NEW CONSTRUCTIONS OF PIECEWISE-CONSTANT WAVELETS*

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Abstract. The classical Haar wavelet system of $L_2(\mathbb{R}^n)$ is commonly considered to be very local in space. We introduce and study in this paper piecewise-constant framelets (PCF) that include the Haar system as a special case. We show that any bi-framelet pair consisting of PCFs provides the same Besov space characterizations as the Haar system. In particular, it has *Jackson-type performance* $s_J = 1$ and *Bernstein-type performance* $s_B = 0.5$. We then construct two PCF systems that are either, in high spatial dimensions, far more local than Haar, or are as local as Haar while delivering better performance: $s_J = s_B = 1$. Both representations are computed and inverted by fast algorithms.

Key words. frames, framelets, wavelets, Haar wavelets, piecewise-constant wavelets, PCF, Besov spaces, Unitary Extension Principle

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1. Introduction.

1.1. Local wavelet representation. In a recent paper, [HR], we develop a novel methodology for wavelet constructions (under the acronym CAMP) that yields very local wavelet systems in arbitrarily high spatial dimensions. The most local construction of [HR] is based on piecewise-linear box splines. In order to develop "the most local possible construction", we employ in this paper a piecewise-constant setup. Since the general "performance analysis" of [HR] does not cover piecewise-constants, we precede the actual construction with a general analysis of redundant piecewise-constant wavelet representations. We then introduce and analyse two types of local piecewise-constant wavelet systems. The high-pass filters in both constructions are mostly 2-tap (in the decomposition, as well as in the inversion), hence are the shortest possible. One of the systems is exact, i.e., bi-orthogonal, and the other one is slightly redundant, i.e., a bi-frame.

1.2. Notations. For $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, we let $|t| := \sqrt{t_1^2 + \cdots + t_n^2}$. The inner product of two vectors t, x in $\mathbb{R}^n$ is denoted by $t \cdot x$. We use the following normalization of the Fourier transform (for, e.g., $f \in L_1(\mathbb{R}^n)$):

$$\widehat{f}(\omega) := \int_{\mathbb{R}^n} f(t) e^{-i\omega\cdot t} dt.$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of test functions, and by $\mathcal{S}'(\mathbb{R}^n)$ its dual, the space of tempered distributions. Given a function space whose elements are defined on $\mathbb{R}^n$, we sometimes omit the domain $\mathbb{R}^n$ in our notation. Also, we denote by $\mathcal{S}'/\mathcal{P}$ the space of equivalence classes of (tempered) distributions modulo polynomials. For any $f, g \in L_2$, we define

$$\langle f, g\rangle := \int_{\mathbb{R}^n} f(t)\overline{g(t)}\, dt.$$

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For any $f \in \mathcal{S}'$ and $g \in \mathcal{S}$, we define $\langle f, g\rangle := f(\overline{g})$ with the usual extensions, by means of duality, to the various subspaces of $\mathcal{S}'$.

Given $f : \mathbb{R}^n \to \mathbb{R}$, we denote

$$f_{j,k} := 2^{j\frac{n}{2}} f(2^j \cdot -k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^n.$$

Also, we use the notation χ for the characteristic function of the unit cube $[0, 1)^n$.

Throughout the paper, c stands for a generic constant that may change with every occurrence. We use the notation $a \lesssim b$ to mean that there is a constant $c > 0$ such that $a \le cb$. We use the notation $a \approx b$ to denote two quantities that satisfy $c_1 a \le b \le c_2 a$, for some positive constants. The specific dependence of the constants c_1, c_2 on the problem's parameters is explained in the text, whenever such an explanation is required.

