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The exponentials in the span of the multiinteger translates of a compactly supported function; quasiinterpolation and approximation order

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ABSTRACT

Given a compactly supported function $\varphi : \mathbb{R}^s \to \mathbb{C}$ and the space S spanned by its integer translates, we study quasiinterpolants which reproduce (entirely or in part) the space H of all exponentials in S. We do this by imitating the action on H of the associated semi-discrete convolution operator $\varphi *'$ by a convolution $\lambda *$, λ being a compactly supported distribution, and inverting $\lambda *_{|H}$ by another local convolution operator $\mu *$. This leads to a unified theory for quasiinterpolants on regular grids, showing that each specific construction now in the literature corresponds to a special choice of λ and μ. The natural choice $\lambda = \varphi$ is singled out, and the interrelation between $\varphi *'$ and $\varphi *$ is analyzed in detail.

We use these observations in the conversion of the local approximation order of an exponential space H into approximation rates from any space which contains H and is spanned by the $h\mathbb{Z}^s$-translates of a single compactly supported function φ. The bounds obtained are attractive in the sense that they rely only on H and the basic quantities $\operatorname{diam}\operatorname{supp}\varphi$ and $h^s \|\varphi\|_\infty / \widehat{\varphi}(0)$.

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1. Introduction

Spaces spanned by integer translates of compactly supported functions have recently received much attention in multivariate spline theory. The main aim was a detailed analysis of special cases (primarily box spline and exponential box spline spaces); however, some of the fundamental issues, like approximation orders and linear independence of the translates, have been studied intensively in general settings as well.

The standard approach in the case of a single compactly supported $\varphi : \mathbb{R}^s \to \mathbb{C}$ can be identified with the Strang-Fix conditions. These characterize the "controlled" and the "local" approximation order from the dilates S_h of the span $S(\varphi)$ of the integer translates of φ in terms of the highest d for which the space Π_d of polynomials of degree $\le d$ lies in $S(\varphi)$. This makes it important to identify $S(\varphi) \cap \Pi$ (with Π the space of all s-variate polynomials). With this space in hand, one constructs bounded local linear maps into $S(\varphi)$, the so-called *quasiinterpolants*, which reproduce $S(\varphi) \cap \Pi$ (or at least Π_d). Dilates of these maps provide approximations from S_h to smooth functions whose error behaves like h^{d+1} (say, in the max norm). By now, the literature has various different constructs of quasiinterpolants to offer, some of them seem to have only little in common (compare, e.g., the "Neumann series approach" of [CD$_1$] with the construction in [SF] or [DM$_1$]).

When the S_h are derived from $S(\varphi)$ by processes other than dilation, the polynomials in $S(\varphi)$ cease to play the above decisive role. Specifically, the approximation order for exponential box splines was established in [DR] by choosing the S_h in such a way that each contains the space $H(\varphi)$:= the space of all exponentials in $S(\varphi)$, with an *exponential* being any linear combination of products of polynomials with the *pure exponentials*

$$e_\theta : x \mapsto e^{\langle \theta, x\rangle}, \qquad \theta \in \mathbb{C}^s. \tag{1.1}$$

This makes it possible to construct a sequence of uniformly bounded uniformly local quasiinterpolants, each of which (maps into the associated S_h and) reproduces $H(\varphi)$, and so allows to convert the local approximation order (= the approximation order at the origin) of $H(\varphi)$ into rates of approximation from $\{S_h\}_h$.

In this paper we provide an analysis which unifies all these various approaches. Our starting point (as it is for most quasiinterpolation arguments) is the semi-discrete convolution

$$\varphi *' : f \mapsto \varphi *' f := \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha) f(\alpha), \tag{1.2}$$

which is well-defined for every f defined (at least) on $\mathbb{Z}^s$, since φ is of compact support. The main idea is to replace this semi-discrete convolution operator by an ordinary (distributional) convolution operator, i.e., to look for distributions λ such that $\lambda *$ matches $\varphi *'$ on some subspace of $(\varphi *')^{-1}(H(\varphi))$.

In Section 2, we single out one natural choice for the distribution λ: the function φ itself. We identify an exponential space H_φ which is mapped by $\varphi*'$ onto $H(\varphi)$ and for which

$$(1.3) \qquad \varphi*_{|H_\varphi} = \varphi*'_{|H_\varphi}.$$

This leads to a simple characterization, in terms of the Fourier transform $\widehat{\varphi}$ of φ, of all admissible distributions λ that satisfy

$$(1.4) \qquad \lambda*_{|H_\varphi} = \varphi*'_{|H_\varphi}.$$

It is also pointed out that the distribution $\lambda = \varphi_|$ defined by

$$\varphi_| : f \mapsto \sum_\alpha \varphi(\alpha) f(\alpha)$$

satisfies (1.4), hence provides another natural choice.

The construction of quasiinterpolants is carried out in Section 3. Under a regularity condition on φ, the two spaces $H(\varphi)$ and H_φ are known to coincide. To construct then a quasiinterpolant, we replace $\varphi*'$ by any $\lambda*$ that satisfies (1.4), invert $\lambda*_{|H(\varphi)}$ with the aid of another compactly distribution μ, and so obtain that $\lambda*\mu*$ is the identity mapping on $H(\varphi)$, and invoke the results of Section 2 for a complete characterization of all possible μ (again in terms of their Fourier transform). The quasiinterpolant Q is then defined as

$$Q : f \mapsto \varphi*'(\mu*f).$$

Computations are much simplified by the fact that standard distributional and Fourier transform methods are available for the study of $\lambda*_{|H}$, in case of an exponential H. For example, general recurrence relations that solve equations of form

$$\lambda*? = f$$

for (almost) any exponential f can be easily derived, and these relations are also valid for the equation

$$\varphi*'? = f,$$

in case $\lambda*$ and $\varphi*'$ agree on f, i.e., in case $f \in H(\varphi)$. We show that the concrete constructions of quasiinterpolants now in the literature are covered by the two basic choices $\lambda = \varphi$ and $\lambda = \varphi_|$, and the various ways for choosing the reciprocal μ can be seen to satisfy the above-mentioned conditions for an admissible μ in terms of these λ.

In the last section, we consider approximation orders for a sequence of spaces $(S_h)_h$, each of which spanned by the $h\mathbb{Z}^s$-translates of a compactly supported function ψ_h and containing an h-independent exponential space H. In order to make good use of the assumption that ψ_h has compact support, it is essential to assume that $\operatorname{diam}\operatorname{supp}\psi_h$ shrinks to 0 linearly in h, as we do. Under a further mild restriction on the sequence $(\psi_h)_h$, we show that the local approximation order of H provides the expected lower bound for the approximation order from $\{S_h\}_h$ to a smooth function.

