Radial Basis Functions: L^p-approximation orders with scattered centres

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Abstract. In this paper we generalize several results on uniform approximation orders with radial basis functions in (Buhmann, Dyn and Levin, 1993) and (Dyn and Ron, 1993) to L^p-approximation orders. These results apply, in particular, to approximants from spaces spanned by translates of radial basis functions by *scattered centres*. Examples to which our results apply include quasi-interpolation and least-squares approximation from radial function spaces.

1. Introduction

Radial basis function methods are tools for multivariable approximation where functions $f : \mathbb{R}^n \to \mathbb{R}$ are approximated from spaces spanned by translates of normally just one function $\varphi : \mathbb{R}^n \to \mathbb{R}$. The points by which φ is translated are usually called "centres". The function φ may or may not be a radially symmetric function, i.e., $\varphi = \tilde{\varphi}(\|\cdot\|_2)$, but for the most common and best studied examples it is, such as the ubiquitous multiquadric function, where $\tilde{\varphi} = \sqrt{(\cdot)^2 + c^2}$, c being a parameter, or the equally well-known thin plate spline, where $\tilde{\varphi} = (\cdot)^2 \log(\cdot)$. This is why these methods are still called radial basis function methods, although their performance depends much more on φ's smoothness and growth properties than on its spherical symmetry.

There are several recent reviews about the state-of-the-art in the investigation of radial basis function methods, but we mention here only the one due to the first author (1993). It is apparent from this survey that, in most articles about radial basis functions and their approximation properties so far, only translates along multiintegers have been studied in detail. Quasi-interpolation and least-squares approximation from the radial basis function spaces, and interpolation at the centres, have been investigated thoroughly for multiinteger centres, but, to the best of our knowledge, extensions of this

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work to infinitely many scattered centres have not been provided until recently, when the works of Buhmann, Dyn and Levin, and by Dyn and Ron (both 1993) were completed. In both papers, quasi-interpolation from radial basis function spaces with infinitely many scattered centres is analyzed. In the former paper, quasi-interpolants for sets of scattered centres that enjoy certain regularity conditions are explicitly constructed. It is shown that they provide the same approximation orders as the quasi-interpolants on gridded centres (which they agree with in the special case when the scattered centres happen to be gridded) if the centres become dense in the underlying $\mathbb{R}^n$. The class of radial basis functions for which this approach applies is restricted by various conditions, but none of the usually considered φs is excluded.

In (Dyn and Ron, 1993), a different track is proposed. Rather than constructing directly approximation schemes that employ scattered translates of φ, this paper suggests and analyzes a method for converting known "gridded centre approximation schemes" to the scattered case. This is done in such a way that the difference of the latter and the former is, at least, of the same order as the approximation order anticipated (when the gridded centres become dense in the underlying space). The approach to these results rests on the idea that the gridded centre approximant can itself be approximated by a function from the space spanned by scattered centre translates. For this, one replaces each multiinteger translate of the radial basis function, $\varphi(\cdot - j)$ for a $j \in \mathbb{Z}^n$ say, by an approximation, call it φ_j, from the space spanned by the scattered translates. The difference $\varphi(\cdot - j) - \varphi_j$ then has to satisfy a certain decay condition and there are other conditions on the gridded centre scheme. This work applies to the class of radial basis functions as defined in (Buhmann, Dyn and Levin, 1993). It is, in fact, more general because not just one quasi-interpolant for scattered centres is manufactured. Instead, a general conversion method is established that allows many approximation schemes to be converted from gridded to scattered centres.

All the error estimates in the literature mentioned so far are in $L^\infty(\mathbb{R}^n)$, i.e., they are uniform estimates. Error analyses of quasi-interpolation schemes in general $L^p(\mathbb{R}^n)$-norms are usually more subtle, and are less common than their $L^\infty(\mathbb{R}^n)$-counterparts. We mention here the work of Lei and Jia (1991), where $L^p(\mathbb{R}^n)$-approximation orders are established from spaces spanned by gridded translates of a compactly supported piecewise-exponential function; in this regard we also mention arguments employed in the book on box splines by de Boor, Höllig and Riemenschneider (1993) for a related theorem on L^p-approximation orders. Later, Jia and Lei (1993) offered approximation schemes and a complementary error analysis for spaces generated by the gridded translates of several non-compactly supported functions. This applies to quasi-interpolation with basis functions of global support, such as is required here for radial basis function approximants, in contrast to the L^p-approximation order result by (de Boor, Höllig and Riemenschneider, 1993) which only applies to compactly supported basis functions φ. On the other hand, the scheme proposed by Jia and Lei is more specific in nature.

In our work, we amalgamate these approaches and provide L^p-error estimates between the gridded and the scattered centre quasi-interpolants by basis functions of global support. Therefore, they apply to quasi-interpolation schemes using radial basis functions. We complement these results by simplified approaches to estimating the L^p-approximation orders of quasi-interpolants even for gridded centres. Examples that show the usefulness of our work are provided in the paper too.

2. L^p-approximation orders on uniform grids

We always assume that the given approximand is an $f \in W_p^k(\mathbb{R}^n)$, $p \in [1,\infty)$, without further mention, $W_p^k(\mathbb{R}^n)$ being the usual L^p-Sobolev space of order k of functions in n variables. We restrict attention to $p \in [1,\infty)$ because the results for $p = \infty$ are readily available, as pointed out in the introduction, and they are of the same nature as our L^p-results, but simpler in various respects. In fact, all our theorems remain true *verbatim* if every occurrence of p is replaced by ∞. The positive integer k is associated with the polynomial recovery of the semi-discrete convolution operator

$$\psi *' : f \mapsto \sum_j \psi(\cdot - j)\, f(j)$$

(all sums are over $\mathbb{Z}^n$ unless otherwise indicated) with a given basis function ψ, i.e., we assume that $\psi *'$ is the identity on Π_{k-1} which is the space of polynomials in n variables of total degree less than k. Such ψs that are linear combinations of radial basis functions exist in abundance, for instance $k = n+1$ is permissible for multiquadrics and $k = n+3$ for thin plate splines when n is odd or even respectively (Buhmann, 1989, for instance). In connection with the L^p-Sobolev spaces, we shall shortly require the following further notation. By $|\cdot|_{k,p}$ we shall denote the homogeneous kth order L^p-Sobolev semi-norm on $W_p^k(\mathbb{R}^n)$, i.e.,

$$|f|_{k,p} = \sum_{\substack{\alpha\in\mathbb{Z}_+^n \\ |\alpha|=k}} \|D^\alpha f\|_p,$$

where $D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$ is the vector of partial derivatives and $\|D^\alpha f\|_p$ is the L^p-norm of $D^\alpha f$. Moreover, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and, as we shall require in the sequel, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$, all for $\alpha \in \mathbb{Z}_+^n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Finally, $\|\cdot\|_{p,\Omega}$ and $|\cdot|_{k,p,\Omega}$ denote the L^p-norm and the homogeneous kth order L^p-Sobolev semi-norm, restricted to a prescribed subset $\Omega \subseteq \mathbb{R}^n$, respectively, and $\|\cdot\|_{p,\mathbb{Z}^n}$ stands for the $\ell^p(\mathbb{Z}^n)$ norm.

