UNIVERSITY OF WISCONSIN-MADISON
COMPUTER SCIENCES DEPARTMENT

Polynomial ideals and multivariate splines

Carl de Boor[1] & Amos Ron

Computer Sciences Department
1210 W. Dayton St.
University of Wisconsin-Madison
Madison, WI 53706

June, 1989

Abstract

The well-established theory of multivariate polynomial ideals (over $\mathbb{C}$) was found in recent years to be important for the investigation of several problems in multivariate approximation. In this note we draw connections between the validity of spectral synthesis in the space of all multivariate sequences and questions in multivariate splines on uniform meshes.

AMS (MOS) Subject Classifications: 13A15, 41A15, 41A63

Key Words and phrases: polynomial ideals, polynomials, box splines, multivariate splines, spectral synthesis.

[1] supported by the National Science Foundation under Grant No. DMS-8701275 and by the United States Army under Contract No. DAAL03-87-K-0030

Polynomial ideals and multivariate splines
Carl de Boor & Amos Ron

1. The issue

An **exponential** is, by definition, any function of the form $\sum_{\theta\in\boldsymbol{\Theta}} e_\theta p_\theta$, with $\boldsymbol{\Theta}$ a finite subset of the set $\mathbb{C}^s$ of complex s-vectors, with each $p_\theta \in \pi\backslash 0$, i.e., a (nontrivial) polynomial in s variables, and with $e_\theta : x \mapsto e^{\langle\theta,x\rangle}$ the **pure exponential for the frequency** θ. We call the pointset $\boldsymbol{\Theta}$ the **spectrum** of the exponential $\sum_{\theta\in\boldsymbol{\Theta}} e_\theta p_\theta$. We take the spectrum of a function space to be the union of the spectra of all exponentials in that space. We think of exponentials as defined on $\mathbb{C}^s$ or merely on $\mathbb{Z}^s$, depending on the situation. In the latter case, we treat the frequency θ of e_θ as an element of $\mathbb{C}^s/2\pi i\mathbb{Z}^s$, since $e_\theta = e_\xi$ on $\mathbb{Z}^s$ whenever $\theta = \xi \bmod 2\pi i\mathbb{Z}^s$. We also use $e^x := (e^{x(1)}, \ldots, e^{x(s)})$.

Let ϕ be a distribution in the space $\mathcal{E}'(\mathbb{R}^s)$ of all compactly supported complex-valued s-dimensional distributions. Let $\mathcal{C}$ be the space of all sequences defined on the lattice $\mathbb{Z}^s$. Define the operator $\phi*$ on $\mathcal{C}$ by

$$\phi* : c \mapsto \phi*c := \sum_{\alpha\in\mathbb{Z}^s} c_\alpha E^{-\alpha}\phi, \tag{1.1}$$

where E^α is the shift operator $E^\alpha : f \mapsto f(\cdot+\alpha)$. The range of $\phi*$ is denoted here by $S(\phi)$, i.e., $S(\phi)$ is the space spanned by the integer translates of ϕ. Important subspaces of $S(\phi)$ are the space $H(\phi)$ of all exponentials in $S(\phi)$ and its polynomial subspace $\pi(\phi) = S(\phi)\cap\pi$. The study of these spaces, or, more generally, any subspace $F \subset S(\phi)$, is facilitated once we know the preimage $(\phi*)^{-1}(F)$ of F. Since $H(\phi)$ and $\pi(\phi)$ are both shift-invariant (i.e., closed under integer translates), their preimage is too. Moreover, since these spaces are finite-dimensional, their preimage is closed under pointwise convergence. For that reason, we equip $\mathcal{C}$ with the topology of pointwise convergence (which makes it into a Fréchet space), and equip $S(\phi)$ with the topology induced by $\phi*$, i.e., the strongest topology that makes $\phi*$ continuous. The main result of this note is as follows.

(1.2) Theorem. *Let F be a closed shift-invariant subspace of $S(\phi)$. Then there exists a finite set $\boldsymbol{\Theta} \subset \mathbb{C}^s/2\pi i\mathbb{Z}^s$ and polynomial spaces $\{P_\theta\}_{\theta\in\boldsymbol{\Theta}}$ such that the (sequence) exponential space $\bigoplus_{\theta\in\boldsymbol{\Theta}} e_\theta P_\theta$ is dense in $(\phi*)^{-1}(F)$.*

We note that this theorem supplies also information about the **kernel** of $\phi*$, which corresponds to the choice $F = 0$.