2. Convolution and Semi-Discrete Convolution

Let φ be a compactly supported function defined on $\mathbb{R}^s$. Given any (complex-valued) function f, defined (at least) on $\mathbb{Z}^s$, we use the notation

$$\varphi *' f := \sum_{\alpha\in\mathbb{Z}^s} \varphi(\cdot - \alpha)\, f(\alpha). \tag{2.1}$$

The special symbol $*'$ was selected to denote the above semidiscrete convolution, in order to distinguish this operation from the distributional convolution product $\lambda*\mu$, which, in case λ and μ are distributions of function-type, is reduced to the standard convolution product

$$\lambda*\mu := \int_{\mathbb{R}^s} \lambda(\cdot - x)\mu(x)dx.$$

In the sequel, we use D for the differentiation operator and E for the shift operator (i.e., for $\alpha \in \mathbb{Z}^s$, $D^\alpha = \prod_{j=1}^s {D_j}^{\alpha_j}$, with D_j the partial differentiation in the jth direction, and $E^\alpha : f \mapsto f(\cdot + \alpha)$), hence, for a polynomial p, $p(D)$ and $p(E)$ denote the evaluation of p at D and E respectively, (for example, $p(D) = \sum_{\alpha\in\mathbb{Z}^s_+} \frac{D^\alpha p(0)}{\alpha!} D^\alpha$). As in the introduction $S(\varphi)$ stands for the space spanned by the integer translates of φ, and $H(\varphi)$ denotes the set of all exponentials in $S(\varphi)$. Since $S(\varphi)$ is closed under integer translates (E-invariant), so is $H(\varphi)$. But, in general, $H(\varphi)$ need not be closed under differentiation (D-invariant), as e.g., Example 2.1 of [R$_1$] shows. Since φ is compactly supported, $H(\varphi)$ is necessarily finite-dimensional, hence

$$H(\varphi) \subset \mathrm{Exp}_{\boldsymbol{\Theta}} := \sum_{\theta\in\boldsymbol{\Theta}(\varphi)} e_\theta \Pi$$

for a minimal finite $\boldsymbol{\Theta} := \boldsymbol{\Theta}(\varphi) \subset \mathbb{C}^s$, called the **spectrum** of $H(\varphi)$ and denoted by $\operatorname{spec} H(\varphi)$. If $H(\varphi)$ is D-invariant, then (cf. [BAR$_1$; Lemma 3.1])

$$H(\varphi) = \bigoplus_{\theta\in\boldsymbol{\Theta}(\varphi)} e_\theta P_\theta,$$

with each P_θ a (nontrivial) D-invariant polynomial space. In that case,

$$\boldsymbol{\Theta}(\varphi) = \{\theta \in \mathbb{C}^s : e_\theta \in H(\varphi)\}.$$

The goal of this section is to establish conditions which guarantee the equality

$$\varphi*f = \varphi*' f \tag{2.2}$$

for $f \in \mathrm{Exp}_{\mathrm{T}}$, with T a finite subset of $\mathbb{C}^s$. We refer to [BrH] and [DM$_4$] for prior results in this direction with regard to a tensor B-spline and a polynomial box spline respectively, hence for a polynomial f.

Since

$$\varphi * \mathrm{Exp}_{\mathrm{T}} \subset \mathrm{Exp}_{\mathrm{T}},$$

an immediate necessary condition for (2.2) is that $\varphi *' f \in \mathrm{Exp}_{\mathrm{T}}$. It turns out that such a condition is already (essentially) sufficient. For the proof of this claim, we need the exponential space

$$H_{\varphi} := \bigoplus_{\theta \in \mathbf{\Theta}(\varphi)} e_{\theta} \Pi_{\theta}(\varphi), \tag{2.3}$$

with $\Pi_{\theta}(\varphi)$ the maximal D-invariant subspace of

$$\{p \in \Pi : p(-iD)\widehat{\varphi} = 0 \text{ on } -i\theta + (2\pi \mathbb{Z}^s \backslash 0)\}$$

(cf. [B;Prop.2.2]).

The argument makes use of the following

(2.4) Decomposition Lemma. *Let $H = \sum_{\theta \in \mathrm{T}} e_{\theta} Q_{\theta}$, where $\mathrm{T} \subset \mathbb{C}^s$ is finite, and each Q_{θ} a finite-dimensional polynomial space. Then the condition*

$$(\mathrm{T} - \mathrm{T}) \cap 2\pi i \mathbb{Z}^s = \{0\} \tag{2.5}$$

implies, for each $\theta \in \mathrm{T}$, the existence of a (θ-dependent) polynomial p so that the difference operator $p(E)$ projects all of H onto $e_{\theta} Q_{\theta}$. If $Q_{\theta} \neq \{0\}$ for all $\theta \in \mathrm{T}$, then (2.5) is necessary as well.

Proof: The necessity of the condition is clear: if $p(E)$ annihilates some non-trivial space $e_{\theta} Q_{\theta}$, then $p(E) e_{\theta} = 0$ since e_{θ} lies in the space generated by shifts (i.e., integer translates) of $e_{\theta} Q_{\theta}$, and these shifts trivially commute with $p(E)$. Also, we always have $p(E) e_{\theta} = c e_{\theta}$, for some constant c, and for any $\vartheta \in \theta + 2\pi i \mathbb{Z}^s$, $p(E) e_{\vartheta} = c e_{\vartheta}$, with the same constant c. Therefore, if $p(E) e_{\theta} = 0$, then also $p(E) e_{\vartheta} = 0$, and hence $p(E)$ cannot be the identity on $e_{\vartheta} Q_{\vartheta}$, unless Q_{ϑ} is trivial.

For the sufficiency, we may assume without loss that the polynomial spaces Q_{θ} are all E-invariant (by replacing each Q_{θ} by its E-closure, if need be). The argument used in the proof of [BAR$_1$; Lem. 3.1] then shows that under (2.5) one can find, for each θ, a difference operator $q(E)$ which maps H onto $e_{\theta} Q_{\theta}$. It follows that the operator $A := q(E)_{|e_{\theta} Q_{\theta}}$ is invertible, hence its inverse is representable as a (univariate) polynomial in A, therefore also as some (multivariate) polynomial $r(E)$ in E. With that, $p(E) := q(E) r(E)$ is the required difference operator. ♠

Condition (2.5) plays an important role in the theorem below. Therefore, it is worthwhile to note the following:

(2.6) Corollary. *If the finite $\mathrm{T} \subset \mathbb{C}^s$ satisfies (2.5), then $H_{\mathrm{T}} := \mathrm{Exp}_{\mathrm{T}} \cap H(\varphi)$ is D-invariant.*

The converse of this corollary is not valid (cf. Example 7.1 in [BAR$_1$]), yet is true for *univariate* splines (cf. [R$_2$; Proposition 4.6]).

Proof: Since $H(\varphi)$ is finite-dimensional, we may consider the smallest polynomial spaces Q_{θ} (necessarily finite-dimensional) for which $H_{\mathrm{T}} \subset \sum_{\theta \in \mathrm{T}} e_{\theta} Q_{\theta}$. For each $\theta \in \mathrm{T}$, let $p_{\theta}(E)$ be the projector (provided by Lemma 2.4) which carries this sum onto its summand $e_{\theta} Q_{\theta}$. Then, for every $g = \sum_{\theta} e_{\theta} q_{\theta} \in H_{\mathrm{T}}$, every $\theta \in \mathrm{T}$ and every $\alpha \in \mathbb{Z}^s$,

$$E^{\alpha}(e_{\theta} q_{\theta}) = E^{\alpha} p_{\theta}(E) g \in E^{\alpha} p_{\theta}(E) H_{\mathrm{T}} \subset E^{\alpha} p_{\theta}(E) H(\varphi) \subset H(\varphi).$$

Using the definition of H_{T}, the above shows that H_{T} contains, for every $\alpha \in \mathbb{Z}_+^s$, $E^\alpha(e_\theta Q_\theta)$ and that each Q_θ is necessarily E-invariant, hence [B;§2] D-invariant. Therefore $H_{\mathrm{T}} = \sum_{\theta\in\mathrm{T}} e_\theta Q_\theta$ and this sum is D-invariant. ♠