1.3. The performance of piecewise-constant framelets. Let Ψ be a finite subset of $L_2(\mathbb{R}^n)$. The *wavelet system* generated by the *mother wavelets* Ψ is the family

$$X(\Psi) := \{\psi_{j,k} :\ \psi \in \Psi,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}.$$

The *analysis operator* is defined as

$$T^*_{X(\Psi)} : f \mapsto (\langle f, x\rangle)_{x\in X(\Psi)};$$

the entries of $T^*_{X(\Psi)} f$ are the *wavelet coefficients* of f (with respect to the system $X(\Psi)$.) The system $X(\Psi)$ is a *frame* if the analysis operator is bounded above and below, viz., if there exist two positive constants A, B such that

$$A\, \|f\|^2_{L_2(\mathbb{R}^n)} \le \sum_{x\in X(\Psi)} |\langle f, x\rangle|^2 \le B\, \|f\|^2_{L_2(\mathbb{R}^n)}, \quad \text{for all } f \in L_2(\mathbb{R}^n). \tag{1.1}$$

The sharpest constants A and B are called the *frame bounds*. $X(\Psi)$ is a *Bessel system* if $T^*_{X(\Psi)}$ is bounded, i.e., the right-hand side of (1.1) is valid.

We are interested in wavelet frames that are derived from a *multiresolution analysis* (MRA) ([Ma],[RS1],[DHRS]). One begins with the selection of a function $\phi \in L_2(\mathbb{R}^n)$. With ϕ in hand, one defines

$$V_0 := V_0(\phi)$$

to be the closed linear span of the shifts of ϕ, i.e., V_0 is the smallest closed subspace of $L_2(\mathbb{R}^n)$ that contains $E(\phi) := \{\phi(\cdot - k) : k \in \mathbb{Z}^n\}$. Then, with D the operator of dyadic dilation:

$$(Df)(t) := 2^{\frac{n}{2}} f(2t),$$

one sets

$$V_j := V_j(\phi) := D^j V_0(\phi), \quad j \in \mathbb{Z}.$$

The primary condition of the MRA setup is that the $(V_j)_j$ sequence is nested:

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots. \tag{1.2}$$

Whenever this condition holds, one refers to ϕ as a *refinable function.* In addition, one requires that the union $\cup_j V_j$ is dense in $L_2(\mathbb{R}^n)$. However, if ϕ is compactly supported and $\widehat{\phi}(0) \neq 0$, the density condition always holds. The simplest example of a refinable function is the support function χ of the unit cube:

DEFINITION 1.1. *Let $(V_j)_j$ be the MRA associated with χ. A wavelet system $X(\Psi)$ is said to be* **piecewise-constant** *if $\Psi \subset V_1$. If, in addition, the system $X(\Psi)$ is a frame, we refer to it, as well as to each of its elements, as a* **piecewise-constant framelet** *(PCF).*
The classical Haar (orthonormal) wavelet is clearly a special case of a PCF.

Next, we illustrate the way the "performance" of a wavelet frame $X(\Psi)$ may be graded, and use the L_2-setup to this end. For $\alpha > 0$, let $W_2^\alpha(\mathbb{R}^n)$ be the usual Sobolev space. We would like first the representation to satisfy

$$\|T^*_{X(\Psi)} f\|_{\ell_2(\alpha)} \leq A_\alpha \|f\|_{W_2^\alpha(\mathbb{R}^n)}, \quad \forall f \in W_2^\alpha(\mathbb{R}^n). \tag{1.3}$$

Here,

$$\|T^*_{X(\Psi)} f\|_{\ell_2(\alpha)} := \left(\sum_{\psi,j,k} \left(1 + 2^{2j\alpha}\right) |\langle f, \psi_{j,k}\rangle|^2 \right)^{\frac{1}{2}}.$$

The supremum

$$s_J := \sup\{\alpha \geq 0 : \quad X(\Psi) \text{ satisfies (1.3) for the given } \alpha\},$$

is one way to quantify the "performance-grade" of $X(\Psi)$. Since the inequality (1.3) is the counterpart of the Jackson-type inequalities in Approximation Theory, we refer to the above s_J as the *Jackson-type performance* of $X(\Psi)$. It is known that the essential condition Ψ needs to satisfy for having "performance-grade" s_J is that each $\psi \in \Psi$ has s_J *vanishing moments* :

$$\widehat{\psi} = O(|\cdot|^{s_J}) \text{ near the origin, } \forall \psi \in \Psi.$$