The approximation to f is provided in the form $\psi *' g$, for a suitably chosen g. In case f is smooth enough (say, $f \in W_p^k(\mathbb{R}^n) \cap C^k(\mathbb{R}^n)$), we may take $g = f$. Otherwise, a smoothing procedure is required, and g is defined as $g := \lambda * f$, where $\lambda \in C^k(\mathbb{R}^n)$ satisfies the two decay conditions

$$\|\|\cdot\|_\infty^k \lambda\|_1 < \infty \quad \text{and} \quad \sum_j \|\lambda\|_{\infty, j+\mathrm{U}} < \infty. \tag{2.1}$$

Here, $\mathrm{U} := (-\frac{1}{2}, \frac{1}{2})^n$. We also demand λ to satisfy the moment conditions

$$\int \lambda(x)x^\alpha \, dx = \delta_{0\alpha} \qquad \forall\ |\alpha| < k. \tag{2.2}$$

Any integrals in this work are over $\mathbb{R}^n$ unless noted otherwise. We note that functions λ that satisfy the above conditions do exist, and any such function will do here.

The approximation scheme $f \approx \psi *' (\lambda * f)$ can be refined by dilation, that is, with σ_h the scaling operator

$$\sigma_h : f \mapsto f(\cdot/h),$$

finer approximations to f are provided by

$$f \approx \sigma_h(\psi *' (\lambda * (\sigma_{1/h} f)).$$

Here and throughout, $*$ denotes convolution, either discrete or continuous, but never semi-discrete, as we use the $*'$ notation for the latter. The exact meaning of $*$ will be clear from the context. Note that this latter approximant is a linear combination of the $h\mathbb{Z}^n$-translates of $\sigma_h\psi$. One should note that in order to derive approximation orders, it suffices to establish a bound of the form

$$\|f - \psi *' (\lambda * f)\|_p \le C|f|_{k,p}, \tag{2.3}$$

(where C denotes here and everywhere in the paper a universal positive constant), since then it is straightforward to derive the estimate

$$\|f - \sigma_h(\psi *' (\lambda * (\sigma_{1/h} f)))\|_p \le Ch^k |f|_{k,p}, \qquad h \to 0. \tag{2.4}$$

Our principal result in this direction is as follows:

Theorem 1. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a function that satisfies the following conditions:*

(i) Decay condition: *for all $x \in \mathbb{R}^n$, it is true that*

$$|\psi(x)| \le C(1 + \|x\|_\infty)^{-\nu_\psi - k},$$

where k is a given positive integer, $\nu_\psi > n$ and C does not depend on x.

(ii) Polynomial reproduction: *$\psi *'$ is the identity on Π_{k-1}.*

Let $f \in W_p^k(\mathbb{R}^n)$, and let $g \in W_p^k(\mathbb{R}^n)$ satisfy the following three conditions:

(a) *g is k-times continuously differentiable.*

(b) *It is true that*

$$\left(\sum_j |g|^p_{k,\infty,j+\mathrm{U}}\right)^{\frac{1}{p}} \le C|f|_{k,p}. \tag{2.5}$$

(c) *It is true that*

$$\|f - g\|_p \le C|f|_{k,p}. \tag{2.6}$$

Then

$$\|f - \psi *' g\|_p \le C|f|_{k,p}. \tag{2.7}$$

Thus, according to Theorem 1, we choose $g = f$, as long as f satisfies the first two conditions (a)–(b) required of g in Theorem 1. Otherwise, the above detailed smoothing process is suitable, i.e., $g := \lambda * f$ satisfies the three required properties. Indeed, we note first that $g = \lambda * f \in C^k(\mathbb{R}^n) \cap W_p^k(\mathbb{R}^n)$ due to standard properties of the convolution (see, e.g., Stein and Weiss, 1971), so property (a) holds always. The fact that the two other required properties, i.e., (b) and (c), also hold for $g := \lambda * f$, is the content of the following two propositions.

Proposition 1. *Let $g := \lambda * f$, with λ satisfying the moment conditions* (2.2) *as well as the decay conditions* (2.1). *Then,*

$$\|f - g\|_p \le C|f|_{k,p}, \tag{2.8}$$

that is, g satisfies condition (c) *of* Theorem 1.

Proof. According to the integral remainder form of the multivariable Taylor theorem and according to (2.2),

$$(g-f)(x) = \int \lambda(u) \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = k}} \frac{(-u)^\gamma}{\gamma!} \int_0^1 (1-t)^{k-1} D^\gamma f(x - ut)\, dt\, du, \quad x \in \mathbb{R}^n. \tag{2.9}$$

Therefore, using the generalized Minkowski inequality, (2.9) implies that the left-hand side of (2.8) is at most a constant multiple of

$$\int \int_0^1 |\lambda(u)||u|^k (1-t)^{k-1}\, dt\, du\, |f|_{k,p}. \tag{2.10}$$

By virtue of (2.1), expression (2.10) is a constant multiple of $|f|_{k,p}$. □

The proof shows that the constant C in (2.8) may depend on λ and k, but is independent of p and f.

Proposition 2. *Let g be as in* Proposition 1. *Then* (2.5) *holds, and, further,*

$$|g|_{k,p} \le C|f|_{k,p}. \tag{2.11}$$

Proof. We use Hölder's inequality and obtain, for *any* $\widetilde{f} \in L^p(\mathbb{R}^n)$, for any $j, i \in \mathbb{Z}^n$, and for every $x \in j + \mathrm{U}$,

$$\left| \int_{i+\mathrm{U}} \lambda(u) \widetilde{f}(x-u)\, du \right| \le \|\widetilde{f}\|_{p, j-i+2\mathrm{U}} \|\lambda\|_{p', i+\mathrm{U}}, \tag{2.12}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Introducing the two sequences

$$a(i) := \|\lambda\|_{p', i+\mathrm{U}}, \quad b(i) := \|\widetilde{f}\|_{p, i+2\mathrm{U}}, \qquad i \in \mathbb{Z}^n,$$

and summing (2.12) over all integer i, we obtain the bound

$$\|\lambda * \widetilde{f}\|_{\infty,j+\mathrm{U}} \le \big(a * b\big)(j), \qquad j \in \mathbb{Z}^n.$$

Therefore, by Young's inequality (cf., e.g., Stein and Weiss, 1971, p. 178),

$$\Big(\sum_j \|\lambda * \widetilde{f}\|^p_{\infty,j+\mathrm{U}}\Big)^{\frac{1}{p}} \le \|a\|_{1,\mathbb{Z}^n} \|b\|_{p,\mathbb{Z}^n}.$$

Because of the second assumption in (2.1), $\|a\|_{1,\mathbb{Z}^n} < \infty$. Also, $\|b\|_{p,\mathbb{Z}^n} \le 2^{n/p}\|\widetilde{f}\|_p$, and we thus have proved that

$$\Big(\sum_j \|\lambda * \widetilde{f}\|^p_{\infty,j+\mathrm{U}}\Big)^{\frac{1}{p}} \le C\|\widetilde{f}\|_p.$$

Substituting $\widetilde{f} := D^\gamma f$, for all possible γ with $|\gamma| = k$, we obtain the desired claim. The second inequality (2.11) is an easy consequence of Young's inequality and the fact that, as a result of (2.1), $\lambda \in L^1(\mathbb{R}^n)$. □

Propositions 1 and 2 assure us of the existence of a suitable g for the sake of constructing the approximation scheme $f \approx \psi *' g$. We can, therefore, turn our attention to the proof of Theorem 1, as, indeed, we do now.