We also note the following useful connection between the action of $\varphi*$ and the structure of H_φ:

(2.7) Proposition. *Let φ be a compactly supported distribution, and $e_\theta p$ ($\theta \in \mathbb{C}^s$, $p \in \Pi$) an exponential. Then, $e_\theta p \in H_\varphi$ if and only if*

$$\varphi*(e_\vartheta p) = 0, \quad \forall \vartheta \in \theta + (2\pi i\mathbb{Z}^s\backslash 0). \tag{2.8}$$

Proof: Since $\varphi*(e_\vartheta p)$ is in $e_\vartheta \Pi$, it is analytic, and hence $\varphi*(e_\vartheta p) = 0$ if and only if, for every $\alpha \in \mathbb{Z}_+^s$,

$$\varphi*(D^\alpha(e_\vartheta p))(0) = D^\alpha(\varphi*(e_\vartheta p))(0) = 0.$$

Since the spaces $\{D^\alpha(e_\vartheta p) : \alpha \in \mathbb{Z}_+^s\}$ and $\{e_\vartheta D^\alpha p : \alpha \in \mathbb{Z}_+^s\}$ are identical, we conclude that $\varphi*(e_\vartheta p) = 0$ if and only if

$$\varphi*(e_\vartheta D^\alpha p)(0) = 0, \quad \alpha \in \mathbb{Z}_+^s.$$

On the other hand (cf. (3.7) below), $\varphi*(e_\vartheta p) = e_\vartheta p(\cdot - iD)\widehat{\varphi}(-i\vartheta)$, and the desired result now follows from the definition (2.3) of H_φ. ♠

(2.9) Theorem. *Let φ be a compactly supported distribution, and let $f \in \mathrm{Exp}_{\mathrm{T}}$, for some finite $\mathrm{T} \subset \mathbb{C}^s$. Consider the following conditions:*

(a) $\quad f \in H_\varphi$;
(b) $\quad \varphi*' f = \varphi* f$;
(c) $\quad \varphi*' f \in \mathrm{Exp}_{\mathrm{T}}$.

Then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). *If, in addition,* $(\mathrm{T}-\mathrm{T}) \cap 2\pi i\mathbb{Z}^s = \{0\}$, *then* (c) $\Longrightarrow$ (a) *as well.*

For simplicity, we first prove the complete theorem for a polynomial f (i.e., for $\mathrm{T} = \{0\}$).

(2.10) Lemma. *The conditions in Theorem 2.9 are equivalent in case* $\mathrm{T} = \{0\}$, *i.e., in case* $f \in \Pi$.

Proof: Since the original argument that we had for the proof was already sketched in [B_2], we feel free to present here a newer proof, based on the argument given in the recent paper [RS]. This proof relies on the fact [RS] that, for any compactly supported φ and any infinitely smooth f,

$$\varphi*' f = \sum_{\alpha\in 2\pi i\mathbb{Z}^s} \varphi*(e_\alpha f), \tag{2.11}$$

where both sides are interpreted as (limits of) elements in the space $\mathcal{D}'$ of s-dimensional complex valued distributions. Here is, for completeness, a proof of (2.11): By Poisson's summation formula [F; p. 104], the sum $\sum_{\alpha\in 2\pi i\mathbb{Z}^s} e_\alpha$ converges in $\mathcal{D}'$ to $\sum_{\alpha\in\mathbb{Z}^s} \delta_\alpha$ (with δ_α being point-evaluation at α). Thus, since multiplication by $f \in C^\infty$ as well as convolution with a compactly supported φ are continuous operations in $\mathcal{D}'$, we obtain

$$\begin{aligned}\varphi*' f = \sum_{\alpha\in\mathbb{Z}^s} f(\alpha)\varphi*\delta_\alpha &= \sum_{\alpha\in\mathbb{Z}^s} \varphi*(f\delta_\alpha) = \varphi*(f \sum_{\alpha\in\mathbb{Z}^s} \delta_\alpha)\\ &= \varphi*(f \sum_{\alpha\in 2\pi i\mathbb{Z}^s} e_\alpha) = \varphi*(\sum_{\alpha\in 2\pi i\mathbb{Z}^s} e_\alpha f) = \sum_{\alpha\in 2\pi i\mathbb{Z}^s} \varphi*(e_\alpha f).\end{aligned}$$

Now let

$$F(f) := \sum_{\alpha\in 2\pi i\mathbb{Z}^s\backslash 0} \varphi*(e_\alpha f).$$

We see from (2.11) that (b) holds if and only if $F(f) = 0$, and (given that $f \in \Pi$) that (c) holds if and only if $F(f) \in \Pi$, while, by Proposition 2.7, (a) holds if and only if $\varphi*(e_\alpha f) = 0$ for all $\alpha \in 2\pi i\mathbb{Z}^s\backslash 0$. It thus follows that (a)$\Longrightarrow$(b)$\Longrightarrow$(c), and further, that the implication (c)$\Longrightarrow$(a) is equivalent to

$$(2.12) \qquad F(f) = \sum_{\alpha\in 2\pi i\mathbb{Z}^s\backslash 0} \varphi*(e_\alpha f) \in \Pi \quad \Longrightarrow \quad \varphi*(e_\alpha f) = 0 \ \ \forall \alpha \in 2\pi i\mathbb{Z}^s\backslash 0.$$

For the proof of (2.12), assume that $F(f) \in \Pi$. Since $f \in \Pi$, the infinite sum $F(f)$ converges in the topology of tempered distributions, while the supports of the Fourier transforms of its summands $\{\varphi*(e_\alpha f) : \alpha \in 2\pi i\mathbb{Z}^s\backslash 0\}$ are pairwise disjoint and also disjoint of the support of the Fourier transform of any particular $p \in \Pi$. Thus $F(f)$ fails to be in Π unless all its summands vanish. ♠

We start the **proof of Theorem 2.9** with the implication (a)$\Longrightarrow$(b). This implication follows from Lemma 2.10 by shifting in the frequency domain, using the fact that (a) implies that $\mathrm{T} \subset \boldsymbol{\Theta}(\varphi)$, hence that $f = \sum_{\theta\in\boldsymbol{\Theta}} f_\theta$ with $f_\theta = e_\theta p_\theta$ for some $p_\theta \in \Pi_\theta(\varphi)$. It follows that each such p_θ lies in $\Pi_0(e_{-\theta}\varphi)$. Therefore, by Lemma 2.10,

$$\varphi*' f_\theta = e_\theta\left((e_{-\theta}\varphi)*' p_\theta\right) = e_\theta\left((e_{-\theta}\varphi)* p_\theta\right) = \varphi* f_\theta$$

for each $\theta \in \boldsymbol{\Theta}$, and (b) follows.

The implication (b) $\Longrightarrow$ (c) is immediate, since, as mentioned before, $\varphi*\mathrm{Exp}_{\mathrm{T}} \subset \mathrm{Exp}_{\mathrm{T}}$.

