for S' can be obtained as

$$\Phi' \ := \ \{\widehat{\phi'} := \det G(\Psi)\,(\widehat{\phi} - \widehat{P}_\Psi \widehat{\phi}) : \ \phi \in \Phi\}, \tag{3.40}$$

where $\Psi \cup \Phi$ is any compactly supported completion of Ψ to a basis for S. Here, $\widehat{P}_\Psi$ is the orthogonal projection onto $\widehat{S(\Psi)}$.

Proof. By Theorem 3.35, Ψ contains a basis for $S(\Psi)$, which we assume to be Ψ itself. With regard to the localness of S', we first note that Corollary 3.34 ensures the existence of a finite set Φ of compactly supported functions so that $\Psi \cup \Phi$ is a basis for S. By Proposition 3.1 and Corollary 3.4, $(1-\widehat{P}_\Psi)\widehat{\phi}_{\|x}$, $\phi \in \Phi$, is a basis for $J_{S'}(x)$ for a.e. x. Since $\det G(\Psi)$ is nonzero a.e., it follows that also $\widehat{\Phi}'_{\|x} = \{\widehat{\phi}'_{\|x} : \phi \in \Phi\}$ is a basis for $J_{S'}(x)$, hence Φ' is a basis for S'. It follows that each ϕ' is compactly supported, since the corresponding ϕ as well as all functions in Ψ are compactly supported, therefore each entry of each determinant in the formula (3.10) for the orthogonal projection $\widehat{P}_\Psi(\widehat{\phi})$ is a trigonometric polynomial (cf. Lemma 1.10) and hence $\widehat{\phi}' - \det G(\Psi)\widehat{\phi} = -\det G(\Psi)\widehat{P}_\Psi(\widehat{\phi})$ equals $\sum_{\psi \in \Psi} \tau_\psi \widehat{\psi}$, with each τ_ψ a trigonometric polynomial. □

Assume now that Φ is a compactly supported basis for the FSI space S. Then, for any $\phi \in \Phi$, we can apply Theorem 3.38 with respect to the choice ϕ and $\Psi := \Phi\backslash\{\phi\}$, to conclude that $S(\Phi)$ is the orthogonal sum of the local spaces $S(\phi')$ and $S(\Psi)$, with $\widehat{\phi'}$ given by (3.40). Assume further that the basis Φ is stable. Then, the basis $\{\phi'\} \cup \Psi$ is stable if and only if ϕ' is a stable generator of $S(\phi')$. In the context of compactly supported generators, Theorem 3.19(c) reduces the stability of a finite generating set F to the linear independence of $\widehat{F}_{\|x}$ for every $x \in \mathbb{T}^d$. Thus, ϕ' is an unstable generator if and only if $\widehat{\phi'}_{\|x} = 0$ for some $x \in \mathbb{T}^d$. Since $\widehat{P_\Psi \phi}_{\|x} \in \widehat{\Psi}_{\|x}$, and $\widehat{\phi}_{\|x}$ is independent of $\widehat{\Psi}_{\|x}$, by virtue of the stability of $\Phi = \{\phi\} \cup \Psi$, this can only happen if $\det G(\Phi)(x) = 0$, which in turn, would contradict the stability of Φ. Consequently, $\{\phi'\} \cup \Psi$ is still stable, and, using an inductive construction, we recover the following result of Jia and Micchelli.

Corollary 3.41 [JM2]. *Any FSI space with a compactly supported stable basis is the orthogonal sum of finitely many PSI spaces, each with a compactly supported stable generator.*

The results in section 2.3 concerning linearly independent generators for a PSI space have obvious extensions to local spaces.

For example, we obtain by repeated applications of Theorem 3.38 that the orthogonal complement of any local subspace S' of a local space S is the orthogonal sum of finitely many PSI spaces, each generated by a compactly supported function. Since Corollary 2.27 guarantees the existence of a linearly independent generator for any *univariate* local PSI space, this proves the following.

Proposition 3.42. *The orthogonal complement of a local subspace within a* **univariate** *local space is the orthogonal sum of finitely many PSI spaces, each generated by a linearly independent (hence compactly supported) function. In particular, each univariate local space is such a sum.*

It follows that a *univariate* local space always has a linearly independent basis in the sense of the following definition.