In case $F = H(\phi)$, F is also finite-dimensional. For a finite-dimensional F, more can be said.

(1.3) Theorem. *Let F be a finite-dimensional shift-invariant subspace of $S(\phi)$. Then there exists an exponential subspace G of $(\phi*)^{-1}(F)$ which is mapped bijectively onto F. Furthermore, G can be so chosen that, for every θ in its spectrum and every $\alpha \in \mathbb{Z}^s$, $e_\theta(\alpha)$ is an eigenvalue of ${E^\alpha}_{|F}$.*

The proof of the above results uses the validity of spectral synthesis in $\mathcal{C}$ (i.e., the fact that every closed shift-invariant subspace of $\mathcal{C}$ contains a dense exponential space) and makes essential use of polynomial ideal theory. In section 2, we review some analytic aspects of polynomial ideals, and in section 3 discuss the kind of ideal that appears in our context. Lefranc's [L] proof of the validity of spectral synthesis in $\mathcal{C}$ is provided (in a slightly modified version) in section 4. In section 5, we prove the above theorems and apply them to the special choices $F = H(\phi)$, $F = \pi(\phi)$. An example concerning box splines is then examined in the last section.

We now make more explicit the connection between $\mathcal{C}$ and polynomial ideals, showing that each closed shift-invariant subspace of $\mathcal{C}$ is characterized by a corresponding ideal. The dual of $\mathcal{C}$ is given by $\mathcal{P}$, the space of all sequences with finite support, with the natural pairing

$$\mathcal{P} \times \mathcal{C} \to \mathbb{C} : (p, c) \mapsto \langle p, c\rangle := \sum_{\alpha \in \mathbb{Z}^s} p_\alpha c_\alpha.$$

In particular,

$$C\perp := \{p \in \mathcal{P} : \ \langle p, c\rangle = 0, \ \text{all } c \in C\} \tag{1.4}$$

is the **annihilator** of the subspace C of $\mathcal{C}$, while the annihilator of a subspace $P \subset \mathcal{P}$ is the analogously defined subspace of $\mathcal{C}$. It is a consequence of the Hahn-Banach theorem that every closed subspace $C \subset \mathcal{C}$ satisfies $C\perp\perp = C$. Furthermore, in case $P \subset \mathcal{P}$ is shift-invariant, it is determined by its subspace P_+ of all elements supported on $\mathbb{Z}^s_+$. Also, a shift-invariant subspace $C \subset \mathcal{C}$ is orthogonal to the shift-invariant $P \subset \mathcal{P}$ if and only if it is orthogonal to P_+. We thus conclude

(1.5) Proposition. *Let S and C be shift-invariant subspaces of $\mathcal{C}$. Then their closures coincide if and only if*

$$S\perp_+ = C\perp_+.$$

Polynomial ideals enter in this way, since each $p \in \mathcal{P}_+$ can be identified with the polynomial $p\check{} := \sum_\alpha p_\alpha ()^\alpha$, while the shift-invariance of P implies (yet is not equivalent to) the fact that $P_+\check{}$ is an ideal of π. Further, with this identification, $p\check{}(e^\xi) = \langle p, e_\xi\rangle$, hence the point $\theta = e^\xi$ lies in the variety of the ideal $C\perp_+\check{}$ if and only if the pure exponential e_ξ lies in C.

2. Polynomial ideals

Let I be a polynomial ideal in the ring π of all polynomials in s variables (over $\mathbb{C}$). Associated with I is its **variety**

$$\mathcal{V}_I := \{v \in \mathbb{C} : \ q(v) = 0, \ \text{all } q \in I\}. \tag{2.1}$$

The fundamental result in polynomial ideal theory is Hilbert's Nullstellensatz which says that a power of $p \in \pi$ lies in I whenever p vanishes on $\mathcal{V}_I$.