The condition (c) on g (in Theorem 1) allows us to bound the error by

$$\|f - \psi *' g\|_p \le \|f - g\|_p + \|g - \psi *' g\|_p \le C|f|_{k,p} + \|g - \psi *' g\|_p, \quad (2.13)$$

and therefore leads to the desired estimate (2.7), as soon as one has an estimate of the form

$$\|g - \psi *' g\|_p \le C|f|_{k,p} \quad (2.14)$$

in hand. One can in fact prove more generally for *any* $g \in C^k(\mathbb{R}^n) \cap W^k_p(\mathbb{R}^n)$, and $p \in [1, \infty)$, that

$$\|g - \psi *' g\|_p \le C|g|_{k,p} \quad (2.15)$$

so long as g satisfies

$$\Big(\sum_j |g|^p_{k,\infty,j+\mathrm{U}}\Big)^{\frac{1}{p}} \le C|g|_{k,p}. \quad (2.16)$$

The arguments below that lead to (2.16) also support the derivation of (2.14) from our condition (2.5): one simply needs to replace in the proofs all applications of (2.16) by (2.5).

We first establish our approximation order result for the (relatively simple case) when ψ is of compact support. Thereafter we will provide a proof for Theorem 1 in its full generality.

Theorem 2. *The conclusions of* Theorem 1 *hold in case ψ is bounded and of compact support. More generally, if $g \in C^k(\mathbb{R}^n) \cap W_p^k(\mathbb{R}^n)$, and ψ is bounded and of compact support and satisfies* (ii), *then* (2.16) *implies* (2.15).

Proof. Let $x \in \mathbb{R}^n$ be arbitrary. If T denotes the Taylor polynomial to $g \in C^k(\mathbb{R}^n)$ about x of degree $k-1$, we get (since $\psi *' T = T$)

$$|(\psi *' g - g)(x)| \le \sum_j |\psi(x-j)||g(j) - T(j)|.$$

Let m be a positive integer that satisfies $\operatorname{supp}\psi \subset (2m-1)\mathrm{U}$. Therefore

$$\|\psi *' g - g\|_p \le \left[\int \left(\sum_{j \in Z_x} |\psi(x-j)||g(j) - T(j)| \right)^p dx \right]^{\frac{1}{p}}, \tag{2.17}$$

where $Z_x := \{ j \in \mathbb{Z}^n \mid \|x - j\|_\infty < \frac{1}{2}(2m-1) \}$. By the integral form of the Taylor remainder, the right-hand side of (2.17) is at most $\|\psi\|_\infty$-times

$$\begin{aligned}
&\left[\int \left(\sum_{j \in Z_x} |g(j) - T(j)| \right)^p dx \right]^{\frac{1}{p}} \\
&= \left[\int \left(\sum_{j \in Z_x} \Big| \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = k}} \frac{(j-x)^\gamma}{\gamma!} \int_0^1 (1-t)^{k-1} D^\gamma g((1-t)x + tj)\, dt \Big| \right)^p dx \right]^{\frac{1}{p}} \\
&\le \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = k}} \frac{m^k}{\gamma!} \left[\int \left(\int_0^1 \sum_{j \in Z_x} (1-t)^{k-1} |D^\gamma g((1-t)x + tj)|\, dt \right)^p dx \right]^{\frac{1}{p}} \\
&\le \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma| = k}} \frac{m^k}{\gamma!} \int_0^1 (1-t)^{k-1} \Big\| \sum_{j \in Z_{(\cdot)}} |D^\gamma g((1-t)(\cdot) + tj)| \Big\|_p dt,
\end{aligned}$$

where we have used the Minkowski inequality and the generalized Minkowski inequality in the penultimate and in the last inequality, respectively.

We estimate the above expression term-by-term in γ and for each $t \in (0,1)$ separately. Hence we consider

$$\Big\| \sum_{j \in Z_{(\cdot)}} |D^\gamma g((1-t)(\cdot) + tj)| \Big\|_p, \tag{2.18}$$

leaving out the factor of $(1-t)^{k-1}$. Note that $Z_x = Z_\ell$ if $x \in \mathrm{U} + \ell$. Hence we have for each $t \in (0,1)$ that (2.18) is at most the p-th root of

$$\begin{aligned}
&\int \Big(\sum_{j\in Z_x} |D^\gamma g(tj+(1-t)x)| \Big)^p dx \\
&= \sum_\ell \int_{\mathrm{U}+\ell} \Big(\sum_{j\in Z_\ell} |D^\gamma g(tj+(1-t)x)| \Big)^p dx \\
&= \sum_\ell \int_{\mathrm{U}+\ell} \Big(\sum_{j\in Z_\ell} |D^\gamma g(j+(1-t)(x-j))| \Big)^p dx \\
&\le \sum_\ell (2m)^{np} \|D^\gamma g\|^p_{\infty, Z_\ell+2m\mathrm{U}} \\
&\le C \sum_\ell |g|^p_{k,\infty,\ell+\mathrm{U}},
\end{aligned}$$

the last inequality being a simple consequence of the fact that $\{\ell+\mathrm{U}\}_{\ell\in\mathbb{Z}^n}$ form a tiling of $\mathbb{R}^n$.

Now, if assumption (2.16) on g holds, we obtain that the last expression is bounded by a constant multiple of $|g|^p_{k,p}$. We then take the p-th root to obtain (2.15). If, instead, we apply condition (b), i.e., replace the usage of (2.16) by using (2.5), we arrive directly at the desired result (2.7). □

We remark that the various constants that appear in the proof of Theorem 2 depend on k, m, $\|\psi\|_\infty$, n and p. These constants, however, do not rely on f.

Proof of Theorem 1. Before we begin the proof of this result, we point out once again that there are many ψs that satisfy the assumptions of this theorem which are linear combinations of radial basis functions (see, for instance, Buhmann, 1989, and see in particular our example at the end of this note).