Finally, we show that, under the additional assumption (2.5), (c) $\Longrightarrow$ (a). Decompose $f = \sum_{\theta\in\mathrm{T}} f_\theta$ into its various frequency components $f_\theta = e_\theta q_\theta$, as before. Assuming (2.5) and (c), Lemma 2.4 provides, for each $\theta \in \mathrm{T}$, a polynomial $p = p_\theta$ so that $p(E)f = f_\theta$ and $p(E)(\varphi*' f) \in e_\theta\Pi$. Therefore

$$e_\theta\Pi \ni p(E)\left(\varphi*' f\right) = \varphi*' p(E) f = \varphi*' f_\theta.$$

But this says that $r := e_{-\theta}(\varphi*' f_\theta) \in \Pi$, i.e., $(e_{-\theta}\varphi)*' q_\theta = r \in \Pi$, therefore $q_\theta \in \Pi_0(e_{-\theta}\varphi) = \Pi_\theta(\varphi)$ by Lemma 2.10, i.e., $f_\theta \in H_\varphi$. ♠

It should be emphasized that (2.5) is essential for the derivation of (a) from (c): If $\theta, \vartheta \in \mathrm{T}$ and $\theta - \vartheta \in 2\pi i\mathbb{Z}^s\backslash 0$, then $f := e_\theta q - e_\vartheta q$ vanishes on $\mathbb{Z}^s$ for any $q \in \Pi$, hence $\varphi*' f = 0$, yet $f \neq 0$, hence does not belong to H_φ if q has sufficiently high degree.

(2.13) Corollary. *For any $f \in H_\varphi$ and $p \in \Pi$,*

$$p(D)\left(\varphi*' f\right) = \varphi*' p(D) f.$$

Proof: Since H_φ is D-invariant, $p(D)f \in H_\varphi$; hence by Theorem 2.9

$$p(D)\left(\varphi*' f\right) = p(D)\left(\varphi* f\right) = \varphi* p(D) f = \varphi*' p(D) f. \qquad ♠$$

The implication (a) $\Longrightarrow$ (c) in Theorem 2.9 shows that H_φ is mapped by $\varphi*'$ *into* $H(\varphi)$. When we want to study specific $H(\varphi)$, it is important to know whether H_φ is mapped *onto* $H(\varphi)$:

(2.14) Theorem. *Let H be a D-invariant subspace of $H(\varphi)$. Then $H \subset \varphi *' H_\varphi$. Furthermore, $H(\varphi) = \varphi *' H_\varphi$ if and only if $H(\varphi)$ is D-invariant .*

Proof: Since H is D-invariant, it is the direct sum of spaces of the form $e_\theta P$, so for the proof we may assume without loss that $H = e_\theta P$, for some D-invariant polynomial space P. We now invoke [BR$_2$; Corollary 5.5] to conclude that there exists a polynomial space Q such that $\varphi *' e_\theta Q = e_\theta P$, and the implication (c) $\Longrightarrow$ (a) of Theorem 2.9 then shows that indeed $H \subset \varphi *' H_\varphi$.

If now $H(\varphi)$ is D-invariant, then, with $H = H(\varphi)$, we conclude that $H(\varphi) \subset \varphi *' H_\varphi$, and since the converse inclusion holds unconditionally (by the implication (a)$\Longrightarrow$(c) in Theorem 2.9), equality holds. The equality $H(\varphi) = \varphi *' H_\varphi$ cannot hold when $H(\varphi)$ is not D-invariant, since, by Corollary 2.13, $\varphi *' H_\varphi$ is always D-invariant. ♠

In cases of interest, one may not know *a priori* whether $H(\varphi)$ or a subspace of it are D-invariant. In such a case, the following corollary, which follows directly from Theorem 2.14 when combined with Corollary 2.6, is of interest.

(2.15) Corollary. *For every $\mathrm{T} \subset \mathbb{C}^s$ satisfying (2.5), $H(\varphi) \cap \mathrm{Exp}_\mathrm{T} \subset \varphi *' H_\varphi$.*

We now explain in a more precise form some of the motivation for the above discussion. Theorem 2.9 allows us to represent $T := \varphi *'_{|H_\varphi}$ by the convolution operator $\varphi *$, hence to employ standard techniques (such as the Fourier transform) for the investigation of T. As a matter of fact, any compactly supported distribution λ for which

$$\lambda *_{|H_\varphi} = \varphi *'_{|H_\varphi} \tag{2.16}$$

may be used for that purpose. The distributions λ that satisfy (2.16) can be characterized in the following simple way:

(2.17) Proposition. *A compactly supported distribution λ satisfies (2.16) if and only if*

$$p(-iD)\widehat{\lambda}(-i\theta) = p(-iD)\widehat{\varphi}(-i\theta), \ \forall e_\theta p \in H_\varphi. \tag{2.18}$$

Proof: By Theorem 2.9, (2.16) is equivalent to

$$\lambda *(e_\theta p) = \varphi *(e_\theta p), \ \forall e_\theta p \in H_\varphi, \tag{2.19}$$

while the equivalence of (2.19) and (2.18) is obtained along the lines of the proof of Proposition 2.7. ♠

Assuming φ to be a function (and not merely a distribution), as we do from now on, a specific distribution λ satisfying (2.16) is

$$\varphi_| : f \mapsto \sum_{\alpha \in \mathbb{Z}^s} \varphi(\alpha) f(\alpha),$$

as we now show. The convolution operator $\varphi_| *$ associated with $\varphi_|$ is the difference operator

$$\varphi_| * : f \mapsto \varphi_| * f = \sum_\alpha f(\cdot - \alpha)\varphi(\alpha).$$

The Fourier transform of $\varphi_|$ is the **symbol** (or the **discrete** Fourier transform)

$$\widetilde{\varphi} := \sum_{\alpha} \varphi(\alpha) e_{-i\alpha} \tag{2.20}$$

of φ. To see that $\lambda = \varphi_|$ satisfies (2.16), we first invoke Theorem 2.9 to conclude that $\varphi *' f \in e_\theta \Pi$ for $f := e_\theta p \in H_\varphi$. It follows that indeed

$$\varphi *' f = \varphi_| * f,$$

since they both lie in $e_\theta \Pi$ and they coincide on $\mathbb{Z}^s$. (We are using the fact that no non-trivial polynomial vanishes on $\mathbb{Z}^s$). We may now appeal to Proposition 2.17 to conclude

(2.21) Corollary. *The operator $\varphi_| *$ agrees with $\varphi *'$ on H_φ, hence*

$$p(-iD)\widetilde{\varphi}(-i\theta) = p(-iD)\widehat{\varphi}(-i\theta), \ \forall e_\theta p \in H_\varphi.$$

3. Quasi-Interpolation

We are interested in constructing quasiinterpolants for $S(\varphi)$. These are linear maps into $S(\varphi)$ which are the identity on some D-invariant subspace H of $H(\varphi)$. Following [BH], it has become standard to construct such maps in the form

$$G_\nu : f \mapsto \sum_{\alpha \in \mathbb{Z}^s} \varphi(\cdot - \alpha) \ (\nu f)(\cdot + \alpha) \tag{3.1}$$

for some suitable compactly supported distribution ν which is well-defined and continuous on a given translation-invariant superspace F of C^∞ (e.g., $F = C^k(\mathbb{R}^s)$ with k the order of ν). The idea is to choose ν as an extension of the linear functional ν_0 given on H by

$$\nu_0(f) = \big(T^{-1} f\big)(0)$$

with

$$T := \varphi *'_{|H}.$$

Of course, we should require T to be injective. As a matter of fact, it is known [B], [R$_1$], that T becomes an automorphism in case φ is assumed to be **regular with respect to** H, which means that H is D-invariant and

$$\widehat{\varphi}(\theta) \neq 0 \ , \quad \forall \theta \in -i \operatorname{spec} H. \tag{3.2}$$

If, indeed, we choose ν as some extension of the above ν_0, then, taking into account that T^{-1} commutes with integer translates (since $\varphi *'$, and hence T, do, and H is E-invariant), we see that, for $f \in H$,

$$\nu(E^\alpha f) = \big(T^{-1}(E^\alpha f)\big)(0) = \big(E^\alpha (T^{-1} f)\big)(0) = (T^{-1} f)(\alpha),$$

and therefore

$$G_\nu f = \varphi *' (T^{-1} f) = f$$

for every $f \in H$. We note that the above regularity assumption also implies (2.5) for $\mathrm{T} := \operatorname{spec} H$ (cf. [R$_1$;Corollary 2.1]). Therefore, the implication (c)$\Longrightarrow$(b) in Theorem 2.9 provides the following.