An ideal is **primary** if some power of p must lie in it whenever the product pq lies in it and q does not. The importance of primary ideals in ideal theory is primarily due to the classical fact that every ideal I is the intersection of finitely many primary ideals. The primary decomposition is, in general, not unique even when assuming irredundancy. Yet, the **prime components** of $\mathcal{V}_I$, i.e, the varieties of the primary ideals in an irredundant decomposition of I, are uniquely determined by I. Furthermore, every primary ideal which corresponds to a maximal prime component is also determined uniquely by I.

The primary decomposition can be used to show that *a polynomial ideal I is characterized by its* **multiplicity spaces**

$$I\perp_v := \{p \in \pi : \; p(D)q(v) = 0, \; \forall q \in I\}. \tag{2.2}$$

Here, $p(D)$ is the differential operator with constant coefficients induced by the polynomial p. As the notation indicates, we think of $I\perp_v$ as the annihilator of I with respect to the pairing

$$\pi \times \pi \to \mathbb{C}^s : (p, q) \mapsto p(D)q(v).$$

$I\perp_v$ is nontrivial exactly when $v \in \mathcal{V}_I$. Since I is an ideal, $I\perp_v$ is D-invariant, i.e., closed under differentiation. Conversely, for any D-invariant polynomial space P and any $v \in \mathbb{C}^s$,

$$P\perp_v := \{q \in \pi : p(D)q(v) = 0, \forall p \in P\}$$

is an ideal.

(2.3)Theorem. *If I is a primary ideal, then $I = I\perp_v\perp_v$ for any $v \in \mathcal{V}_I$.*

Outline of the proof of (2.3)Theorem([L]): Assume without loss that $v = 0$ (which can always be achieved by a translation). The ideal I_A generated by I in the ring A of formal power series is closed (in the natural topology of A as a local ring, i.e., f_n converges to f iff, for every k, all terms of order $< k$ of $f - f_n$ are zero eventually; cf., e.g., [N;Proposition 2 on page 85]), yet $I_A\perp_0 = I\perp_0$, therefore $I_A = I\perp_0\perp := \{f \in A : p(D)f(0) = 0 \; \forall p \in I\perp_0\}$, using the fact that the pairing

$$\pi \times A \to \mathbb{C}^s : (p, f) \mapsto p(D)f(0)$$

makes it possible to identify π with the continuous dual of A and A with the dual of π. On the other hand, since I is primary, the Noether-Lasker Theorem (cf., e.g., [K;p.61]) ensures that $I = I_A \cap \pi$. $\square$

The primary decomposition available for an arbitrary polynomial ideal provides the following

(2.4)Corollary. *Let $I = \cap_i Q_i$ be a primary decomposition for the ideal I. Then, for any V which intersects each $\mathcal{V}_{Q_i}$,*

$$I = \cap_{v \in V} I \perp_v \perp_v. \tag{2.5}$$

Indeed, with $v \in \mathcal{V}_{Q_i} \cap V$, we have $I \perp_v \supset Q_i \perp_v$ since $I \subset Q_i$, hence $I \perp_v \perp_v \subset Q_i \perp_v \perp_v = Q_i$ by (2.3)Theorem. This shows that the right side of (2.5) is contained in I, while the opposite inclusion is trivial.

3. q-ideals

If the linear subspace P of $\mathcal{P}$ is shift-invariant, then $P_+{}^\vee$ is an ideal, but not every ideal in π arises in this way. A polynomial ideal I is of the form $P_+{}^\vee$ for some shift-invariant subspace P of $\mathcal{P}$ if and only if it satisfies the condition

$$p \in I \Longleftrightarrow ()^\alpha p \in I \tag{3.1}$$

for every $\alpha \in \mathbb{Z}_+^s$. This condition is equivalent to the requirement that

$$p \in I \Longleftrightarrow qp \in I \tag{3.2}$$

for the polynomial $q := ()^{1,1,\ldots,1}$. Provided I is non-trivial, we call such an ideal a **q-ideal**. We define the **q-reduced variety** $\mathcal{V}_I^q$ of I by

$$\mathcal{V}_I^q := \{\theta \in \mathcal{V}_I : \ q(\theta) \neq 0\}.$$

An **E-ideal** corresponds to the choice $q = ()^{(1,\ldots,1)}$, hence its reduced variety becomes $\mathcal{V}_I^* := \mathcal{V}_I \cap \mathbb{C}_*^s$, where $\mathbb{C}_*^s := (\mathbb{C}\backslash 0)^s$.