To embark on the proof, we first follow the argument in the proof of Theorem 1, until (2.18) is reached. Here, due to the global support of ψ, there is no need to introduce the set Z_x. Instead, we use the decay assumption (i) on ψ, and obtain the bound

$$\begin{aligned}
&\|\psi *' g - g\|_p \le \Big[\int \Big(\sum_j |\psi(x-j)||g(j)-T(j)| \Big)^p dx \Big]^{\frac{1}{p}} \\
&\le C \sum_{\substack{\gamma\in\mathbb{Z}^n_+ \\ |\gamma|=k}} \tfrac{1}{\gamma!} \int_0^1 (1-t)^{k-1} \Big\| \sum_j (1+\|\cdot-j\|_\infty)^{-\nu_\psi} |D^\gamma g(tj+(1-t)(\cdot))| \Big\|_p dt,
\end{aligned}$$

where we have used Minkowski's inequality and its generalization as in the previous proof. We observe that the integrand of this expression is for each $t \in (0,1)$ less than or equal to

$$\Big(\sum_i \Big\| \sum_j (1+\|\cdot-j\|_\infty)^{-\nu_\psi} |D^\gamma g(tj+(1-t)(\cdot))| \Big\|^p_{p,\mathrm{U}+i} \Big)^{\frac{1}{p}}. \tag{2.19}$$

Expression (2.19) equals

$$\begin{aligned}
&\Bigl(\sum_i \Bigl\|\sum_j (1+\|\cdot-j\|_\infty)^{-\nu_\psi}|D^\gamma g(i+tj+(1-t)(\cdot))|\Bigr\|_{p,\mathrm{U}}^p\Bigr)^{\frac{1}{p}} \\
&\le \Bigl(\sum_i \Bigl\|\sum_j (\tfrac{1}{2}+\|j\|_\infty)^{-\nu_\psi}|D^\gamma g(i+tj+(1-t)(\cdot))|\Bigr\|_{p,\mathrm{U}}^p\Bigr)^{\frac{1}{p}} \qquad (2.20)\\
&\le \Bigl(\sum_i \Bigl(\sum_j (\tfrac{1}{2}+\|j\|_\infty)^{-\nu_\psi}\bigl\|D^\gamma g(i+tj+\cdot)\bigr\|_{\infty,\mathrm{U}}\Bigr)^p\Bigr)^{\frac{1}{p}},
\end{aligned}$$

where the last estimate depends on the triangle inequality for the uniform norm and on $t\in(0,1)$.

For $t=1$, the last expression can be identified as $\|B*G\|_{p,\mathbb{Z}^n}$, with $B(j):=(\frac{1}{2}+\|j\|_\infty)^{-\nu_\psi}$, $j\in\mathbb{Z}^n$, and $G(j):=\bigl\|D^\gamma g(j+\cdot)\bigr\|_{\infty,\mathrm{U}}$, $j\in\mathbb{Z}^n$. For that case, Young's inequality can be invoked to yield

$$\|B*G\|_{p,\mathbb{Z}^n}\le\|B\|_{1,\mathbb{Z}^n}\|G\|_{p,\mathbb{Z}^n}.$$

While for $t<1$ the expression in question is *not* a convolution product, it can still be majorised by an appropriate convolution, as the following argument shows. For each $t\in(0,1)$ fixed, we define the map $N(j):=\lfloor(1-t)j\rfloor$, $j\in\mathbb{Z}^n$, with $\lfloor\cdot\rfloor$ denoting (componentwise) the greatest integer less than $\bigl(\cdot+\frac{1}{2}\bigr)$. We can then write the above sequence B as the sum $\sum_\ell B_\ell$ of sequences of disjoint supports defined as follows:

$$B_\ell(j):=\begin{cases} B(j), & \ell=N(j),\\ 0, & \text{otherwise,}\end{cases}\qquad j,\ell\in\mathbb{Z}^n.$$

By using the bound

$$\bigl\|D^\gamma g(i+tj+\cdot)\bigr\|_{\infty,\mathrm{U}}\le\bigl\|D^\gamma g(i+j-N(j)+\cdot)\bigr\|_{\infty,2\mathrm{U}},$$

defining $\widetilde{G}(j):=\bigl\|D^\gamma g(j+\cdot)\bigr\|_{\infty,2\mathrm{U}}$, $j\in\mathbb{Z}^n$, and using the nonnegativity of the entries of the sequences B and $\widetilde{G}$, we can bound the last expression in the chain of inequalities (2.20) from above by

$$\Bigl\|\sum_\ell B_\ell*E^\ell\widetilde{G}\Bigr\|_{p,\mathbb{Z}^n}=\Bigl\|\Bigl(\sum_\ell E^\ell B_\ell\Bigr)*\widetilde{G}\Bigr\|_{p,\mathbb{Z}^n},$$

where $E^\ell: f\mapsto f(\cdot-\ell)$. Therefore, Young's inequality applies once again to yield the upper bound

$$\Bigl\|\sum_\ell E^\ell B_\ell\Bigr\|_{1,\mathbb{Z}^n}\|\widetilde{G}\|_{p,\mathbb{Z}^n}=\|B\|_{1,\mathbb{Z}^n}\|\widetilde{G}\|_{p,\mathbb{Z}^n}= \sum_j(\tfrac{1}{2}+\|j\|_\infty)^{-\nu_\psi}\Bigl(\sum_j\widetilde{G}(j)^p\Bigr)^{\frac{1}{p}}.$$

Finally, the sum $\left(\sum_j \widetilde{G}(j)^p\right)^{1/p}$ can certainly be bounded by a constant multiple of $\left(\sum_j G(j)^p\right)^{1/p}$, and this latter expression is bounded by $C|f|_{k,p}$, by our assumption (b) (or by $C|g|_{k,p}$, if, instead of assuming (b), we adopt (2.16)). The theorem is proved. □

If we apply Propositions 1 and 2, we arrive at the following

Corollary 1. *If $f \in W_p^k(\mathbb{R}^n)$, if the assumptions* (i)–(ii) *of* Theorem 1 *hold and if $\lambda \in C^k(\mathbb{R}^n)$ satisfies* (2.2) *and* (2.1), *then* (2.3), *and thus* (2.4) *too, hold. Furthermore, if f is k-times continuously differentiable and satisfies*

$$\left(\sum_j |f|^p_{k,\infty,j+\mathrm{U}}\right)^{\frac{1}{p}} \leq C|f|_{k,p},$$

then

$$\|f - \sigma_h(\psi *' \sigma_{1/h} f)\|_p \leq Ch^k |f|_{k,p}$$

for h tending to zero. □

The second assertion in the above theorem says in particular that for sufficiently smooth f one needs to know only the values $f_{|h\mathbb{Z}^n}$ in order to compute its L^p-approximant. This is especially important from a practical point of view, since computing the values on $h\mathbb{Z}^n$ of $\lambda * f$ can be hard.

Our final note in this section concerns the conditions imposed on ψ in Theorem 1. The decay condition (viz., (i)) is essential to the approximation orders derived. Though approximation orders can be derived under weaker decay conditions, such derivations either involve some complementary information of the exact decay rates of ψ (as in Buhmann, 1989, or in Dyn, Jackson, Levin and Ron, 1992), or require a substantial modification of the approximation schemes as well as their error analyses (as is the case in (de Boor and Ron, 1992a) and (de Boor, DeVore and Ron, 1993)). However, the second condition, i.e., that $\psi *'$ be the identity on Π_{k-1} can be relaxed to (the more standard one) that $\psi *'$ induces a linear isomorphism on Π_{k-1}. An appropriate approximation scheme in this case has the form

$$f \approx \psi *' (\rho * \lambda * f),$$

with λ, as before, satisfying (2.2) and (2.1), and ρ, say, compactly supported and satisfying

$$\psi *' (\rho * p) = p, \quad p \in \Pi_{k-1}.$$

Such constructions are discussed in detail in the literature, and we refer the reader to de Boor and Ron (1992b) where such schemes were thoroughly analysed for *compactly supported* ψ. The modifications required in our arguments in order to cover that more general setup are mostly straightforward. Moreover, one should notice that, if ρ is selected to have its support lying in $\mathbb{Z}^n$, then $\psi *' (\rho * \cdot) = (\psi * \rho) *'$, and, for the new function $\Psi := \psi * \rho$, $\Psi *'$ is the identity on Π_{k-1}. Therefore, our results apply directly to this important case.