We call the finite set Φ of compactly supported functions **linearly independent** if the linear map

$$\Phi*' : c \mapsto \sum_{\phi\in\Phi,\alpha\in\mathbb{Z}^d} \phi(\cdot - \alpha)c(\phi, \alpha)$$

is 1-1 even if we allow here arbitrary $c \in \mathbb{C}^{\Phi\times\mathbb{Z}^d}$. The sum here is taken pointwise.

This raises the question whether, in Corollary 3.41, it is possible to replace 'stable' by 'linearly independent', i.e., whether Proposition 3.42 also holds in more than one variable under the additional assumption that the local space has a linearly independent basis. We don't know the answer, but guess that, for a general local FSI, it is in the negative. In this regard, it is useful to quote the following (non-trivial) generalization of Result 2.24.

Result [JM1; Cor.3]. *The finite collection Φ of compactly supported integrable functions is linearly independent if and only if $\widehat{\Phi}_{\|x}$ is linearly independent for all $x \in \mathbb{C}^d$.*

4. An application to approximation theory

Orthogonal projectors are optimal approximation maps (at least as far as the error is concerned), and we became interested in projectors onto shift-invariant spaces in order to understand the approximation properties of these spaces. Indeed, [BDR] contains an extensive discussion of the approximation orders and density orders from principal and from arbitrary shift-invariant spaces.

Here is some terminology. For a closed subspace S of $L_2(\mathrm{IR}^d)$, we use the notation

$$E(f,S) := \|f - P_S f\|, \quad f \in L_2(\mathrm{IR}^d),$$

i.e., $E(f,S)$ is the distance between f and S. Also,

$$S^h := \{f(\cdot/h) : f \in S\}$$

is the h-dilate of the space S. We say that S **provides approximation order** k, in case

$$E(f,S^h) \le \mathrm{const}_S h^k \|f\|_{W_2^k(\mathrm{IR}^d)} \quad \mathrm{for} f \in W_2^k(\mathrm{IR}^d),$$

with

$$W_2^k(\mathrm{IR}^d) := \{f \in L_2(\mathrm{IR}^d) : \ \|f\|_{W_2^k(\mathrm{IR}^d)} := (2\pi)^{-d/2}\|(1+|\cdot|)^k \widehat{f}\| < \infty\}.$$

If, in addition,

$$E(f,S^h) = o(h^k) \quad \mathrm{for} f \in W_2^k(\mathrm{IR}^d),$$

then we say that S **provides density order** k.

We proved in [BDR] that the approximation order of a general (not necessarily finitely generated) shift-invariant space is attained by one of its principal subspaces, but were not prepared there to indicate the nature of this subspace. With the aid of Theorem 3.9, we derive below more concrete results with respect to quasi-regular FSI spaces.

Our first result concerns local FSI spaces.

Theorem 4.1. *Let S be a FSI space and assume that the compactly supported functions in S form a dense subspace of it. Let g be any compactly supported function (not necessarily in S). Then, there exists a compactly supported function $\phi \in S$ such that, for every $f \in L_2(\mathrm{IR}^d)$,*

$$E(f,S(\phi)) \le E(f,S) + 2E(f,S(g)). \tag{4.2}$$

Furthermore, for any compactly supported generating set Φ for S, ϕ can be chosen from $S_0(\Phi)$.

This theorem settles a long standing question in the area of spline theory, namely under what circumstances the approximation power of a local FSI space S is already realized by one of its local PSI subspaces. The theorem says that, in $L_2(\mathrm{IR}^d)$ at least, this is always possible: For, it is not hard to show that a local FSI space can only have finite approximation order (since it is locally finite-dimensional). Thus it is possible to choose a compactly supported function g whose corresponding PSI space $S(g)$ has approximation order at least as good as that of S. The theorem then asserts the existence of a compactly supported ϕ in S for which $S(\phi)$ has the same approximation order as S.

The question of when a local space contains a principal subspace with the same approximation orders was first studied in [SF], following a suggestion by Babuška that this should always be possible. Introducing a restrictive notion of approximation, the so-called "controlled approximation" (which only considers approximations of the form $\Phi *' c$ with $\|c\|_{\ell_2}$ bounded in terms of the approximand), the authors there claimed that the local space $S = S(\Phi)$ has "controlled" approximation order k if and only if there is some $\phi \in S_0(\Phi)$ (with some special properties) so that already $S(\phi)$ has that "controlled" approximation order. This claim was shown to be false by Jia, in [J1], but the claim was shown to be true (in [BJ]) if "controlled" approximation is replaced by "local" approximation (which is another restrictive notion of approximation).