(3.3) Proposition. *If I is a q-ideal and the polynomial p vanishes on $\mathcal{V}_I^q$, then a power of p lies in I.*

Indeed, if p vanishes on $\mathcal{V}_I^q$, then pq vanishes on $\mathcal{V}_I$. Therefore, by the Nullstellensatz, $p^k q^k \in I$ for some k, and repeated application of (3.2) then yields $p^k \in I$.

The following theorem is a special case of [N;Thm. 6, p. 23]:

(3.4) Theorem. *An ideal I is a q-ideal if and only if it admits a primary decomposition $I = \cap_i Q_i$ with $\mathcal{V}^q_{Q_i} \neq \emptyset$ for all i.*

Proof: Assume first that I is a q-ideal, let $\cap_i Q_i$ be a primary decomposition of I and suppose that for some j, $\mathcal{V}_{Q_j}$ lies in the zero set of q. Then $J := \cap_{i\neq j} Q_i \supset I$. On the other hand q vanishes on $\mathcal{V}_{Q_j}$, hence $q^n \in Q_j$ for some n, and therefore $q^n J \subset Q_j J \subset Q_j \cap J = I$. Since I is a q-ideal, it follows that $J \subset I$ and consequently $J = I$. We conclude that Q_j is a redundant component in the primary decomposition of I.

For the converse, we assume that $I = \cap_i Q_i$ and that no $\mathcal{V}_{Q_i}$ lies in the zero set of q. Then, for every i, no power of q can lie in Q_i, hence, since Q_i is primary,

$$pq \in I \Longrightarrow pq \in Q_i \Longrightarrow p \in Q_i.$$

We conclude that $p \in I$ and thus I is a q-ideal. □

(3.5) Corollary. *A q-ideal can be decomposed into primary q-ideals.*

The following two corollaries will be used in the sequel:

(3.6) Corollary. *If I is a q-ideal, then V in (2.4)Corollary can be chosen from $\mathcal{V}^q_I$. In particular, if I is an E-ideal, V can be chosen from $\mathcal{V}^*_I \subset \mathbb{C}^s_*$.*

(3.7) Corollary. *Assume $\mathcal{V}^q_I$ is finite. Then I is a q-ideal if and only if $\mathcal{V}_I = \mathcal{V}^q_I$.*

Proof: If $\mathcal{V}_I = \mathcal{V}^q_I$, then I is a q-ideal by (3.4)Theorem, since q vanishes nowhere on $\mathcal{V}_I$. Conversely, if $\mathcal{V}_I \backslash \mathcal{V}^q_I$ is not empty, it contains a maximal prime component of $\mathcal{V}_I$ which lies entirely in the zero set of q, which means that there exists a primary ideal Q which appears in every primary decomposition of I and whose variety lies entirely in the zero set of q. Consequently, by (3.4)Theorem, I is not a q-ideal. □

4. Spectral synthesis in $\mathcal{C}$

The following lemma is the technical link between ideal theory and spectral synthesis in $\mathcal{C}$. It uses the **normalized factorial function** $[]^\alpha$ defined by

$$[x]^\alpha := \prod_j [x(j)]^{\alpha(j)}, \quad \text{with } [t]^n := t(t-1)\cdots(t-n+1)/n!.$$

This function's chief virtue lies in the fact that $\Delta^\alpha []^\beta = []^{\beta-\alpha}$, with Δ the forward difference operator. This provides the pretty identity

$$\Delta p([]) = (Dp)([]), \tag{4.1}$$

in which $p([]) := \sum_\beta []^\beta D^\beta p(0)$, and which is meant to signify that $\Delta^\alpha p([]) = (D^\alpha p)([])$ for all $\alpha \in \mathbb{Z}^s_+$.

(4.2) Lemma. *Let* $p \in \pi$, $q \in \mathcal{P}_+$, $v \in \mathbb{C}_*^s$, *and let* $\theta = \log v$, *i.e.,* $e^{\theta_j} = v_j$, $j = 1, ..., s$. *Then* $p(vD)q\check{}(v) = 0$ *if and only if* $\langle q, e_\theta p([])\rangle = 0$.