Another way to measure the performance of $X(\Psi)$ is to insist that, in addition to (1.3), the inverse inequality holds as well:

$$\|T^*_{X(\Psi)} f\|_{\ell_2(\alpha)} \geq B_\alpha \|f\|_{W_2^\alpha(\mathbb{R}^n)}, \quad \forall f \in W_2^\alpha(\mathbb{R}^n). \tag{1.4}$$

We denote

$$s_B := \sup\{\alpha \geq 0 : \quad X(\Psi) \text{ satisfies (1.3) and (1.4) for the given } \alpha\}.$$

The inequality (1.4) is the counterpart of the Bernstein-type inequalities in Approximation Theory, and therefore we refer to the above s_B as the *Bernstein-type performance* of $X(\Psi)$. Obviously, $s_B \leq s_J$, and usually strict inequality holds. The value of s_B is not connected directly to any easy-to-check property of the system $X(\Psi)$. As a matter of fact, the value of s_B is related to the smoothness of the *dual frame* $X(\Psi^{\mathrm{d}})$, which we now introduce.

First, one defines a map $\Psi \ni \psi \mapsto \psi^{\mathrm{d}} \in L_2(\mathbb{R}^n)$, and extends it naturally to $X(\Psi)$ (i.e., $(\psi_{j,k})^{\mathrm{d}} := (\psi^{\mathrm{d}})_{j,k}$). Assume that $X(\Psi^{\mathrm{d}})$ with $\Psi^{\mathrm{d}} := \{\psi^{\mathrm{d}} : \psi \in \Psi\}$ is also a frame. The frame $X(\Psi^{\mathrm{d}})$ is then said to be *dual* to $X(\Psi)$ if one has the *perfect reconstruction property* :

$$f = T_{X(\Psi^{\mathrm{d}})} T^*_{X(\Psi)} f = \sum_{x \in X(\Psi)} \langle f, x \rangle x^{\mathrm{d}}, \quad f \in L_2(\mathbb{R}^n).$$

Here, $T_{X(\Psi)}$ is the *synthesis operator*:

$$T_{X(\Psi)} : \mathbb{C}^{X(\Psi)} \ni a \mapsto \sum_{x \in X(\Psi)} a(x) x.$$

Thus, one strives to build wavelet frames that have a high number of vanishing moments, and can be associated with smooth dual frames. This brings us to the question of how wavelet systems are constructed. The most general recipe in this regard is known as the Oblique Extension Principle (OEP, [DHRS]). However, in this paper we will need its special, and simpler, case, the Unitary Extension Principle (UEP). Both lead to the simultaneous construction of a frame and its dual frame. We describe now the UEP.

The refinability assumption on the function ϕ is equivalent to the condition that

$$\widehat{\phi}(2\cdot) = \tau \widehat{\phi},$$

for some 2π-periodic function τ, called the *refinement mask.* Let us assume for simplicity that τ is a trigonometric polynomial. The assumption that $\Psi := \{\psi_1, \ldots, \psi_L\} \subset V_1(\phi)$ amounts to the existence of 2π-periodic functions (=:*wavelet masks*) $(\tau_i)_{i=1}^L$ such that

$$\widehat{\psi}_i(2\cdot) = \tau_i \widehat{\phi}, \quad i = 1, \cdots, L.$$

Again, we assume for simplicity that $(\tau_i)_{i=1}^L$ are trigonometric polynomials. Next, let us assume that the dual refinable function ϕ^{d} has a trigonometric polynomial refinement mask τ^{d}. The assumption that $\Psi^{\mathrm{d}} := \{\psi_1^{\mathrm{d}}, \ldots, \psi_L^{\mathrm{d}}\} \subset V_1(\phi^{\mathrm{d}})$ amounts to the existence of 2π-periodic functions $(\tau_i^{\mathrm{d}})_{i=1}^L$ such that

$$\widehat{\psi}_i^{\mathrm{d}}(2\cdot) = \tau_i^{\mathrm{d}} \widehat{\phi}^{\mathrm{d}}, \quad i = 1, \cdots, L.$$

Again, we assume that the masks (τ_i^{d}) are trigonometric polynomials.

