The quasi–interpolant as a tool in elementary polynomial spline theory

Carl de Boor[1]

This talk is intended to demonstrate with the help of some examples that the quasi–interpolant of [2] is very convenient when it comes to proving even very elementary old and new facts about polynomial splines. The key is a formula which gives each B–spline expansion coefficient for a given spline in terms of the value of its derivatives at a point.

1. Definitions

Let $k \in \mathbb{N}$, let $\mathbf{t} := (t_i)_{-\infty}^{\infty}$ be real, nondecreasing $t_i < t_{i+k}$, all i, and set

$$a := \inf_i t_i,$$

and

$$b := \sup_i t_i.$$

For $i \in \mathbb{Z}$, **the i^{th} B–spline of order k with** (or, for the) **knot sequence t** is given by the rule

$$N_{ik}(t) := g_k(t_i, \ldots, t_{i+k};\, t)\ (t_{i+k} - t_i)$$
$$g_k(s;t) := (s - t)_+^{k-1}$$

taking, for each fixed t, the k^{th} divided difference of $g(s) := g_k(s;t)$ at $t_i, \ldots t_{i+k}$ in the usual manner even when some or all of the t_j's coincide. I leave unresolved any possible ambiguity when $t = t_j$ for some j, and concern myself only with left and right limits at such a point; i.e., I replace each $t = t_j$ by the "two points" t_j^- and t_j^+.

As is well known,

$$N_{ik} > 0 \text{ on } (t_i, t_{i+k}), \text{ and } N_{ik} = 0 \text{ off } [t_i^+, t_{i+k}^-]$$

so that (since $t_i < t_{i+k}$, by assumption) N_{ik} is not identically zero, while on the other hand, no more than k of the N_{jk}'s are nonzero at any particular point. Consequently, for an arbitrary $\mathbf{a} \in \mathbb{R}^{\mathbb{Z}}$, the rule

$$f(t) := \sum_i a_i N_{ik}(t)$$

defines a function on (a, b) if we take the sum to be *pointwise*. I call every such function a **polynomial spline of order k with knot sequence t**, and denote their collection by

$$\mathcal{S}_{k,\mathbf{t}}.$$

The "quasi–interpolator" Q of interest here is given by the rule

$$Qf := \sum_i (\lambda_i f) N_{ik}$$

where

$$\lambda_i f := \lambda_{\tau_i, \psi_{ik}} f := \sum_{j<k} (-)^{k-1-j} \psi_{ik}^{(k-1-j)}(\tau_i) f^{(j)}(\tau_i)$$
$$\psi_{ik}(t) := (t_{i+1} - t)\ \ldots\ (t_{i+k-1} - t)/(k-1)!$$

and τ_i is an *arbitrary* point in (t_i, t_{i+k}). One verifies directly that [2]

$$\lambda_i N_{jk} = \delta_{ij}, \quad \text{all } i, j.$$

Consequently,

(i) Q is a linear projector with range $\mathcal{S}_{k,\mathbf{t}}$;
(ii) every $f \in \mathcal{S}_{k,\mathbf{t}}$ has a unique representation as a B–spline series;
(iii) if $f = \sum_i a_i N_{ik}$, then
$$a_i = \lambda_{\tau_i, \psi_{ik}} f \text{ for arbitrary } \tau_i \in (t_i, t_{i+k}).$$

2. Existence and uniqueness of the B–spline expansion

The rather curious freedom in the choice of τ_i above leads to the following short proof of

Theorem (Curry et Schoenberg [3]). *$\mathcal{S}_{k,\mathbf{t}}$ consists of exactly those f on (a,b) for which*
(i) *for all i, $f|_i \in \mathcal{P}_k$(:= polynomials of degree $< k$); and*
(ii) *if $t_s < t_{s+1} = \cdots = t_{s+r} < t_{s+r+1}$, then* $\operatorname{jump}_{t_{s+1}} f^{(k-j)} = 0$ *for all $j > r$.*

In particular, any such f has exactly one B–spline expansion (in terms of the B–splines of order k with knots $\mathbf{t}$).