Proof: For every $\alpha, \beta \in \mathbb{Z}_+^s$,

$$(vD)^\beta ()^\alpha (v) = \frac{\alpha!}{(\alpha - \beta)!} v^\alpha = \beta! [\alpha]^\beta v^\alpha. \tag{4.3}$$

Hence

$$p(vD)q\check{}(v) = \sum_\alpha q_\alpha p([\alpha]) v^\alpha = \langle q, e_\theta p([])\rangle.$$

♠

(4.4) Theorem([L]). *Every closed shift-invariant subspace of* $\mathcal{C}$ *contains a dense exponential subspace of finite spectrum.*

Proof: Let C be the space in question. Then $I := C\perp_+\check{}$ is an E-ideal. By (2.4)Corollary and (3.6)Corollary, there exists $V \subset \mathbb{C}_*^s$ such that

$$C\perp_+\check{} = I = \bigcap_{v \in V} I\perp_v\perp_v. \tag{4.5}$$

We conclude from (4.2)Lemma that, with $\mathbf{\Theta} := \log V$,

$$C\perp_+ = \{q \in \mathcal{P}_+ : \ \langle q, e_\theta r([])\rangle = 0, \ \text{all } \theta \in \mathbf{\Theta}, \ r \in P_\theta\}, \tag{4.6}$$

where $P_\theta = \{p(\cdot/v) : p \in I\perp_v\}$. Since $I\perp_v$ is D-invariant, so is P_θ, and hence $P_\theta([]) := \{p([]) : p \in P_\theta\}$ is shift-invariant by (4.1). This implies that the subspace F of $\mathcal{C}$ defined by $F := \bigoplus_{\theta \in \mathbf{\Theta}} e_\theta P_\theta([])$ is shift-invariant, while $C\perp_+ = F\perp_+$ by (4.6). Therefore, an application of (1.5)Proposition shows that C and F have the same closure, and since C is closed, F is dense in C. □

The proof just given supports the following corollary:

(4.7) Corollary. *Let* C *be a closed shift-invariant subspace of* $\mathcal{C}$. *Let* $\mathbf{\Theta}$ *be a subset of* $\mathbb{C}^s$. *If* $e^{\mathbf{\Theta}} := \{e^\theta : \theta \in \mathbf{\Theta}\}$ *intersects each prime component of* $\mathcal{V}_{C\perp_+\check{}}$, *then the space of all exponentials in* C *with spectrum in* $\mathbf{\Theta}$ *is dense in* C.

With the aid of (3.7)Corollary, we also conclude

(4.8) Corollary. *A closed shift-invariant subspace* C *of* $\mathcal{C}$ *is finite-dimensional if and only if it has finite spectrum.*

Proof: The "only if" claim is trivial. For the converse, we note that if the spectrum of C is finite, then the reduced variety of $C\perp_+\check{}$ is finite, hence by (3.7)Corollary, so is $\mathcal{V}_{C\perp_+\check{}}$. Now, for an ideal of finite variety, it follows from the Nullstellensatz that each of the multiplicity spaces associated with the variety is finite-dimensional. Application of (4.2)Lemma then yields that the space of all exponentials in C is finite-dimensional, and by virtue of (4.4)Theorem, so is C. □

5. Main results

Unless stated otherwise, the exponentials considered in the rest of the paper will always be defined on $\mathbb{Z}^s$, hence the associated spectra are meant in $\mathbb{C}^s/2\pi i\mathbb{Z}^s$.

We prove here (1.2)Theorem, (1.3)Theorem and draw other conclusions from the spectral synthesis in $\mathcal{C}$.

Proof of (1.2)Theorem: Since F is closed and shift-invariant, $(\phi*)^{-1}(F)$ is a closed shift-invariant subspace of $\mathcal{C}$. Now apply (4.4)Theorem. □

Proof of (1.3)Theorem: The first part of the theorem follows directly from (1.2)Theorem and the fact that F, being finite-dimensional, contains no proper dense subspaces. To prove the second part, we note that, for every $\alpha \in \mathbb{Z}^s$, E^α is an endomorphism on F, hence it indeed makes sense to consider the spectrum $\sigma(\alpha)$ of ${E^\alpha}_{|F}$.