3. L^p-approximation orders using scattered centres

We use the results of the previous section in the derivation of approximation schemes that employ scattered translates of the basis function φ. The approach as well as the error analysis follow the work of (Dyn and Ron, 1993), where L^∞-approximation orders were obtained.

We let ψ be a linear combination

$$\psi = \varphi *' \mu, \tag{3.1}$$

where φ is a given function in $C(\mathbb{R}^n)$ such as one of our radial basis functions and where $\mu : \mathbb{Z}^n \to \mathbb{R}$ is such that the infinite sum in (3.1) converges uniformly on compacta. We shall as before consider the approximation operator $f \mapsto Lg := \psi *' g$, where $g = \lambda * f$ with $\lambda \in C^k(\mathbb{R}^n)$ satisfying (2.2) and (2.1). We make assumptions according to those in (Dyn and Ron, 1993): we always assume from now on that the thus constructed $L : C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \to C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a bounded operator without further mention. As in the previous section, we use here *stationary schemes*, i.e., define the h-refinement L_h by dilation:

$$L_h := \sigma_h L \sigma_{1/h}.$$

We shall further assume that

$$\left.\begin{array}{ll} |\mu(\ell)| \le C(1+\|\ell\|_\infty)^{-\nu_\mu}, \quad \ell \in \mathbb{Z}^n, & \\ |\psi(x)| \le C(1+\|x\|_\infty)^{-\nu_\psi}, \quad x \in \mathbb{R}^n, & \nu_\mu, \nu_\psi > n. \end{array}\right\} \tag{3.2}$$

We take, for a given set of (scattered) centres $\Xi \subset \mathbb{R}^n$, approximations to each translate $\varphi(\cdot - j)$ of the form

$$\varphi_j := \sum_{\xi \in \Xi} A_{j,\xi} \varphi(\cdot - \xi),$$

where $\{A_{j,\xi}\}_{j \in \mathbb{Z}^n, \xi \in \Xi}$ are suitable coefficients which provide that

$$|\varphi_j(x) - \varphi(x-j)| \le C(1+\|x-j\|_\infty)^{-\nu_A} \left\{ \begin{array}{l} x \in \mathbb{R}^n, \\ j \in \mathbb{Z}^n, \end{array}\right. \tag{3.3}$$

holds for a $\nu_A > n$ and a positive constant C which is uniform in x and j. We call the functions $\{\varphi_j\}_j$ *pseudo-shifts* (as they are approximants to the "true" shifts $\{\varphi(\cdot - j)\}_j$ from our space), and use these pseudo-shifts to construct another set of approximations, now to the localized functions $\{\psi(\cdot - j)\}_j$. We carry out this second task by imitating the localization process of (3.1). Precisely, we define

$$\psi_j := \sum_\ell \varphi_\ell \, \mu(\ell - j). \tag{3.4}$$

Finally, the approximation scheme L_A, that employs the scattered translates $\{\varphi(\cdot - \xi)\}_{\xi\in\Xi}$, is obtained from L by replacing each $\psi(\cdot - j)$ by ψ_j:

$$L_A : f \mapsto \sum_j f(j)\psi_j.$$

As before, there are functions f for which convolution with λ need be performed:

$$f \approx L_A(\lambda * f).$$

The h-refinement here is the same as before for L, that is

$$L_{h,A} := \sigma_h L_A \sigma_{1/h}.$$

One verifies that $L_{h,A}$ maps into the span of the $h\Xi$-translates of φ. We remark that other, more flexible refinements may be allowed, too, and refer the interested reader to the discussion of that point in §2.6 of (Dyn and Ron, 1993) and to the specific strategy suggested for the purpose of treating sets of scattered centres becoming dense in the underlying space in the paper (Buhmann, Dyn and Levin, 1993).

Theorem 3. *Let φ, μ, ψ, ψ_j, L_h and $L_{h,A}$ be as above. Assume that* (3.2) *and* (3.3) *hold. Suppose also that*

$$\nu_\mu \geq n + k + \varepsilon - 1 \quad \text{for an } \varepsilon \in (0,1] \tag{3.5}$$

and

$$\mu : f \mapsto \sum_\ell \mu(-\ell) f(\ell) \quad \text{satisfies} \quad \mu\Pi_{k-1} = \{0\}. \tag{3.6}$$

Then, for $h \in (0,1)$ and $f \in W_p^k(\mathbb{R}^n)$,

$$\|(L_h - L_{h,A})(\lambda * f)\|_p \leq Ch^k(|f|_{k-1,p} + |f|_{k,p}) \begin{cases} h^{\varepsilon-1} & \text{if } \varepsilon < 1, \\ -\log h & \text{if } \varepsilon = 1. \end{cases} \tag{3.7}$$

Moreover, if $\varepsilon = 1$ and $\sum_j |\mu(j)||j|^k < \infty$, then

$$\|(L_h - L_{h,A})(\lambda * f)\|_p \leq Ch^k|f|_{k,p}. \tag{3.8}$$

Implicit in the statement of theorem is the assertion that $L_A(\lambda * f)$ is well-defined, i.e., that the series $L_A(\lambda * f)$ converges absolutely for every, say, bounded f. This well-definedness has been established in Dyn and Ron (1993); cf. Lemma 2.2.6 in that article.

Proof. We begin with establishing (3.8). We may consider L and L_A instead of L_h and $L_{h,A}$, because, as we have pointed out earlier, the powers of h are introduced by a suitable scaling. Set $g := \lambda * f$. Since the series that define L and L_A converge absolutely, we may sum $(L - L_A)g$ by parts to obtain

$$\|(L - L_A)g\|_p \leq \Big\| \sum_j |\varphi(\cdot - j) - \varphi_j| \Big| \sum_\ell \mu(-\ell) g(j+\ell) \Big| \Big\|_p.$$

Since μ annihilates Π_{k-1} by (3.6), we have, for each $j \in \mathbb{Z}^n$,

$$\sum_{\ell} \mu(-\ell) g(j+\ell) = \sum_{\ell} \mu(-\ell)(g(j+\ell) - T_j(j+\ell)),$$

with T_j the $(k-1)$-degree Taylor polynomial of g about j. Using, as in the previous section, the integral remainder formula for the Taylor expansion, followed by the triangle inequality and summation by parts, we obtain the following upper bound on $\|(L-L_A)g\|_p$:

$$\Big\| \sum_{\ell} \sum_{\substack{\gamma \in \mathbb{Z}_+^n \\ |\gamma|=k}} \int_0^1 t^{k-1} \Big|\mu(-\ell)\frac{\ell^\gamma}{\gamma!}\Big| \sum_j |\varphi(\cdot - j) - \varphi_j| \Big| D^\gamma g(j+(1-t)\ell)\Big|\, dt \Big\|_p .$$