Suppose now that the two systems $X(\Psi)$ and $X(\Psi^{\mathrm{d}})$ are known to be, each, a Bessel system, and they satisfy the Mixed Unitary Extension Principle (MUEP) :

$$\overline{\tau}\tau^{\mathrm{d}}(\cdot + \mu) + \sum_{i=1}^{L} \overline{\tau}_i \tau_i^{\mathrm{d}}(\cdot + \mu) = \begin{cases} 1, & \mu = 0, \\ 0, & \mu \in \{0, \pi\}^n \backslash 0, \end{cases} \tag{1.5}$$

and $\widehat{\phi}(0) = \widehat{\phi}^{\mathrm{d}}(0) = 1$. Then $X(\Psi)$ and $X(\Psi^{\mathrm{d}})$ form a pair of a wavelet frame and a dual wavelet frame [RS2]. We refer then to the pair $(X(\Psi), X(\Psi^{\mathrm{d}}))$ as a (UEP) bi-framelet.

In §2, we explore the function space characterizations that are provided by PCFs. As an illustration, we list here our result for the special case of the Sobolev spaces $W_2^{\alpha}(\mathbb{R}^n)$, $\alpha > 0$. The result follows from Theorems 2.12 and 2.13, and is essentially known, at least for the case of when the frames in the bi-framelet pair are Riesz bases and not only frames (in this case, the pair is more customarily referred to as a *bi-orthogonal* pair.)

THEOREM 1.2. *Suppose that* $(X(\Psi), X(\Psi^{\rm d}))$ *is a bi-framelet, and both* $X(\Psi)$ *and* $X(\Psi^{\rm d})$ *are PCFs. Then* $s_J = 1$ *and* $s_B = 0.5$.

In §3, we describe two PCF constructions that are valid in all spatial dimensions and are, both, extremely local: Perhaps as local as any wavelet construction can be. One of the constructions is of a bi-orthogonal system (hence uses $L = 2^n - 1$ mother wavelets), while the other, closely related, one is an honest PCF (and employs $L = 2^n$ mother wavelets). We use then the results of §2 to identify the performance of the two systems. Their Jackson-type performance is proved to be same, while the redundant system is proved to yield a higher Bernstein-performance grade: its s_B equals 1, too.

2. Characterization of Besov spaces using PCFs.

2.1. Besov spaces. We recall first (one of) the (equivalent) definition(s) of Besov spaces [T]. Let $\varphi \in \mathcal{S}$ be such that

$$
\begin{aligned}
&\operatorname{supp} \widehat{\varphi} \subset \{\frac{1}{2} \le |\omega| \le 2\}, \\
&|\widehat{\varphi}(\omega)| \ge c > 0, \quad \frac{3}{5} \le |\omega| \le \frac{5}{3}, \\
&|\widehat{\varphi}(\omega)|^2 + |\widehat{\varphi}(\frac{\omega}{2})|^2 = 1, \quad 1 < |\omega| < 2.
\end{aligned} \tag{2.1}
$$

Let $\varphi_j := 2^{jn}\varphi(2^j\cdot)$, for $j \in \mathbb{Z}$.

For $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$, the (homogeneous) Besov space $\dot{B}^s_{pq} := \dot{B}^s_{pq}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that

$$
\|f\|_{\dot{B}^s_{pq}(\mathbb{R}^n)} := \left(\sum_{j\in\mathbb{Z}} (2^{js}\|\varphi_j * f\|_{L_p(\mathbb{R}^n)})^q \right)^{\frac{1}{q}} < \infty,
$$

with the usual modification for $q = \infty$.

In [FJ1], M. Frazier and B. Jawerth showed that the convolution operator in the above definition of $\dot{B}^s_{pq}$ can be discretized. In order to present their result, we will need the discrete analog, $\dot{b}^s_{pq}$, of the Besov space which is defined as the space of all sequences $h := (h(j,k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n)$ that satisfy

$$
\|h\|_{\dot{b}^s_{pq}} := \left(\sum_{j\in\mathbb{Z}} \left(2^{-jn} \sum_{k\in\mathbb{Z}^n} |2^{j(s+\frac{n}{2})} h(j,k)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty.
$$