We now take an arbitrary exponential $g := \sum_{\theta\in\mathbf{\Theta}} e_\theta p_\theta$ in the preimage of F (under $\phi*$) and, following the argument of [BR; Prop. 7.1], show that, for every $\vartheta \in \mathbf{\Theta}$ for which $e_\vartheta(\alpha) \notin \sigma(\alpha)$ for some $\alpha \in \mathbb{Z}^s$, the summand $e_\vartheta p_\vartheta$ is in the kernel of $\phi*$, hence can be omitted from the sum.

For any α, the characteristic polynomial χ_α of ${E^\alpha}_{|F}$ gives

$$\phi*\chi_\alpha(E^\alpha)g = \chi_\alpha(E^\alpha)(\phi*g) \in \chi_\alpha(E^\alpha)(F) = \{0\}.$$

For arbitrary $p \in \pi$, $\beta \in \mathbb{Z}^s$ and $\lambda \in \mathbb{C}^s$

$$E^\beta(e_\theta p) - \lambda e_\theta p = e_\theta\ (e_\theta(\beta)E^\beta p - \lambda p),$$

hence $(E^\beta - \lambda)(e_\theta p) = e_\theta q$, with q a polynomial that satisfies

$$\deg q = \deg p \Longleftrightarrow e_\theta(\beta) - \lambda \neq 0.$$

Assume now that $e_\vartheta(\alpha) \notin \sigma(\alpha)$. Then $\chi_\alpha(E^\alpha)$ is 1-1 on $e_\vartheta\pi$. Also, we can find a polynomial q for which $q(E)$ annihilates $e_\theta p_\theta$ for all $\theta \in \mathbf{\Theta}\backslash\vartheta$ but is 1-1 on $e_\vartheta\pi$. Consequently,

$$0 = q(E)(\phi*\chi_\alpha(E^\alpha)g) =: \phi*r(E)(e_\vartheta p_\vartheta). \tag{5.1}$$

Since $r(E)$ is 1-1 on $e_\vartheta\pi$, it carries each $e_\vartheta\pi_k$ onto itself, hence, with $k \geq \deg p_\vartheta$, there is some polynomial s so that $(sr)(E)$ is the identity on $e_\vartheta\pi_k$. Thus, from (5.1), $0 = s(E)0 = \phi*(sr)(E)(e_\vartheta p_\vartheta) = \phi*e_\vartheta p_\vartheta$, which is what we set out to prove. □

To make use of the second part of (1.3)Theorem, one needs to know the spectrum of sufficiently many $E^{\alpha}{}_{|F}$, a task that might appear to be difficult in general. Yet, if we assume that F is an exponential space and denote its spectrum by $\mathbf{\Theta}$, then F contains each e_θ with $\theta \in \mathbf{\Theta}$, hence

$$\sigma(\alpha) = \{e_\theta(\alpha)\}_{\theta\in\mathbf{\Theta}}, \text{ all } \alpha \in \mathbb{Z}^s. \tag{5.2}$$

This implies that the points $\theta \in \mathbf{\Theta}$ are the only frequencies that satisfy

$$e_\theta(\alpha) \in \sigma(\alpha), \text{ all } \alpha \in \mathbb{Z}^s. \tag{5.3}$$

So we obtain

(5.4) Corollary. *Let H be a shift-invariant exponential subspace of $S(\phi)$ with spectrum $\mathbf{\Theta} \subset \mathbb{C}^s$. Then there exists a finite-dimensional shift-invariant exponential space of spectrum $\mathbf{\Theta}/2\pi i\mathbb{Z}^s$ which is mapped by $\phi*$ onto H.*

Of particular interest is the following

(5.5) Corollary. *For every $\theta \in \mathbb{C}^s$, $S(\phi) \cap e_\theta\pi$ is the image of some finite-dimensional shift-invariant space $C \subset \mathcal{C} \cap e_\theta\pi$ under $\phi*$.*

6. An example: box splines

We discuss here an example in which we identify the spectrum of the preimage of $H(\phi)$ for a box spline ϕ. For background about box splines we refer to [BR], from where most of the notations are borrowed.