Using the Minkowski inequality we can take the summations over ℓ and γ out of the p-norm expression. Furthermore, using the generalized Minkowski inequality we can also interchange integration over t and the L^p-norm. This leaves us with the task of estimating the expression

$$\Big\| \sum_j |\varphi(\cdot - j) - \varphi_j| \Big| D^\gamma g(j+(1-t)\ell)\Big| \Big\|_p . \tag{3.9}$$

Because of (3.3), we further can bound expression (3.9) by a constant multiple of

$$\Big\| \sum_j (1+\|\cdot - j\|_\infty)^{-\nu_A} |D^\gamma g(j+(1-t)\ell)| \Big\|_p . \tag{3.10}$$

The estimation of expression (3.10) follows a similar estimation done in the proof of Theorem 1, only the present case is in fact slightly simpler. Indeed,

$$\begin{aligned}
&\Big\| \sum_j (1+\|\cdot - j\|_\infty)^{-\nu_A} |D^\gamma g(j+(1-t)\ell)| \Big\|_p^p \\
&= \sum_i \Big\| \sum_j (1+\|\cdot - j\|_\infty)^{-\nu_A} |D^\gamma g(j+(1-t)\ell)| \Big\|_{p,\mathrm{U}+i}^p \\
&\le \sum_i \Big(\sum_j (\tfrac{1}{2}+\|j\|_\infty)^{-\nu_A} |D^\gamma g(i-j+(1-t)\ell)| \Big)^p \\
&\le C \sum_i |D^\gamma g(i+(1-t)\ell)|^p \\
&\le C \sum_i |g|_{k,\infty,\mathrm{U}+i}^p \\
&\le C |f|_{k,p}^p,
\end{aligned}$$

where we have used the Young inequality in the second estimate and Proposition 2 in the last. Now we may use the fact that $\sum_\ell |\mu(-\ell)||\ell|^k$ is finite and take p-th roots to deduce the estimate (3.8) from the above chain of inequalities.

The estimate (3.7) is proved similarly, but we can no longer rely on the convergence of $\sum_\ell |\mu(-\ell)||\frac{\ell^\gamma}{\gamma!}|$, and need to split the sum over ℓ into two partial sums: $\sum_{\ell\in\mathbb{Z}^n\cap h^{-1}\mathrm{U}}$ and the remaining sum. The first sum is estimated exactly as above, and the bound obtained then takes the form

$$C|f|_{k,p} \sum_{\ell\in\mathbb{Z}^n\cap h^{-1}\mathrm{U}} |\mu(-\ell)|\Big|\frac{\ell^\gamma}{\gamma!}\Big|$$

for each γ with $|\gamma| = k$. In the remaining sum, the zero at j of $g(j+\ell) - T_j(j+\ell)$, though being of order k, is treated as being of order $k-1$, thus leading to a Taylor remainder formula of the form

$$\sum_{\substack{\gamma\in\mathbb{Z}^n_+\\|\gamma|=k-1}} \int_0^1 t^{k-2}\Big|\mu(-\ell)\frac{\ell^\gamma}{\gamma!}\Big|\sum_j |\varphi(\cdot-j)-\varphi_j|\Big|D^\gamma(g-T_j)(j+(1-t)\ell)\Big|\,dt.$$

Since $D^\gamma T_j = D^\gamma g(j)$ for all admissible γ, we can follow the argument that was used to prove (3.8), to obtain a bound of the form

$$C|f|_{k-1,p} \sum_{\ell\in\mathbb{Z}^n,\,\ell\notin h^{-1}\mathrm{U}} |\mu(-\ell)|\Big|\frac{\ell^\gamma}{\gamma!}\Big|$$

for each γ with $|\gamma| = k-1$. (In the derivation of this latter estimate, one invokes Proposition 2, with k there replaced by $k-1$.) Now, we can use the fact, that, whenever the required decay estimate on μ holds for $\varepsilon\in(0,1)$,

$$\sum_{\ell\in\mathbb{Z}^n\cap h^{-1}\mathrm{U}} |\mu(-\ell)|\Big|\frac{\ell^\gamma}{\gamma!}\Big| \le Ch^{\varepsilon-1}, \tag{3.11}$$

$$\sum_{\ell\in\mathbb{Z}^n,\,\ell\notin h^{-1}\mathrm{U}} |\mu(-\ell)|\Big|\frac{\ell^\gamma}{\gamma!}\Big| \le Ch^{\varepsilon}, \tag{3.12}$$

whereas if $\varepsilon = 1$, the right-hand side of (3.11) has to be replaced by $-C\log h$. All these estimates were proved for instance in (Buhmann, Dyn and Levin, 1993), and are also well-documented in Dyn and Ron (1993). In summary, the total error bound for, say, $\varepsilon < 1$ takes the form

$$\|(L-L_A)(\lambda * f)\|_p \le C(h^{\varepsilon-1}|f|_{k,p} + h^\varepsilon|f|_{k-1,p}). \tag{3.13}$$

It is straightforward to insert the $\log h$-term into this expression when $\varepsilon = 1$. When finally estimating $(L_h - L_{h,A})(\lambda * f)$, we simply invoke (3.13) with f replaced by $\sigma_{1/h}f$ for the given value of h, and then scale to obtain (3.7). □

In analogy to Theorem 2.3.1 in (Dyn and Ron, 1993) (where the case $p=\infty$ was discussed) we establish the following result. Its purpose is to relax the decay conditions on μ in the previous result which requires μ to decay fast enough so as to be summable against polynomials. It is now replaced by a condition on the Fourier series of μ, a condition that may be valid even when μ decays in a milder fashion. Incidentally, we will see that the rates derived in that result may be superior to those of Theorem 3.

In order to present the next theorem, we have to introduce some additional notation. Given $1 \le p < \infty$, let $\widetilde{W}_p^k(\mathbb{R}^n)$ be the space of all functions f whose Fourier transform $\widehat{f}$ satisfies $(1+\|\cdot\|_2)^k \widehat{f} \in L^{p'}(\mathbb{R}^n)$, where $\frac{1}{p}+\frac{1}{p'}=1$. The $L^{p'}$-norm of $\|\cdot\|_2^k \widehat{f}$ is denoted by $|f|'_{k,p}$. Note that, for a nonnegative integer k and $p \ge 2$, the Hausdorff–Young Theorem shows that the space $\widetilde{W}_p^k(\mathbb{R}^n)$ equipped with the norm $\|f\|'_{k,p} := \|(1+\|\cdot\|_2)^k \widehat{f}\|_{p'}$ is continuously embedded into $W_p^k(\mathbb{R}^n)$. Further, for $p=2$ and any nonnegative integer k, the two spaces coincide. We have the following

Theorem 4. *Assume* $1 \le p < \infty$. *Let all the assumptions of the previous theorem be in force (including in particular* (3.2)*) except for* (3.5) *and* (3.6). *Let* $f \in \widetilde{W}_p^k(\mathbb{R}^n)$ *be bounded and assume that the inequality*

$$|\mu f| \le C|f|'_{k,p} \tag{3.14}$$

holds. Then the estimate

$$\|(L-L_A)f\|_p \le C|f|'_{k,p} \tag{3.15}$$

holds as well. The estimate (3.15) *remains valid even if we replace* f *on its left-hand side by* $(\lambda * f)$, *provided that, in addition to all other requirements on* λ, $\widehat{\lambda} \in L^\infty(\mathbb{R}^n)$.