(3.3) Corollary. *Assume φ is regular with respect to $H(\varphi)$. Then*

$$\varphi *_{|H(\varphi)} = \varphi *'_{|H(\varphi)}.$$

In particular, if $f \in H(\varphi) \cap \mathrm{Exp}_{\mathrm{T}}$, then so is $\varphi *' f$.

For notational convenience, we choose $H = H(\varphi)$, and simply use "regular" in the sense of "regular with respect to $H(\varphi)$".

Quasiinterpolants are discussed in great detail in the literature ([SF], [BH], [$\mathrm{DM}_{1-3,5}$], [BJ], [CJW], [$\mathrm{CD}_{1,2}$], [CL], [B], [DR], [$\mathrm{R}_{1,3}$], [$\mathrm{J}_{1,2}$], [BAR_2]), and various concrete constructions of ν are suggested.

We want to take here a different tack, based on the fact (Corollary 3.3) that $\varphi *'$ agrees with $\varphi *$ on $H(\varphi)$. For, this result suggests that we construct the quasiinterpolant in the form

$$Q_\mu := \varphi *' \mu * \tag{3.4}$$

with $\mu *$ any convenient convolution which agrees with T^{-1} on H. We recover the earlier formulation with the choice $\mu : f \mapsto \nu f(-\cdot)$ (since $\mu * f(x) = \nu(E^x f)$), but find the formulation (3.4) so much more straightforward that we abandon (3.1) and concentrate instead on the problem suggested by the formulation (3.4): For given D-invariant $H \subset H(\varphi)$, find distributions μ (of some desirable form) so that $\mu *_{|H} = (\varphi *'_{|H})^{-1} = (\varphi *_{|H})^{-1}$. In fact, there is useful additional freedom here: It is sufficient to construct μ so that $\mu * = (\lambda *_{|H})^{-1}$ for some distribution λ for which $\lambda * = \varphi *$ on H. For example, one might choose to use the particular distribution

$$\varphi_| : f \mapsto \sum_\alpha \varphi(\alpha) f(\alpha)$$

in place of φ (as discussed at the end of the last section).

We now describe some concrete approaches to the construction of suitable μ.

(i) **Matching of Fourier transform** In effect, we are looking for a solution to the convolution equation

$$\lambda * ? = f \tag{3.5}$$

with f some exponential. We would like to express ? in terms of $\widehat{\lambda}$ and f for an *arbitrary* exponential f, and then, for a fixed finite-dimensional exponential space (i.e., the underlying H), would like to write $(\lambda *)^{-1}$ in the form $\mu *$. Substituting $? = \mu * f$ for ? in (3.5), and Fourier transforming, we obtain

$$\widehat{\lambda}\widehat{\mu}\widehat{f} = \widehat{f},$$

which shows that we could use here any μ for which $\widehat{\lambda}\widehat{\mu} - 1$ vanishes to sufficiently high order on the (necessarily finite) spectrum of the exponential f. We now make this observation precise.

Equation (3.5) has a well-known solution in case the space in question is the single exponential space $e_\theta P$ for some D-invariant polynomial space P. For, one computes for the **normalized power function** $[\![]\!]^\alpha : x \mapsto x^\alpha/\alpha!$ that

$$\lambda * (e_\theta [\![]\!]^\alpha) = e_\theta \left((\lambda e_{-\theta}) * [\![]\!]^\alpha \right) = e_\theta [\![\cdot - iD]\!]^\alpha \widehat{\lambda}(-i\theta) = e_\theta \sum_\gamma [\![]\!]^{\alpha-\gamma} [\![-iD]\!]^\gamma \widehat{\lambda}(-i\theta). \tag{3.6}$$

Given that P is D-invariant, this implies that, for any polynomial $p \in P$,

$$(3.7) \quad \lambda{*}(e_\theta p) = e_\theta p(\cdot - iD)\widehat{\lambda}(-i\theta) = e_\theta \sum_\gamma D^\gamma p \; [\![-iD]\!]^\gamma \widehat{\lambda}(-i\theta) \;\in\; e_\theta(\widehat{\lambda}(-i\theta)p + \; P \cap \Pi_{<\deg p}).$$

In particular, $\lambda*$ maps $e_\theta P$ into itself, and is invertible on $e_\theta P$ if and only if $\widehat{\lambda}(-i\theta) \neq 0$. Further, since $p(\cdot - iD)1 = p$, the first equality in (3.7) (with λ replaced by $\lambda{*}\mu$) shows that $\lambda{*}\mu{*} = 1$ on $e_\theta P$ if and only if $p(\cdot - iD)(\widehat{\lambda}\widehat{\mu} - 1)(-i\theta) = 0$ for all $p \in P$. This last condition is equivalent to $\widehat{\lambda}\widehat{\mu} - 1$ having a $P(-i\cdot)$-fold zero at $-i\theta$ (i.e., $p(-iD)(\widehat{\lambda}\widehat{\mu} - 1)(-i\theta) = 0$ for all $p \in P$), since

$$p(\cdot - iD) = \sum_\gamma [\![\,]\!]^\gamma (D^\gamma p)(-iD)$$

and P is D-invariant. This proves the following.

(3.8) Proposition. *If the compactly supported distribution λ satisfies $\widehat{\lambda}(-i\theta) \neq 0$, and the finite-dimensional polynomial space P is D-invariant, then $\lambda*$ maps $e_\theta P$ 1-1 onto itself, and any convolution $\mu*$ with*

$$(3.9) \qquad p(-iD)\widehat{\mu}(-i\theta) = p(-iD)(1/\widehat{\lambda})(-i\theta) \qquad \forall p \in P$$

provides the inverse of $\lambda*$ on $e_\theta P$.

In applications, the polynomial space P is often not known precisely, but its degree can be ascertained, i.e., a k with $P \subset \Pi_k$ can be found. In that case, one would satisfy (3.9) for Π_k rather than P, i.e., one would make certain that all derivatives of order $\leq k$ of $\widehat{\mu}$ at $-i\theta$ match those of $1/\widehat{\lambda}$ there.

By choosing μ so that (3.9) is satisfied with $\lambda = \varphi$ and with $P = P_\theta$ for every $\theta \in \mathbf{\Theta}(\varphi) = \operatorname{spec} H(\varphi)$, one obtains a suitable distribution μ and a quasiinterpolant $\varphi{*}'\mu{*}$. For example, if we choose $\mu*$ to be a differential operator $q(-iD)$ for some polynomial q, then $\widehat{\mu} = q$, while if we choose $\mu*$ to be a finite difference operator $q(E)$, we get the 'trigonometric' polynomial $\widehat{\mu}(w) = q(e_{iw})$. In these cases μ can be chosen by requiring the polynomial q or the trigonometric polynomial $q(e_{iw})$ to provide an osculatory interpolation to $1/\widehat{\varphi}$ at $-i\mathbf{\Theta}(\varphi)$. In the first case, μ is a linear combination of values and derivatives at the origin, while, in the second case, μ employs only function values at some points from $\mathbb{Z}^s$. More generally, one could use μ of the form $\mu : f \mapsto \sum_{x \in X} q_x(-iD)f(x)$, where X is some finite subset of $\mathbb{R}^s$, and each $q_x \in \Pi$; for this case, the polynomials q_x are to be chosen so that the exponential $\widehat{\mu} = \sum_{x \in X} q_x e_{-ix}$ osculates to $1/\widehat{\varphi}$ appropriately at $-i\mathbf{\Theta}(\varphi)$.