Let Γ be a finite index set. The (exponential) box spline B_Γ is defined via its Fourier transform as

$$\widehat{B}_\Gamma(x) = \prod_{\gamma\in\Gamma} \frac{e^{\lambda_\gamma - i\langle x_\gamma, x\rangle} - 1}{\lambda_\gamma - i\langle x_\gamma, x\rangle},$$

where for each γ, $\lambda_\gamma \in \mathbb{C}$ and $x_\gamma \in \mathbb{Z}^s\backslash 0$. We assume that $\operatorname{span}\{x_\gamma\}_{\gamma\in\Gamma} = \mathbb{R}^s$.

Since $C_\Gamma := (B_\Gamma*)^{-1}(H(B_\Gamma))$ is shift-invariant, its spectrum coincides (mod $2\pi i\mathbb{Z}^s$) with the set

$$\mathbf{\Theta} := \{\theta \in \mathbb{C}^s : \ B_\Gamma * e_\theta \in H(B_\Gamma)\}. \tag{6.1}$$

An important subset of $\mathbf{\Theta}$ was identified in [DM] and [BR] as the set

$$\widetilde{\mathbf{\Theta}} := \{\theta \in \mathbb{C}^s : \ \operatorname{span}\{x_\gamma\}_{\gamma\in\widetilde{\Gamma}_\theta} = \mathbb{C}^s\},$$

with

$$\widetilde{\Gamma}_\theta := \{\gamma \in \Gamma : \ \nabla^\gamma(e_\theta) = 0\},$$

and $\nabla^\gamma := 1 - e^{\lambda_\gamma} E^{-x_\gamma}$.

(6.2) Proposition([DM],[BR]). *The set $\widetilde{\boldsymbol{\Theta}}$ is finite mod $2\pi i\mathbb{Z}^s$, and for each $\theta \in \widetilde{\boldsymbol{\Theta}}$*

$$B_\Gamma * e_\theta \in H(B_\Gamma).$$

We therefore conclude that indeed $\widetilde{\boldsymbol{\Theta}} \subset \boldsymbol{\Theta}$. In the following theorem we show that the spectrum of C_Γ is a finite union of linear manifolds, each of which intersects $\widetilde{\boldsymbol{\Theta}}$. For $K \subset \Gamma$, we use here the notations

$$\langle K \rangle := \operatorname{span}\{x_\gamma\}_{\gamma\in K}, \quad K^\perp := \{x \in \mathbb{C}^s :\ x \perp \langle K \rangle\}.$$

(6.3) Theorem. *For an exponential box spline B_Γ, the spectrum of the space C_Γ of the preimage of the exponential space $H(B_\Gamma)$ is*

$$\bigcup_{\theta,K} \theta + K^\perp, \tag{6.4}$$

where θ runs over $\widetilde{\boldsymbol{\Theta}}$ and, for each θ, K runs over all subsets of Γ which are minimal with respect to the property

$$B_K * e_\theta \in H(B_\Gamma).$$

Proof: We show first that each point in (6.4) lies indeed in the desired spectrum $\boldsymbol{\Theta}$ (as given by (6.1)). So assume that $\theta \in \widetilde{\boldsymbol{\Theta}}$, that $B_K * e_\theta \in H(B_\Gamma)$, and K is minimal. Then it is sufficient to prove that $B_K * e_{\theta+\eta} \in H(B_\Gamma)$, for all $\eta \in K^\perp$. By (6.2)Proposition, this is true for $\eta = 0$, and hence there is nothing to prove in case $\langle K \rangle = \mathbb{C}^s$, since then $K^\perp = 0$. Otherwise, since B_K is supported on $\langle K \rangle$, we must have $B_K * e_\theta = 0$. In fact, already $\sum_{\alpha\in\langle K\rangle\cap\mathbb{Z}^s} e_\theta(\alpha)E^{-\alpha}B_K = 0$. Since $e_{\theta+\eta}$ coincides on $\langle K \rangle$ with e_θ, we conclude that indeed $B_K * e_{\theta+\eta} = 0$, and hence the union in (6.4) lies in $\boldsymbol{\Theta}$.