Note that the dash-semi-norm $|f|'_{k,p}$ reacts to scaling in the correct manner: upon scaling of approximand and approximant suitably by h, one does get in (3.15) the power h^k into the bound, as required. Note also that the smoothing process $f \approx \lambda * f$ is permitted but not required in Theorem 4. It may, however, actually be required for guaranteeing that the gridded centre approximant to the function f approximates it to the best order established in Theorem 1.

We will discuss the nature of condition (3.14) after proving this theorem. In particular, we will compare that condition to the polynomial annihilation condition (3.6) assumed in Theorem 3.

Proof of Theorem 4. The proof follows the argument used in Theorem 3, but is much simpler. In Theorem 3, we bounded the error by

$$\sum_j |\varphi(\cdot - j) - \varphi_j| \Big| \sum_\ell \mu(-\ell) f(j+\ell) \Big|. \tag{3.16}$$

Since no smoothing is required, we replaced g by f in (3.16). We bound the error in the same fashion in the present proof, and remark that, since f is bounded and $\mu \in \ell_1(\mathbb{Z}^n)$ by (3.2), the sum $\mu(f(\cdot + j)) = \sum_\ell \mu(-\ell) f(j+\ell)$

converges absolutely. By our assumption (3.14) in the theorem and the fact that the semi-norm $|\cdot|'_{k,p}$ is translation-invariant, we have $|\mu(f(\cdot+j))| \le C|f|'_{k,p}$. We thus obtain the bound

$$C|f|'_{k,p}\sum_j |\varphi(\cdot-j)-\varphi_j|.$$

The desired result is now a consequence of the assumption (3.3). The proof so far also provides that

$$\|(L-L_A)(\lambda * f)\|_p \le C|\lambda * f|'_{k,p} \le C\|\widehat{\lambda}\|_\infty |f|'_{k,p}, \tag{3.17}$$

thus showing that the difference $(L-L_A)(\lambda * f)$ is bounded to the asserted order too. The first inequality in (3.17) is a consequence of (3.15) and the second one is a consequence of the definition of our dash-semi-norm. The theorem is proved. □

Of course, Theorem 4 is useful only if one can establish verifiable conditions on f which ensure (3.14). In this regard, we remind the reader that, if f fails to satisfy the conditions of the theorem, for instance because f is not bounded, we may still invoke the result with respect to a mollified $g := \lambda * f$, and combine (3.15) with an inequality

$$|g|'_{k,p} \le C|f|'_{k,p} \tag{3.18}$$

to obtain the desired estimate. This is of course what we already did at the end of the above proof, where (3.18) was a consequence of the boundedness of λ's Fourier transform.

The crucial bound (3.14) holds under a variety of possible conditions on f, k and μ, and it is beyond the scope of this article to discuss all possible versions. The following seems to be the most useful variant:

Proposition 3. *Assume f is bounded and $\widehat{f} \in L^1(\mathbb{R}^n) \cap \widetilde{W}^k_p(\mathbb{R}^n)$. If*

$$\|\cdot\|_2^{-k}\widehat{\mu} \in L^p(\mathbb{R}^n), \tag{3.19}$$

$\widehat{\mu}$ being the Fourier series of μ, then (3.14) *holds.*

Proof. Recall that $\mu \in \ell_1(\mathbb{Z}^n)$, which renders μf well-defined. The crux in the proof is the identity

$$\mu f = \int \widehat{\mu}\widehat{f}. \tag{3.20}$$

This can be derived as in (Dyn and Ron, 1993) (cf. the proof of the lemma in Theorem 2.3.1 in that article), where the Poisson Summation Formula (Stein and Weiss, 1971, p. 252) is the salient ingredient to the proof and which else requires only $\widehat{f} \in L^1(\mathbb{R}^n)$ and boundedness of f. Now it follows from an application of Hölder's inequality that

$$|\mu f| \le \left(\int (\|\cdot\|_2^{-k}\widehat{\mu})^p\right)^{\frac{1}{p}} \left(\int (\|\cdot\|_2^{k}\widehat{f})^{p'}\right)^{\frac{1}{p'}}. \tag{3.21}$$

Combining (3.19) and (3.21) leads to (3.14). □

The assumption $\widehat{f} \in L^1(\mathbb{R}^n)$ is not very restrictive. Indeed, for $k > n/p$ that condition is implied by the requirement $f \in \widetilde{W}_p^k(\mathbb{R}^n)$. When k is too small and $\widehat{f} \notin L^1(\mathbb{R}^n)$, we still have $\widehat{\lambda * f} \in L^1(\mathbb{R}^n)$ if λ is sufficiently smooth. This smoothness requirement of λ can be shown to be mild, and it decreases with the increase of p. For example, if $p \geq 2$, we only need λ to be in $L^2(\mathbb{R}^n)$ which we assume anyway. In short, for an appropriately chosen λ and under the conditions assumed in Theorem 4, save the boundedness of f and the inequality (3.14), the bound

$$\|(L - L_A)(\lambda * f)\|_p \leq C|f|'_{k,p} \tag{3.22}$$

is valid, as soon as (3.19) holds.

We see that the important assumption in the proposition is the one with respect to μ, and we would like to compare it to the polynomial annihilation condition (3.6) assumed in Theorem 3. If μ decays fast enough to satisfy, say, the condition $\sum_j |\mu(j)||j|^k < \infty$ which was assumed in Theorem 3, then $\widehat{\mu}$ is k-times continuously differentiable everywhere and in particular at the origin. It follows easily that (3.6) implies that $\widehat{\mu}$ has a zero of order k at the origin. If $k > n/p$, this certainly implies (3.19). Moreover, this latter condition is valid for $\|\cdot\|_2^{-k'}\widehat{\mu}$, with $k' < k + n/p$. By choosing $n/p < k' < k + n/p$, we therefore obtain from Theorem 4 the bound (3.22), which yields that, upon scaling, the difference between the scattered centre approximant and the gridded approximant decays to zero with order $h^{k'}$. This is very satisfactory, since the approximation order we expect L_h to provide (in the best circumstances) is only k (not to mention the fact that we had relaxed the decay condition on μ). On the other hand, we should mention that, for example, when $p > 2$, the space of approximands to which Theorem 4 applies (viz., $\widetilde{W}_p^k(\mathbb{R}^n)$) is only a proper subspace of the Sobolev space $W_p^k(\mathbb{R}^n)$ to which Theorem 3 applies.