PROPOSITION 2.1. *Let* $\varphi \in \mathcal{S}$ *be as in (2.1). If* $f \in \mathcal{S}'/\mathcal{P}$, *then*

$$
f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle \varphi_{j,k}
$$

in the sense of $\mathcal{S}'/\mathcal{P}$. *Moreover,*

$$
\|f\|_{\dot{B}^s_{pq}(\mathbb{R}^n)} \approx \|T^*_{X(\varphi)} f\|_{\dot{b}^s_{pq}} (= \|(\langle f, \varphi_{j,k}\rangle : j \in \mathbb{Z}, k \in \mathbb{Z}^n)\|_{\dot{b}^s_{pq}}).
$$

For $s > 0$, $1 \le p \le \infty$ and $0 < q \le \infty$, we define the inhomogeneous Besov space $B^s_{pq} := B^s_{pq}(\mathbb{R}^n)$ to be the set of all $f \in \mathcal{S}'$ such that $\|f\|_{B^s_{pq}(\mathbb{R}^n)} := \|f\|_{L_p(\mathbb{R}^n)} + \|f\|_{\dot{B}^s_{pq}(\mathbb{R}^n)} < \infty$.

We note that many of the traditional smoothness spaces can be captured by choosing suitably the parameters in a Besov space. The L_2-space is $\dot{B}^0_{22}$. The Sobolev space W^s_2, $s > 0$, is B^s_{22}. Also, $B^1_{\infty,\infty}$ is the Zygmund space, while, more generally, for $s > 0$, $B^s_{\infty\infty}$ is the Hölder space.

2.2. Auxiliary results. We develop in this subsection the technical backbone for the PCF function space characterizations. The main results on this subject are proved in the next subsection. We start with the definition of a regularity class :

Definition 2.2. *Let $\gamma > 0$. We define $\mathcal{R}^0_\gamma := \mathcal{R}^0_\gamma(\mathbb{R}^n)$ to be the set of all functions f such that*

$$|f(t)| \lesssim (1 + |t|)^{-\gamma}.$$

For $0 < \beta < 1$, we define $\mathcal{R}^\beta_\gamma$ to be the set of all functions $f \in \mathcal{R}^0_\gamma$, such that

$$|f(z) - f(t)| \lesssim |z - t|^\beta \sup_{|u| \le |z-t|} (1 + |u - t|)^{-\gamma}, \quad |z - t| \le 3.$$

The set of all the compactly supported functions in $\mathcal{R}^\beta_\gamma$ is denoted by $\mathcal{R}^\beta$ (and is trivially independent of γ).

Throughout the entire subsection, we assume that $\xi \in L_2(\mathbb{R}^n)$ is a piecewise-constant mother wavelet (more precisely, a finite linear combination of integer translates of $\chi(2\cdot)$) with one (or more) vanishing moment(s), and that η is a function with one (or more) vanishing moment(s), satisfying $\eta \in \mathcal{R}^\beta_\gamma$ for all $0 \le \beta < 1$ and for all $\gamma \in \mathbb{N}$. We let

$$\mathbf{M} := (\mathbf{M}_{j,l}(k, m) := \mathbf{M}(j, k; l, m) : j, l \in \mathbb{Z},\ k, m \in \mathbb{Z}^n),$$

with $\mathbf{M}(j, k; l, m) := \delta_{j,k;l,m} \langle \xi_{j,k}, \eta_{l,m} \rangle$, $\delta_{j,k;l,m} \in \{\pm 1\}$. Our objective here is to prove that $\mathbf{M}$, as well as its adjoint operator $\mathbf{M}^*$, are well-defined bounded endomorphisms of $\dot{b}^s_{pq}$ for suitable choices of s, p, and q. To this end, we recall two pertinent results from [HR]:

Proposition 2.3. *Let $j, l \in \mathbb{Z}$, $s \in \mathbb{R}$, $0 < p \le \infty$ and*

$$\mu := \frac{n}{\min\{1, p\}}.$$

Suppose that there exist constants $\gamma > \mu$ and $\beta \in \mathbb{R}$ such that for all $k, m \in \mathbb{Z}^n$,

$$|\mathbf{A}_{j,l}(k, m)| \lesssim \frac{2^{(l-j)(s+\frac{n}{2})} 2^{-|l-j|\beta}}{2_+^{(l-j)\mu}} \left(1 + \frac{|2^{l-j}k - m|}{2_+^{l-j}}\right)^{-\gamma}, \quad 2^a_+ := \max\{2^a, 1\}.$$