If $\mu*$ is chosen to be a **difference operator**, (i.e., if μ is a linear combination of point-evaluations from $\mathbb{Z}^s$), then it commutes with $\varphi{*}'$ and thus

$$\varphi{*}'(\mu{*}f) = \mu{*}(\varphi{*}'f) = (\mu{*}\varphi){*}'f.$$

This provides us with a quasiinterpolant of the simple form $\psi{*}'$, with $\psi = \mu{*}\varphi \in S(\varphi)$, and with the support of ψ not exceeding the sum of the supports of μ and φ. In fact [R$_1$], the support of the difference operator $\mu*$ can be chosen so that

$$\operatorname{diam} \operatorname{supp} \psi \leq 2 \operatorname{diam} \operatorname{supp} \varphi,$$

in contrast to the minimal polynomial procedure below in which the inverting difference operator is supported on a relatively large domain.

We summarize the Fourier transform approach in the following

(3.10) Theorem. *Let φ be a regular compactly supported function, H a D-invariant subspace of $H(\varphi)$, and μ a compactly supported distribution. Then*

$$Q_\mu := \varphi *' \mu *$$

is an H-quasiinterpolant (i.e., is the identity on H) if and only if

$$p(-iD)\widehat{\mu}(-i\theta) = p(-iD)(1/\widehat{\lambda})(-i\theta), \ \forall e_\theta p \in H,$$

where λ is any (every) compactly supported distribution whose associated convolution operator $\lambda*$ coincides with $\varphi*'$ on H. Suitable choices for λ are $\lambda = \varphi$ and $\lambda = \varphi_|$.

(ii) **Minimal polynomial** Here, one would choose an 'easily computable' distribution λ for which $\lambda* = \varphi* = \varphi*'$ on H and observe that, since $T = (\varphi*')_{|H}$ is invertible, we can represent T^{-1} as $q(\lambda*)$ for some univariate polynomial q (i.e., obtain the inverse of the operator $\lambda*_{|_H}$ as a linear combination of powers of $\lambda*$). We obtain such a polynomial (up to a normalizing factor) in the form

$$q(t) := (m_T(0) - m_T(t))/t \tag{3.11}$$

with m_T the **minimal (annihilating) polynomial** for T. It may be hard, in general, to produce this polynomial, particularly if the space $H(\varphi)$ is not known precisely. But, we conclude from (3.7) that, for $\theta \in \operatorname{spec} H$, $\varphi*$, and hence any $\lambda*$ which coincides with $\varphi*$ on H, is **degree-preserving** on the exponential space $e_\theta\Pi$, in the sense that, for any polynomial p,

$$\lambda* e_\theta p \in e_\theta(\widehat{\lambda}(-i\theta)p + \Pi_{<\deg p}).$$

In fact, (3.7) implies that

$$\lambda* e_\theta p \in e_\theta(\widehat{\lambda}(-i\theta)p + \Pi_{\deg p - k})$$

in case $\widehat{\lambda} - \widehat{\lambda}(-i\theta)$ has a zero of order k at $-i\theta$. For example, if λ is a radially symmetric function, i.e., $\lambda(-x) = \lambda(x)$, then $\int_{\mathbb{R}^s} p\lambda = 0$ for any homogeneous polynomial p of odd degree and hence in particular

$$\lambda* p \in \widehat{\lambda}(0)p + \Pi_{\deg p - 2} \tag{3.12}$$

in that case, as was already pointed out in [CD_1]. In any case, the map M_θ defined by

$$M_\theta : f \mapsto \widehat{\lambda}(-i\theta)f - \lambda* f,$$

is **degree-reducing** on $e_\theta\Pi$, and thus, for any H in $e_\theta\Pi$, there is a suitable power M_θ^n of M_θ which annihilates H. Also, writing M_θ^n as a linear combination of powers of $\lambda*$, we observe that the constant term here is $\widehat{\lambda}(-i\theta) \neq 0$ (since T is assumed to be invertible), and thus we can use the facts that (a) on H, $1 - M_\theta^n/\widehat{\lambda}(-i\theta)$ is the identity, and (b) $1 - M_\theta^n/\widehat{\lambda}(-i\theta) = \lambda*...$, to write T^{-1} as a polynomial in $\lambda*$. For a general finite-dimensional $H \subset \mathrm{Exp}_\mathrm{T}$, one can take $M = \prod_{\theta\in\mathrm{T}} M_\theta^{n_\theta}$, for sufficiently high powers n_θ. The resulting extension of T^{-1} is an operator of large support (as compared to the support of φ or $\varphi*'$), especially when the spectrum of H is a large set (e.g., when H is spanned by pure exponentials), yet results in a very explicit and easily computable quasiinterpolant (provided that λ is easily computable, which is not always the case). For the choice $\lambda = \varphi_|$, this type of quasiinterpolation has been first suggested in [CD_1] for a polynomial H. It has also been used in [DR] (for an exponential H) in the derivation of the approximation order for exponential box splines, and more recently in [R_3].

(iii) **Recurrence** Equation (3.7) suggests the solution of the equation $\lambda{*}? = f \in e_\theta P$ by backsubstitution, i.e., by recurrence, since it implies that, for $f = e_\theta p \in e_\theta P$,

$$\lambda{*}f = \sum_{\gamma\ge 0} (e_\theta D^\gamma p)\, [\![-iD]\!]^\gamma \widehat{\lambda}(-i\theta),$$

therefore (using the invertibility of $\lambda*$ on $e_\theta P$ and the D-invariance of $e_\theta P$)

$$f = \sum_{\gamma\ge 0} (\lambda*)^{-1}(e_\theta D^\gamma p)\, [\![-iD]\!]^\gamma \widehat{\lambda}(-i\theta),$$

hence

$$(\lambda*)^{-1}(e_\theta p) = (\lambda*)^{-1} f = \left(f - \sum_{\gamma\ne 0} (\lambda*)^{-1}(e_\theta D^\gamma p)\, [\![-iD]\!]^\gamma \widehat{\lambda}(-i\theta) \right) / \widehat{\lambda}(-i\theta). \tag{3.13}$$

For a general exponential f, the resulting solution depends of course on the choice of $\lambda*$, but necessarily, since $\lambda{*}_{|H} = \varphi{*}'_{|H}$, this solution is independent of λ for $f \in H$ (as it is simply $(\varphi*')^{-1} f$).