As an application of our results, we consider the orthogonal projection on a space spanned by translates of a radial basis function. A comprehensive analysis of the best least squares approximations onto radial functions spaces and their approximation orders can be found in Ron (1992), but we will use our theorems in this paper to deduce L^p-convergence orders of best least squares approximations. We take φ to belong to a class specified in (Buhmann and Micchelli, 1992) which contains all the functions that are mentioned in the current literature, see, e.g., the survey by Buhmann (1993). We do not state here the full set of assumptions required for φ to be in that class, but mention the salient condition that $\widehat{\varphi}$ must have a singularity at the origin, viz., $\widehat{\varphi}(t) \sim \|t\|_2^{-k+1-\delta}$ for small $\|t\|_2$, where $\delta \in (0,1]$ and k is a positive integer. It must also satisfy a decay condition $|\widehat{\varphi}(t)| = O(\|t\|_2^{-n-\varepsilon})$ for $\varepsilon > 0$ and $\|t\|_2 \to \infty$. The space of approximants is then defined as

$$S_2(\varphi) := \Big\{ \sum_j d_j \varphi(\cdot - j) \mid$$
$$|d_j| \le C\|j\|_\infty^{-k+1-\delta-\delta'} \text{ for some positive } \delta' \Big\} \cap L^2(\mathbb{R}^n).$$

In fact, spaces other than the above $S_2(\varphi)$ can be used as the underlying approximation space but we restrict attention to $S_2(\varphi)$ in this example for the sake of simplicity. We have set the notations in such a way that k here can be identified with the k of Theorem 1. The approximation scheme we use is the *orthogonal projector* $P : L^2(\mathbb{R}^n) \to S_2(\varphi)$. It follows from the required properties of φ that P can be written as

$$Pf = \sum_j (f, \psi(\cdot - j))\psi(\cdot - j), \qquad f \in L^2(\mathbb{R}^n), \tag{3.23}$$

as we shall explain in the sequel. The inner product in (3.23) is the standard inner product

$$(f_1, f_2) = \int f_1 \overline{f}_2, \qquad f_1, f_2 \in L^2(\mathbb{R}^n). \tag{3.24}$$

Further, the function ψ in (3.23) is a function in $S_2(\varphi)$ whose multiinteger translates are mutually orthonormal with respect to (3.24) and which can be defined by its Fourier transform

$$\widehat{\psi} = \frac{\widehat{\varphi}}{\sqrt{\sum_j |\widehat{\varphi}(\cdot - 2\pi j)|^2}}. \tag{3.25}$$

The orthonormality property we claim ψ to have is easily verified. Further, it is established in Theorem 3.1 on p. 330 and Theorem 3.3 on p. 333 of the same paper by the first author and Micchelli that for φ from the said class of radial basis functions, ψ satisfies the decay assumption on ψ imposed in the statement of Theorem 1 for $\nu_\psi = n + \delta$. It therefore decays fast enough to admit actually any $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, in (3.23), namely, for every $f \in L^p(\mathbb{R}^n)$, the series in (3.23) converges unconditionally and uniformly on compacta to an $L^p(\mathbb{R}^n)$ function. The sum in (3.23) can be rearranged so that the coefficients associated with the multiinteger translates of the radial function φ decay fast enough to satisfy the requirement in the definition of our space $S_2(\varphi)$, cf. Theorem 3.2 on p. 332 in the mentioned paper. Therefore it is settled that (3.23) is indeed the desired projection operator.

We need to verify that the remaining conditions on ψ of Theorem 1 and 3 hold and that (3.23) is actually of the form of the approximation operators we study in this work. To begin with, we observe that the coefficients μ that form ψ as the semi-discrete convolution $\psi = \varphi *' \mu$ satisfy the decay assumptions of Theorem 3 for $\varepsilon = \delta$, and the singularity of $\widehat{\varphi}$ at the origin implies (3.6) because

$$\widehat{\mu} = \frac{1}{\sqrt{\sum_j |\widehat{\varphi}(\cdot - 2\pi j)|^2}}.$$

Next, we need to know that the coefficients of $\psi(\cdot - j)$, $j \in \mathbb{Z}^n$, in the present approximation scheme (3.23) are derived from a convolution process with an appropriate convolutor λ. Indeed, taking $\lambda := \psi(-\cdot)$, we certainly have $(f, \psi(\cdot - j)) = (\lambda * f)(j)$. Also, as proved in Theorem 3.1 on p. 330 of the mentioned paper, ψ decays in such a way to provide (2.1). It satisfies the moment conditions (2.2) for the present k due to $\widehat{\varphi}$'s singularity at the origin and due to $\widehat{\psi}$'s form (3.25). Consequently, Theorems 1 and 3 are applicable to the present case as soon as we show that $\psi *'$ is the identity on Π_{k-1}. That final requirement follows, by a standard argument, from the fact that ψ satisfies the so-called Strang-Fix conditions, namely

$$D^\alpha \widehat{\psi}(2\pi\beta) = \delta_{\beta 0}\delta_{\alpha 0}, \quad \beta \in \mathbb{Z}^n, \quad |\alpha| < k.$$

These conditions can be verified directly from the definition of $\widehat{\psi}$ in (3.25) and from the singularity of the radial function's Fourier transform at the origin. Therefore, our Theorems 1 and 3 apply to this example and provide the desired L^p-approximation orders.

References

1. de Boor, C., K. Höllig and S. Riemenschneider (1993) Box Splines, Springer (New York).
2. de Boor, C., R.A. DeVore and A. Ron (1993) "Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$", to appear in Trans. Amer. Math. Soc.
3. de Boor, C., and A. Ron (1992a) "Fourier analysis of approximation power of principal shift-invariant spaces", Constr. Approx. **8** (1992), 427–462.
4. de Boor, C., and A. Ron (1992b) "The exponentials in the span of the integer translates of a compactly supported function: approximation orders and quasi-interpolation", J. London Math. Soc. **45** (1992), 519–535.
5. Buhmann, M.D. (1989) Multivariable interpolating using radial basis functions, Ph.D. Dissertation, University of Cambridge.
6. Buhmann, M.D. (1993) "New developments in the theory of radial basis function interpolation", in Multivariate Approximation: From CAGD to Wavelets, K. Jetter and F.I. Utreras, eds., World Scientific (Singapore), 35–75.
7. Buhmann, M.D., N. Dyn and D. Levin (1993) "On quasi-interpolation with radial basis functions on non-regular grids", to appear in Constr. Approx.
8. Buhmann, M.D. and C.A. Micchelli (1992) "On radial basis approximations on periodic grids", Mathematical Proceedings of the Cambridge Philosophical Society **112**, 317–334.

9. Dyn, N., I.R.H. Jackson, D. Levin and A. Ron (1992) "On multivariate approximation by the integer translates of a basis function, Israel J. Math. **78**, 95–130.
10. Dyn, N. and A. Ron (1993) "Radial basis function approximation: from gridded centres to scattered centres", CMS Technical Summary Report #94–3, University of Wisconsin–Madison.
11. Jia, R.-Q. and J.-J. Lei (1993) "Approximation by multiinteger translates of functions having global support", J. Approx. Th. **72**, 2–23.
12. Lei, J.-J., and R.-Q. Jia (1991) "Approximation by piecewise exponentials", SIAM J. Math. Anal. **22**, 1776–1789.
13. Ron, A. (1992) "The L_2-approximation orders of principal shift-invariant spaces generated by a radial basis function", in Numerical Methods of Approximation Theory, Vol. 9, D. Braess and L.L. Schumaker, eds., Birkhäuser (Basel), 245–268.
14. Stein, E.M. and G. Weiss (1971) Introduction to Fourier Analysis in Euclidean Spaces, Princeton University Press (Princeton).

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