For the converse, assume that $\theta \in \boldsymbol{\Theta}$ and let K be a minimal subset of Γ with respect to the property

$$B_K * e_\theta \in H(B_\Gamma).$$

By the preceding arguments, $\theta + K^\perp \subset \boldsymbol{\Theta}$. In what follows, we show that $\theta + K^\perp$ intersects $\widetilde{\boldsymbol{\Theta}}$, and hence $\theta + K^\perp = \vartheta + K^\perp$ for some $\vartheta \in \widetilde{\boldsymbol{\Theta}}$.

For that we introduce, for each $\gamma \in \Gamma$, the differential operator $D^\gamma := D_{x_\gamma} - \lambda_\gamma$, and note [BR] that $D^\gamma(B_K * e_\theta) = B_{K\backslash\gamma} * \nabla^\gamma(e_\theta)$ for $\gamma \in K$. Since $B_K * e_\theta$ is an exponential, so is $D^\gamma(B_K * e_\theta)$, and thus, since $\nabla^\gamma(e_\theta)$ is a constant multiple of e_θ, the minimality of K shows that $\nabla^\gamma(e_\theta) = 0$, and since $\gamma \in K$ was arbitrary, $\nabla^\gamma(e_\zeta) = \nabla^\gamma(e_\theta) = 0$, for all $\gamma \in K$, $\zeta \in \theta + K^\perp$.

Now, let η be the unique solution in $K^\perp$ of the equations

$$\langle x_\gamma, \theta + ?\rangle = \lambda_\gamma,\ \text{all } \gamma \in J,$$

where $J \subset \Gamma \backslash K$ is chosen so that $\#J = \dim K^\perp$ and $\langle K \cup J \rangle = \mathbb{C}^s$. Then $\theta + \eta \in \theta + K^\perp$, and also $\nabla^\gamma(e_{\theta+\eta}) = 0$ for every $\gamma \in K \cup J$, which implies that $\vartheta := \theta + \eta \in \widetilde{\mathbf{\Theta}}$, and consequently $\theta + K^\perp$ intersects $\widetilde{\mathbf{\Theta}}$, as claimed.

Finally, if K is also minimal with respect to the property $B_M * e_\vartheta \in H(B_\Gamma)$, then $\theta + K^\perp = \vartheta + K^\perp$ is one of the sets in (6.4); otherwise, a set of the form $\vartheta + M^\perp$ with $M \subset K$ appears in (6.4), and since $K^\perp \subset M^\perp$, $\theta \in \vartheta + M^\perp$, and our claim follows. □

With the aid of (4.4)Theorem we conclude the following

(6.5) Corollary. *Let $C_{\theta,K}$ be the closure of the space of all exponentials in C_Γ with spectrum in $\theta + K^\perp$. Then*

$$C_\Gamma = \sum_{\theta,K} C_{\theta,K}, \tag{6.6}$$

where θ and K vary as in (6.3)Theorem.

Proof: Note first that the right hand side of (6.6) is closed, as the sum of finitely many closed spaces. Furthermore, by (6.3)Theorem, this sum contains all the exponentials in C_Γ. Now apply (4.4)Theorem. □

We conjecture that there is 1-1 correspondence between sets of the form $\theta + K^\perp$ and the components of the variety of $C_\Gamma \perp\check{}$. If so, it will follow that the finite set in (4.4)Theorem can be chosen as $\widetilde{\mathbf{\Theta}}/2\pi i \mathbb{Z}^s$.

Combining (6.3)Theorem with (4.8)Corollary, we obtain a result which was proved in [DM] by other means:

(6.7) Corollary. *C_Γ if finite dimensional if and only if its spectrum is $\widetilde{\mathbf{\Theta}}/2\pi i \mathbb{Z}^s$.*

References

[BR] A. Ben-Artzi and A. Ron, Translates of exponential box splines and their related spaces, Trans. Amer. Math. Soc. **309** (1988), 683–710.

[DM] W. Dahmen and C. A. Micchelli, Multivariate E-splines, Advances in Math., to appear.

[K] W. Krull, *Idealtheorie*, Ergebnisse der Math. **iv**, no.3, (1935), pp.vii+152.

[L] M. Lefranc, Analyse Spectrale sur $\mathrm{Z_n}$, C. R. Acad. Sc. **246** (1958), 1951-1953.

[N] D. G. Northcott, *Ideal Theory*, Cambridge University Press, 1960.