When H is a polynomial space, it is sufficient to know how to solve the equation $\lambda{*}? = f$ for the normalized powers, and these solutions are provided by the **Appell polynomials** (p_α) for λ. By definition, p_α is characterized by the fact that $(\lambda{*}D^\beta p_\alpha)(0) = \delta_{\alpha,\beta}$. Since both differentiation and convolution map polynomials to polynomials and since differentiation commutes with convolution, we have the equivalent characterization of p_α as the unique polynomial solution of the convolution equation

$$\lambda{*}? = [\![]\!]^\alpha.$$

It follows that $D^\beta p_\alpha = p_{\alpha-\beta}$ and that, for any polynomial p, the solution of the equation $\lambda{*}? = p$ is given by

$$\sum_\alpha p_\alpha\ D^\alpha p(0),$$

thus reducing the problem of solving the equation $\lambda{*}? = p$ to solving it for the specific righthand sides $[\![]\!]^\alpha$. For these, the recurrence (3.13) reads

$$p_\alpha = \left([\![]\!]^\alpha - \sum_{\gamma\ne 0} p_{\alpha-\gamma}\ [\![-iD]\!]^\gamma \widehat{\lambda}(0) \right) / \widehat{\lambda}(0), \tag{3.14}$$

since, for $f = p = [\![]\!]^\alpha$, we have $D^\gamma p = [\![]\!]^{\alpha-\gamma}$, hence $(\lambda*)^{-1} D^\gamma p = p_{\alpha-\gamma}$. The derivatives at zero of $\widehat{\lambda}$ needed here can be calculated directly from λ because of the identity

$$[\![-iD]\!]^\gamma \widehat{\lambda}(0) = (\lambda{*}[\![]\!]^\gamma)(0).$$

4. Approximation order for piecewise-exponentials

In this section, we make use of the earlier discussion of quasiinterpolants to provide lower bounds for the approximation order of a family (S_h) of approximating spaces, each of which is spanned by the $h\mathbb{Z}^s$-translates of one locally supported function, i.e.,

$$S_h = S_h(\psi_h) := \operatorname{ran} \psi_h *_h'$$

for some ψ_h, with

$$\psi_h *_h' f := \sum_{\alpha \in \mathbb{Z}^s} \psi_h(\cdot - \alpha h) f(\alpha h).$$

The discussion is novel in that we allow the scaled function $\psi_h(h\cdot)$ to depend on h. We do assume, though, the existence of a fixed (h-independent and necessarily) D-invariant space H contained in each S_h and show that, under mild conditions on the ψ_h, the local approximation order of H is a lower bound for the approximation order of the family (S_h).

Concrete results of this nature were already obtained in [DR], using quasiinterpolant schemes based on the Neumann series approach outlined in the preceding section. By contrast, the approach presented here deviates but slightly from the standard quasiinterpolant argument, hence seems more direct.

In the spirit of the preceding section, the quasiinterpolant setup for approximation from $S_h = S_h(\psi_h)$ is as follows. A function ψ_h with bounded support is given and approximation maps of the form

$$Q := \psi_h *_h' \mu *$$

are sought. Although ψ_h may vary with h in quite an arbitrary way, the constants in the associated bounds depend essentially only on $h^{-1} \operatorname{diam} \operatorname{supp} \psi_h$, and hence it is desirable to assume that

$$\operatorname{supp} \psi_h \subseteq hrB,$$

with B the unit ball and r independent of h. (Otherwise, one needs to make assumptions on the rate at which ψ_h decreases away from the origin.) The (h-dependent) distribution μ is chosen so that Q reproduces a given space H and so that it is of small support; e.g.,

$$\operatorname{supp} \mu \subset hnB,$$

for some h-independent n. In addition, we find it convenient to restrict μ to be a continuous linear functional on $C(\mathbb{R}^s)$ or $L_\infty(\mathbb{R}^s)$, and, correspondingly, denote the norm of this linear functional (which in effect is a measure) by $\|\mu\|_1$. It then follows that Q is local in the sense that

$$Qf(x) = \sum_{\alpha \in \mathbb{Z}^s} \psi_h(x - \alpha h)(\mu * f)(\alpha h) \leq |Q| \, \|f_{|x+(r+n)hB}\|_\infty,$$

with

$$|Q| := \|\psi_h\|_h \|\mu\|_1, \quad \|\psi_h\|_h := \| \sum_{\alpha \in \mathbb{Z}^s} |\psi_h(\cdot - \alpha h)| \, \|_\infty.$$

This implies that, for any $p \in H$,

$$|(f - Qf)(x)| = |(f-p)(x) - Q(f-p)(x)| \le (1 + |Q|)\|(f-p)_{|x+(r+n)hB}\|_\infty,$$

hence allows the conclusion that

$$|(f - Qf)(x)| \le (1 + |Q|)\,\mathrm{dist}_{\infty,x+(r+n)hB}(f, H), \tag{4.1}$$

where, for $\Omega \subset \mathbb{R}^s$,

$$\mathrm{dist}_{\infty,\Omega}(f,H) = \inf_{g\in H} \|(f-g)_{|\Omega}\|_\infty.$$

This shows that the approximation order of Q, as a function of h, is bounded below by the local approximation order of H, provided the product $|Q| = \|\psi_h\|_h \|\mu\|_1$ can be bounded independently of h.

Here, the **local approximation order** (at the origin) of the space H of functions analytic at the origin is defined as the largest d for which there is, for each smooth f, some $p \in H$ for which

$$\|(f-p)_{|tB}\| = O(t^d) \quad \text{as } t \to 0$$

(with $\|\cdot\|$ the max norm, say). The local approximation order of H is characterized [BR$_1$] as the maximal d for which $\Pi_{<d} \subset T_d(H)$, with $T_d f$ being the Taylor expansion of order d (i.e., degree $d-1$) of f about the origin. The local approximation order of H at an arbitrary $x \in \mathbb{R}^s$ is defined analogously, and is independent of x in case H is translation-invariant. Furthermore, in this case $\mathrm{dist}_{\infty,x+tB}(f,H) \le c_f t^d$, with c_f independent of x, and finite whenever f and all its derivatives up to order d are bounded.

In order to derive conditions which provide h-independent bounds for $|Q|$, we normalize the situation by rescaling, i.e., by considering the quasiinterpolant $Q_h := \sigma_{1/h} Q \sigma_h$, as this leaves $|Q_h| = |Q|$ and unscales the scaled semi-discrete convolution. Here, σ_t is the scaling map $(\sigma_t f) : x \mapsto f(x/t)$. We compute that

$$Q_h := \sigma_{1/h} Q \sigma_h = \varphi_h *' \mu_h *, \tag{4.2}$$

with

$$\varphi_h := h^s \sigma_{1/h} \psi_h, \qquad \mu_h := \sigma_{1/h}\mu.$$

Since $\mathrm{diam\,supp}\,\varphi_h \le h^{-1}\,\mathrm{diam}\, hrB = r$, we see that

$$|Q| = |Q_h| = O(\|\varphi_h\|_\infty)\|\mu_h\|_1. \tag{4.3}$$

Thus the point is to show that, under certain conditions on ψ_h, the μ_h can be chosen with support bounded independently of h and so that $\|\varphi_h\|_\infty \|\mu_h\|_1 = O(1)$.