Here and below, we denote by $f|_i$ the restriction of f to (t_i, t_{i+1}). For the proof, I show that $Qf = f$ for all such f :
(a) For all such f, and all i,

$$g(\tau) := \lambda_{\tau,\psi_{ik}} f = \sum_{j<k} (-)^{k-1-j} \psi_{ik}^{(k-1-j)}(\tau) f^{(j)}(\tau)$$

is constant on $\tau \in (t_i, t_{i+k}) =$ support N_{ik}, since
(α) for $\psi \in \mathcal{P}_k$ and smooth f,

$$(\lambda_{\tau,\psi} - \lambda_{\sigma,\psi}) f = \int_\sigma^\tau \psi d f^{(k-1)} \quad (= 0 \text{ if } f|_{[\sigma,\tau]} \in \mathcal{P}_k)$$

hence, as $f|_{(t_j,t_{j+1})} \in \mathcal{P}_k$, g *is constant on each* (t_j, t_{j+1}); and
(β) if $t_i \le t_s < t_{s+1} = \cdots = t_{s+r} < t_{s+r+1} \le t_{i+k}$, then t_{s+1} is an r-fold zero of ψ_{ik}, hence

$$\psi_{ik}^{(k-1-j)}(t_{s+1}) = 0, \quad \text{for } j = k-1,\ k-2, \ldots, k-r,$$

while, by assumption on f,

$$\operatorname{jump}_{t_{s+1}} f^{(j)} = 0, \quad \text{for } j = k-r-1,\ \ldots\ ,\ 0;$$

hence g *is continuous across each* t_{s+1} *with* $t_i < t_{s+1} < t_{i+k}$.
(b) For all such f, and all j with $t_j < t_{j+1}$,

$$(Qf)|_j = f|_j.$$

For, $(Qf)|_j = \sum_{i=j+1-k}^{j} (\lambda_{\tau_i,\psi_{ik}} f)(N_{ik})|_j$. But I can assume by (a) without loss that $\tau_i \in (t_j, t_{j+1})$, $i = j+1-k, \ldots, j$; hence

$$(Qf)|_j = \sum_{i=j+1-k}^{j} \lambda_{\tau_i,\psi_{ik}} (f|_j)\,(N_{ik})|_j,$$

while

$$\delta_{ir} = \lambda_{\tau_i,\psi_{ik}} N_{rk} = \lambda_{\tau_i,\psi_{ik}} (N_{rk}|_j), \quad r = j+1-k, \ldots, j$$

shows the k–sequence $N_{ik}|_j$, $i = j+1-k, \ldots, j$, in $\mathcal{P}_k$ to be independent, hence a basis for P_k. Consequently,

$$\sum_{i=j+1-k}^{j} (\lambda_{\tau_i,\psi_{ik}} h)\,(N_{ik})|_j = h, \quad \text{for all } h \in \mathcal{P}_k.$$

3. Uniqueness of odd–degree spline interpolation

In discussing the smooth extension of a real valued function defined on some closed subset of $\mathbb{R}$ to all of $\mathbb{R}$, Golomb et Schoenberg [4] prove that, for $\mathbf{t}$ strictly increasing, every $f \in \mathcal{S}_{2k,\mathbf{t}}$ which vanishes at the points of $\mathbf{t}$ and has square–integrable k^{th} derivative must vanish identically. Their proof is not simple. In particular, the straightforward argument

> $\forall_i f(t_i) = 0$, hence, $\forall_i\, 0 = f(t_i, \ldots, t_{i+k}) = \int N_{ik}(t) f^{(k)}(t)\, dt / c_{ik}$ with $c_{ik} := (k-1)!(t_{i+k} - t_i)$; i.e., $f^{(k)}$ is *orthogonal* to every N_{ik}, while at the same time being in $S_{k,\mathbf{t}}$ which is spanned by the N_{ik}'s; hence $f^{(k)} = 0$, and so $f = 0$.

was not open to them since it requires (N_{ik}) to be a Schauder basis for $\mathcal{S}_{k,\mathbf{t}} \cap L_2$, a fact they did not know.

