# On Ptak's derivation of the Jordan normal form

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Some readers of [1] might appreciate the following comments that make more explicit how Ptak's beautiful insight there leads to a trivial proof of (the basics of) the Jordan normal form.

The proof of Theorem 1 of [1] can also be based on the observation that, $X$ being finite-dimensional, the sequence $\{0\} \subseteq \ker A \subseteq \ker A^2 \subseteq \cdots$, must eventually be stationary, i.e., $\ker A^q = \ker A^{q+p}$ for some $q$ and all $p > 0$. For such $q$, let $X_r$ and $X_s$ be the range and the kernel, respectively, of $A^q$, hence $\dim X = \dim X_r + \dim X_s$. Further, for any $x \in X_r \cap X_s$, $x = A^q z$ for some $z$, and so $z \in \ker A^{2q} = \ker A^q$, hence $x = 0$. Therefore, $X$ is the direct sum of the two $A$-invariant subspaces $X_s$ and $X_r$, and $A$ is regular on $X_r$ (since $A^q$ is) and is nilpotent on $X_s$.

In the setup and notation of Theorem 2 of [1], there must be, by duality, some $y_0$ in $Y$ for which $\langle x_0 A^{q-1}, y_0\rangle \neq 0$, hence the $q$-order matrix $(\langle x_0 A^{j-1}, y_0 A^{*q-i}\rangle : i, j = 1, \ldots, q)$ is triangular with nonzero diagonal entries, therefore invertible, and this guarantees that $X$ is the direct sum $X_0 + X'$, with $X_0$ the linear span of $(x_0 A^{j-1} : j = 1, \ldots, q)$ and $X'$ the annihilator of $\{y_0 A^{*q-i} : i = 1, \ldots, q\}$, both of which are $A$-invariant. Moreover, it shows $(x_0 A^{j-1} : j = 1, \ldots, q)$ to be a basis for $X_0$, and the matrix representation, with respect to this basis, of $A$ restricted to $X_0$ has the familiar form of a Jordan block (for the eigenvalue 0).

Now, $X$ being finite-dimensional, there are $A$-invariant direct sum decompositions $X = X_1 + \cdots + X_m$ that are minimal in the sense that none of its summands is the direct sum of two nontrivial $A$-invariant subspaces. Take any one such. Then the matrix representation for $A$ with respect to any basis made up from bases for the summands $X_i$ is block diagonal, with the $i$th block the matrix representation of the restriction $A_i$ of $A$ to $X_i$ with respect to the chosen basis for $X_i$.

Assuming the underlying field to be algebraically closed, the restriction $A_i$ of $A$ to $X_i$ has some eigenvalue, $\lambda_i$, and, in view of the minimality of $X_i$, Theorem 1 ensures that $B_i := A_i - \lambda_i$ is nilpotent, while Theorem 2 then ensures that, for some $x \in X_i$ and some $q$, $(xB_i^{j-1} : j = 1, \ldots, q)$ is a basis for $X_i$, and the matrix representation of $A_i$ with respect to that basis is a Jordan block with $\lambda_i$ as its diagonal element.

Theorems 1 and 2 of [1] don't seem to assist in the proof that the Jordan normal form is unique (up to reordering of the blocks), although such uniqueness is readily established by the observation that

$$n_j := \dim \ker(A - \lambda)^j = \sum_{\lambda_i = \lambda} \min(\dim X_i, j),$$

hence $\Delta n_j := n_{j+1} - n_j$ equals the number of blocks for $\lambda$ of order $> j$, giving the decomposition-independent number $-\Delta^2 n_{j-1}$ for the number of Jordan blocks for $\lambda$ of order $j$.

[1] V. Ptak, A remark on the Jordan normal form of matrices, Linear Algebra Appl. (this issue).