Proof. By Corollary 3.36, we know that S contains a compactly supported basis, and furthermore, Theorem 3.35 ensures that every compactly supported generating set can be reduced to a basis. Therefore, in the proof of the theorem, we may assume that we are given a compactly supported basis Ψ for S. Given $f \in L_2(\mathbb{R}^d)$, let Pf denote its orthogonal projection into S. From Theorem 3.3 of [BDR], we know that

$$E(f, S(Pg)) \le E(f, S) + 2E(f, S(g)), \tag{4.3}$$

and hence, for the proof here it suffices to show that $S(Pg)$ is generated by a function which is a finite linear combination of the shifts of Ψ (hence necessarily compactly supported).

Let $G_\psi(g)$ be as in Theorem 3.9. Since g and Ψ are compactly supported, every $\det G_\psi(g)$, $\psi \in \Psi$, as well as $\det G(\Psi)$ are trigonometric polynomials (Lemma 3.33). By Theorem 3.9,

$$\det G(\Psi)\widehat{Pg} = \sum_{\psi \in \Psi} \det G_\psi(g)\widehat{\psi}. \tag{4.4}$$

Defining ϕ as the inverse transform of $G(\Psi)\widehat{Pg}$, we conclude from (4.4) that ϕ is a finite linear combination of the shifts of Ψ. By Result 2.3, $\phi \in S(Pg)$, and by Corollary 2.6, ϕ generates $S(Pg)$. Replacing Pg by ϕ in (4.3), we obtain the desired result. $\square$

Following up on a result in [BD] according to which a locally finite-dimensional shift-invariant space of *univariate* functions has density order 0 (in $C_0(\mathbb{R})$ rather than $L_2(\mathbb{R})$) if and only if it contains a local PSI space, Jia in [J3] proved that a locally finite-dimensional *univariate* shift-invariant space S provides approximation order k (in any particular $L_p(\mathbb{R})$) if and only if some local PSI space in it does so. Further, in [J2], Jia conjectured the corresponding result for functions of several variables. For the case $p = 2$, the following corollary provides an alternative proof for the main step in the proof of the aforementioned result of [J3]:

Corollary 4.5. *Let S be a univariate local space which provides approximation order $k \ge 1$. Then there exists a compactly supported function $\phi \in S$ such that $\widehat{\phi}(0) \ne 0$, but $\widehat{\phi}$ has a zero of order k at each of the points $j \in 2\pi\mathbb{Z}$.*

It is well-known that, for ϕ as in the above corollary, $S(\phi)$ provides approximation order k. This, however, is obtained in the proof below as a by-product.

Proof. By Theorem 4.1, there exists a compactly supported function $\psi \in S$ such that $S(\psi)$ provides approximation order k. By Result 2.26, $S(\psi)$ contains a linearly independent generator ϕ. Since $S(\psi) = S(\phi)$, $S(\phi)$ provides approximation order k. Invoking Theorem 1.14 of [BDR], we obtain that $\widehat{\phi}$ has a zero of order k at each $j \in 2\pi\mathbb{Z}$, in particular, $\widehat{\phi}$ vanishes on $2\pi\mathbb{Z}\backslash 0$. This forces $\widehat{\phi}(0) \ne 0$, since otherwise Result 2.24 would imply that shifts of ϕ are linearly dependent. $\square$

There is work, by Cheney and Light [LC], by Jia and Lei [JL], and by Halton and Light [HL] (see also Light's survey [L]) which extends the results of [BJ] to nonlocal FSI spaces with suitably decaying generators, using a suitable notion of "local controlled" approximation. The argument is based on quasi-interpolants, hence requires the generators $\psi \in \Psi$ to decay at ∞ faster than $|x|^{-d-k}$ if it is to be shown that "local controlled" approximation order k from $S(\Psi)$ (in $L_2(\mathbb{R}^d)$, say) implies the existence of some $\phi \in S(\Psi)$ such that already $S(\phi)$ gives that approximation order.

Motivated by the above-cited references, we derive below a generalization of Theorem 4.1 to spaces generated by functions which are not compactly supported, but still decay at ∞ in some manner. Rather than dealing with specific decay rates, we prefer to describe it in the following convenient axiomatic way.