Theorem. *Let $1 \le p \le \infty$, and $N_{ikp} := (k/(t_{i+k} - t_i))^{1/p} N_{ik}$. Then*

$$\sum_i b_i N_{ikp} \in L_p(a,b) \text{ iff } \|\mathbf{b}\|_p < \infty.$$

Precisely, there exists $D_{kp} > 0$ (independent of $\mathbf{t}$) so that

$$D_{kp}^{-1} \|\mathbf{b}\|_p \le \| \sum_i b_i N_{ikp} \|_p \le \|\mathbf{b}\|_p, \quad \text{for all } \mathbf{b} \in \mathbb{R}^{\mathbb{Z}}.$$

The second inequality is straightforward. As to the first, let $f := \sum_i a_i N_{ik} = \sum_i b_i N_{ikp}$, so that $a_i((t_{i+k} - t_i)/k)^{1/p} = b_i$, all i. Then, from Sec. 1, $|a_i| \le \sum_{j<k} |\psi_{ik}^{(k-1-j)}(\tau_i)|\, |f^{(j)}(\tau_i)|$.

Take I to be a largest interval among $(t_i, t_{i+1}), \ldots, (t_{i+k-1}, t_{i+k})$, and choose $\tau_i \in I$. Then $|\psi_{ik}^{(k-1-j)}(\tau_i)| < A_{jk} |I|^j$ for some constants A_{jk}, while $|f^{(j)}(\tau_i)| \le B_{jkp} |I|^{-j-1/p} \cdot \big(\int_I |f(t)|^p dt\big)^{1/p}$ since $f|_I \in \mathcal{P}_k$. Hence

$$\begin{aligned} |b_i|^p = |a_i|^p (t_{i+k} - t_i)/k \le |a_i|^p |I| &\le \Big(\sum_j A_{jk} B_{jkp}\Big)^p \int_I |f|^p \\ &\le C_{kp} \int_{t_i}^{t_{i+k}} |f|^p \end{aligned}$$

which, after summing over i, gives the required inequality with $D_{kp} = (k C_{kp})^{1/p}$.

For a *uniform* knot sequence $\mathbf{t}$, this theorem has already been proved by Schoenberg in [5] using a special case of the above formula for the B–spline coefficients.

Corollary. *For $1 \le p < \infty$, $(N_{ikp})_{-\infty}^{\infty}$ is a Schauder basis for $\mathcal{S}_{k,\mathbf{t}} \cap L_p(a,b)$.*

Bolstered by this Corollary, the earlier argument establishes uniqueness of odd–degree spline interpolation even in the limiting case of repeated or osculatory interpolation at multiple knots.

4. Bounds for least–squares approximation by splines

An attempt to bound the error in odd–degree spline interpolation to a smooth function in the uniform norm leads to the problem of bounding least–squares approximation by splines, considered as a map on L_∞, independently of the knot sequence (cf. [1]), a question of interest in itself.

Let $n \in \mathbb{N}$, $\mathcal{S} = \operatorname{span}\{N_{1k}, \ldots, N_{nk}\}$, and denote by Lf the least–squares approximation to an $f \in L_\infty[t_1, t_{n+k}]$ by elements of $\mathcal{S}$. Then, L is a linear projector, characterized by the fact that

$$(*) \qquad Lf \in \mathcal{S}, \text{ and, for all } \lambda \in \Lambda,\ \lambda L f = \lambda f$$

with the "interpolation conditions"

$$\Lambda := \{\lambda \in L_\infty^* |\ \text{for some } \varphi \in \mathcal{S} \text{ and all } f,\ \lambda f = \int \varphi f\}.$$

one verifies that $(*)$ implies

$$\|L\| = \sup_{x \in \mathcal{S}} \inf_{\lambda \in \Lambda} \|\lambda\| \, \|x\| / |\lambda x|.$$