For this, recall that Q is supposed to reproduce a certain finite-dimensional space H. This is equivalent, by (4.2), to having Q_h reproduce the space

$$H_h := \sigma_{1/h} H.$$

If we further assume that H is a D-invariant exponential space, then so is H_h. Finally, if we assume that $H \subset S_h(\psi_h)$, then $H_h \subset S(\varphi_h)$ and, by Corollary 3.3, $\varphi_h*' = \varphi_h*$ on H_h, provided that φ_h is regular with respect to H_h, i.e.,

$$\widehat{\varphi}_h(\theta) \neq 0, \qquad \forall \theta \in -i \operatorname{spec} H_h = -ih \operatorname{spec} H. \tag{4.4}$$

In such a case, Q_h reproduces H_h if and only if μ_h* agrees on H_h with the well-defined

$$(\varphi_h*_{|H_h})^{-1}.$$

Since φ_h* commutes with translations, so does $(\varphi_h*_{|H_h})^{-1}$, thus it is a convolution operator, namely there exists a linear functional μ_h^0 on H_h such that $\mu_h^0* = (\varphi_h*_{|H_h})^{-1}$.

We wish to extend μ_h^0 to a suitably bounded linear functional μ_h on $C(\mathbb{R}^s)$ or $L_\infty(\mathbb{R}^s)$ and with support in an h-independent ball nB. This task is relatively simple if we assume that $H \subset \Pi_k$ for some k. Making the normalizing assumption that $\widehat{\varphi}_h(0) = 1$, we recall from (3.7) (with $\theta = 0$) that then $(1 - \varphi_h*)_{|\Pi}$ is degree-reducing and therefore $(1 - \varphi_h*)^{k+1} = 0$ on Π_k. This implies that $1 - \varphi_h*(q(\varphi_h*)) = 0$ on Π_k, with $q(t)$ the univariate polynomial $(1-(1-t)^{k+1})/t$ of degree k. Consequently, $(\varphi_h*_{|\Pi_k})^{-1} = q(\varphi_h*_{|\Pi_k})$. This suggests viewing φ_h* as a map from $C(mB)$ to $C((m-r)B)$, since in these norms (and for any $m > r$) $\|\varphi_h*\| \leq \text{const}\|\varphi_h\|_\infty$, and so that one can view $q(\varphi_h*)$ as map from $C((kr+1)B)$ to $C(B)$ which is bounded by $\text{const}_k\|\varphi_h\|_\infty^k$. Therefore, we obtain the bound

$$\|\tau_h\| \leq \text{const}_k \|\varphi_h\|_\infty^k, \tag{4.5}$$

with $\tau_h := (\varphi_h*_{|\Pi_k})^{-1}$ considered as a map from $\Pi_k \subset C((kr+1)B)$ to $\Pi_k \subset C(B)$, and where const_k depends on k and r. With this, we have

$$\mu_h^0 f = (\mu_h^0 * f(-\cdot))(0) = (\tau_h f(-\cdot))\,(0),$$

and thus,

$$\|\mu_h^0\| \leq \|\tau_h\| \leq \text{const}_k\|\varphi_h\|_\infty^k.$$

We can therefore obtain the functional μ_h in (4.3) with support in krB and with norm bounded by $\text{const}_k\|\varphi_h\|_\infty^k$. Substituting this into (4.3), we obtain for an unnormalized φ_h the inequality

$$|Q| \leq \text{const}_k(\|\varphi_h\|_\infty/|\widehat{\varphi}_h(0)|)^{k+1}, \tag{4.6}$$

where const_k may increase with r (in effect in $\operatorname{diam}\operatorname{supp}\varphi_h = h^{-1}\operatorname{diam}\operatorname{supp}\psi_h$), but does not otherwise depend on φ_h.

In order to extend this observation to a general H, we recall from [BR$_1$] that, as $h \to 0$, H_h converges to a certain polynomial space $H_\downarrow$, in the sense that any given basis $\{b_j\}_j$ for $H_\downarrow$ is associated with a corresponding basis $\{b_{j,h}\}_j$ for H_h so that, for each j, $b_{j,h}$ converges to b_j, as $h \to 0$, uniformly on compact sets. Choosing k so that $H_\downarrow \subset \Pi_k$, this means that, for sufficiently small h and for the const_k of (4.6), we have

$$\|(\varphi_h*_{|H_h})^{-1}\| \leq 2\text{const}_k(\|\varphi_h\|_\infty/|\widehat{\varphi}_h(0)|)^{k+1}, \tag{4.7}$$

where "sufficiently small" depends only on H and r (since it is related to the convergence of H_h to $H_\downarrow$ on $(kr+1)B$).

We arrived at the following

(4.8) Theorem. *Assume that ψ_h is a compactly supported function with $\widehat{\psi}_h(0) \neq 0$, and that H is a finite-dimensional D-invariant exponential space in $S_h(\psi_h) := \operatorname{ran}(\psi_h *'_h)$, for which $\psi_h *'_{h|H}$ is 1-1. Then there exists a continuous linear functional μ on $L_\infty(\mathbb{R}^s)$, with support in a ball of diameter $O(\operatorname{diam}\operatorname{supp}\psi_h)$ so that $Q := \psi_h *'_h \mu *$ reproduces H and so that, for h sufficiently small,*

$$|Q| := \| \sum_{\alpha\in\mathbb{Z}^s} |\psi_h(\cdot - \alpha h)| \, \|_\infty \|\mu\|_1 \leq \operatorname{const}(h^s \|\psi_h\|_\infty / |\widehat{\psi}_h(0)|)^m, \tag{4.9}$$

with the positive integer m dependent only on H, const *depending on H, increasing in $h^{-1}\operatorname{diam}\operatorname{supp}\psi_h$, but otherwise independent of ψ_h and h, and "sufficiently small" depending only on H and $h^{-1}\operatorname{diam}\operatorname{supp}\psi_h$. Consequently, as $h \to 0$, we have*

$$\|f - Qf\| = O(h^d), \tag{4.10}$$

provided that $\operatorname{diam}\operatorname{supp}\psi_h = O(h)$, *that $h^s\|\psi_h\|_\infty / \widehat{\psi}_h(0) = O(1)$, that H has local approximation order d, and that f has bounded continuous derivatives up to order d.*

Proof: We may assume, without loss, that $\widehat{\psi}_h(0) = 1$, hence also $\widehat{\varphi}_h(0) = 1$, with $\varphi_h := h^s \sigma_{1/h}\psi_h$. Then (4.9) is equivalent to

$$\| \sum_{\alpha\in\mathbb{Z}^s} |\varphi_h(\cdot - \alpha)| \|_\infty \|\mu_h\|_1 \leq \operatorname{const}\|\varphi_h\|_\infty^m. \tag{4.11}$$

This has been obtained by the previous arguments, with $m = k+1$.

The equation (4.10) follows directly from (4.9) and (4.1), provided that we confirm the regularity of φ_h (for small h), i.e., that

$$\widehat{\varphi}_h(\theta) \neq 0, \qquad \forall \theta \in -i \operatorname{spec} H_h = -ih \operatorname{spec} H.$$

Since by the assumption here $\operatorname{diam}\operatorname{supp}\varphi_h = O(1)$, this follows from the fact that, for any fixed θ, $e_{h\theta}$ converges uniformly to 1 on $\operatorname{supp}\varphi_h$ (as $h \to 0$), since it implies that for any fixed θ (and in the current normalization)

$$\widehat{\varphi}_h(h\theta) - 1 = \widehat{\varphi}_h(h\theta) - \widehat{\varphi}_h(0) = [(e_{-ih\theta} - 1)\varphi_h]\widehat{\ }(0) \to 0,$$

using the fact that, in this normalization, $\|\varphi_h\|_\infty = h^s\|\psi_h\|_\infty = O(1)$, by assumption. ♠

In reaction to a preprint of the present paper, [LJ] show that Theorem 4.8 is sharp, in the sense that the approximation order from a piecewise H-space cannot be better than the local approximation order from H.

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