Definition 4.6. *We say that the pair (F,C) is* **compatible**, *in case F is a subspace of $L_2(\mathbb{R}^d)$, C is a subalgebra of $\ell_1(\mathbb{Z}^d)$, satisfying the following:*
(a) $\{\langle f, g(\cdot+\alpha)\rangle\}_{\alpha\in\mathbb{Z}^d} \in C$ *for every* $f, g \in F$.
(b) $f*'c \in F$ *for every* $f \in F$ *and* $c \in C$.

For example, the set of compactly supported functions in $L_2(\mathbb{R}^d)$, together with the set of finitely supported functions on $\mathbb{Z}^d$, is such a pair. Another example is given by the pair

$$F_r := \{f : |f(x)| \le \mathrm{const}_f (1+|x|)^{-r}\}, \quad C_r := \{c : \mathbb{Z}^d \to \mathbb{C} : \ |c(\alpha)| \le \mathrm{const}_c (1+|\alpha|)^{-r}\} \tag{4.7}$$

with $r > d$. In this case, F_r and C_r are subalgebras of $L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ and $\ell_1(\mathbb{Z}^d)$ respectively, hence (a) holds, while (b) can be verified directly. As further examples, we may take F (C) to be the space of all L_2 (ℓ_1) rapidly decaying functions (sequences), or take F (C) to be the space of all L_2-functions (ℓ_1-sequences) which decay exponentially at ∞.

Theorem 4.8. *Let (F,C) be a compatible pair (as defined in Definition 4.6). Let $\Psi \subset F$ be a quasi-basis for the FSI space S. Then, for every $g \in F$, there exists $\phi \in S \cap F$ that satisfies (4.2). Furthermore, $\phi = \Psi*'c$, with $c \in C^\Psi$.*

The point of the theorem is as follows: if F is known to contain a function g whose corresponding $S(g)$ provides approximation order k, then, with $\phi \in S$ the function associated with g by the theorem, $S(\phi)$ provides approximation order k as soon as S does so. Note that in all the examples discussed prior to the theorem, the space F contains $\mathcal{D}(\mathbb{R}^d)$ (the space of compactly supported C^∞-functions), and we can choose g to be an appropriately selected element of $\mathcal{D}(\mathbb{R}^d)$.

Proof. Since Ψ is a quasi-basis, $\det G(\Psi)$ vanishes almost nowhere on $\sigma(S)$, and hence, by Corollary 2.4, for every $\phi \in S$, the inverse transform of $\det G(\Psi)\widehat{\phi}$ generates $S(\phi)$, provided that $\det G(\Psi)\widehat{\phi} \in L_2(\mathbb{R}^d)$. Therefore, following the proof of Theorem 4.1 (and in view of Theorem 1.7), we need only to prove that the function ϕ, defined by

$$\widehat{\phi} = \sum_{\psi\in\Psi} \det G_\psi(g)\widehat{\psi},$$

is in F. Recall from Lemma 1.10 that each entry $[\widehat{\psi}_1, \widehat{\psi}_2]$ of any $G_\psi(g)$ is the Fourier series of the sequence $\{\langle \psi_1, \psi_2(\cdot+\alpha)\rangle\}_{\alpha\in\mathbb{Z}^d}$. Since $\{g\} \cup \Psi \subset F$, then, by (4.6)(a), each $[\widehat{\psi}_1, \widehat{\psi}_2]^\vee$ is in C. Thus, $\det G_\psi(g)$ is the transform of linear combinations of convolution products of elements from C. Since C is an algebra, it follows that $(\det G_\psi(g))^\vee$ is in C as well. Finally, $\det G_\psi(g)\widehat{\psi}$ is the Fourier transform of the semi-discrete convolution product $\psi*'(\det G_\psi(g))^\vee$, hence is in F, by virtue of (4.6)(b).

The fact that $\phi = \Psi*'c$ follows from the above argument (with $c_\psi = (\det G_\psi(g))^\vee$.) □

From the first example discussed after Definition 4.6, we see that Theorem 4.8 generalizes Theorem 4.1. Furthermore, for the choice $(F,C) := (F_r, C_r)$, Theorem 4.8 provides the following:

Corollary 4.9. *For some $r > d$, let (F_r, C_r) be the compatible pair defined in (4.7). Let $\Psi \subset F_r$ be a basis for the FSI space S. If S provides approximation order k, then there exists $c \in C_r^\Psi$ such that $\phi := \Psi*'c \in F_r \cap S$, and the PSI subspace $S(\phi)$ of S provides approximation order k, as well.*

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