But, in order to compute, one needs to coordinatize. Letting (λ_i) and (φ_i) be bases for Λ and $\mathcal{S}$, respectively, we get that

$$\|L\| = \sup_{\mathbf{a}} \inf_{\mathbf{b}} \| \sum_i b_i \lambda_i \| \| \sum_j a_j \varphi_j \| / \big| \sum_{ij} b_i \lambda_i \varphi_j a_j \big|.$$

Take $\varphi_i := N_{ik}, \lambda_i := k \int \cdot N_{ik}/(t_{i+k} - t_i), i = 1, \ldots n$. From the earlier theorem,

$$D_{k1}^{-1} D_{k\infty}^{-1} \|\mathbf{b}\|_1 \|\mathbf{a}\|_\infty \leq \| \sum_i b_i \lambda_i \| \, \| \sum_j a_j \varphi_j \| \leq \|\mathbf{b}\|_1 \|\mathbf{a}\|_\infty$$

while

$$\sup_{\mathbf{a}} \inf_{\mathbf{b}} \|\mathbf{b}\|_1 \|\mathbf{a}\|_\infty / \big| \sum_{ij} b_i \lambda_i \varphi_j a_j \big| = \|(\lambda_i \varphi_j)^{-1}\|_\infty$$

with $\|A\|_p$ denoting the norm for the matrix A induced by the p–norm on vectors. This proves

Proposition. *For some positive C_k (independent of $\mathbf{t}$ and n),*

$$C_k \|(\lambda_i \varphi_j)^{-1}\|_\infty \leq \|L\| \leq \|(\lambda_i \varphi_j)^{-1}\|_\infty$$

(considering L as a map on $L_\infty[t_1, t_{n+k}]$), with

$$(**) \qquad \lambda_i \varphi_j = k \int N_{ik} N_{jk} / (t_{i+k} - t_i), \quad i, j = 1, \ldots, n.$$

It has been known for some time that L could be bounded if only the Gramian $(\lambda_i \varphi_j)$ could be bounded below (in the max–norm). This proposition adds that such bounding below of the Gramian is also necessary for bounding L. For this reason, I offer the modest sum of m–1972 ten dollar bills to the first person who communicates to me a proof or a counterexample (but not both) of his or her own making for the following conjecture (known to be true when $k = 2$ or $k = 3$):

Conjecture. *For given n and $\mathbf{t}$, let $(\lambda_i \varphi_j)$ be the $n \times n$ matrix whose entries are given by $(**)$. Then*

$$\sup_{n, \mathbf{t}} \|(\lambda_i \varphi_j)^{-1}\|_\infty < \infty.$$

Here, m is the year A.D. of such communication.

5. Estimates for $\operatorname{dist}(f, \mathcal{S}_{k,\mathbf{t}})$

Let Qf be the quasi–interpolant to f as defined in Section 1. For a sufficiently smooth f,

$$f(t) - (Qf)(t) = \int E(t, s) \, df^{(k-1)}(s)$$

with $E(t, \cdot)$ a *nonnegative* function of *small* support. This makes Qf a convenient approximation when it comes to estimating the distance of such f from splines with fixed and with variable knots. Lack of space precludes, unfortunately, any discussion of this important aspect of the quasi–interpolant here.

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References

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[4] M. Golomb and I. J. Schoenberg (1968), "On $\mathcal{H}^m$–extension of functions and spline interpolation", MRC, U.Wisconsin-Madison.

This paper exists only as a reference in other papers, e.g., in [I. J. Schoenberg, On spline interpolation at all integer points of the real axis, Mathematica **10**(33), (1968), 151–170] where its proposed content is outlined, and in [M. Golomb and J. Jerome, Linear ordinary differential equations with boundary conditions on arbitrary point sets, Trans. AMS **153**, (1971), 235–264] in which it is incorrectly specified as a MRC TSR and where its proposed content is generalized.

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