Then we have, for all $0 < p \le \infty$,

$$\|\mathbf{A}_{j,l}\|_p := \|\mathbf{A}_{j,l}\|_{\ell_p(\mathbb{Z}^n) \to \ell_p(\mathbb{Z}^n)} \lesssim 2^{(l-j)(s+\frac{n}{2}-\frac{n}{p})} 2^{-|l-j|\beta}.$$

Proposition 2.4. *Let $s \in \mathbb{R}$ and $0 < p \le \infty$. Let $\mathbf{A}$ be a complex-valued matrix whose rows and columns are indexed by $\mathbb{Z} \times \mathbb{Z}^n$:*

$$\mathbf{A} := (\mathbf{A}_{j,l}(k, m) := \mathbf{A}(j, k; l, m) : j, l \in \mathbb{Z}, k, m \in \mathbb{Z}^n).$$

Suppose that there exists a constant $\varepsilon := \varepsilon(p,s) > 0$ *such that, for all* j,l,

$$\|\mathbf{A}_{j,l}\|_p \lesssim 2^{(l-j)(s+\frac{n}{2}-\frac{n}{p})}2^{-|l-j|\varepsilon}.$$

Then $\mathbf{A}$ *is a bounded endomorphism of* $\dot{b}^s_{pq}$ *for all* $0 < q \le \infty$.

We further need a result from [FJ2]. We actually list below a special and simplified version of that result which suffices for our purposes.

PROPOSITION 2.5. *Let* $j \le l$ *and* $\gamma > n+1$. *Suppose that* $\theta, \zeta \in \mathcal{R}^0_\gamma$. *Then*

$$|\langle \theta_{j,k}, \zeta_{\,l,m}\rangle| \lesssim 2^{-(l-j)\frac{n}{2}} \left(1 + \frac{|2^{l-j}k - m|}{2^{l-j}}\right)^{-\gamma}. \tag{2.2}$$

If, in addition, ζ *has one vanishing moment and* $\theta \in \mathcal{R}^\beta_\gamma$ *for some* $0 < \beta < 1$, *then*

$$|\langle \theta_{j,k}, \zeta_{\,l,m}\rangle| \lesssim 2^{-(l-j)(\beta+\frac{n}{2})} \left(1 + \frac{|2^{l-j}k - m|}{2^{l-j}}\right)^{-\gamma}.$$

Finally, we need the following simple result (see e.g. [K]) :

PROPOSITION 2.6. *Let* $l \in \mathbb{Z}$. *If* $k \in \mathbb{Z}^n$ *and* $\gamma > n$, *then*

$$\sum_{m\in\mathbb{Z}^n} \left(1 + \frac{|2^l k - m|}{2^l_+}\right)^{-\gamma} \lesssim 2^{ln}_+.$$

In the rest of this subsection, the letter γ is used for a suitably large integer. More precisely, given $0 < p \le \infty$, one can choose γ to be any integer $> \frac{n}{\min\{1,p\}} + n + 1$.

Our immediate objective is to provide estimates on the p-norm of the operator $\mathbf{M}_{j,l}$. To this end, we observe that when $j \ge l$, the magnitude of $|\mathbf{M}_{j,l}(k,m)| = |\langle \xi_{j,k}, \eta_{l,m}\rangle|$ is governed by the vanishing moment of ξ and the smoothness of η. Since η is (minimally) smooth, the single vanishing moment of ξ delivers to us the bound we need:

LEMMA 2.7. *Let* $s < 1$ *and* $j \ge l$. *Then we have, with* $\varepsilon_1 := \varepsilon_1(s) > 0$, *for all* $0 < p \le \infty$,

$$\|\mathbf{M}_{j,l}\|_p \lesssim 2^{(l-j)(s+\frac{n}{2}-\frac{n}{p})}2^{-|l-j|\varepsilon_1}. \tag{2.3}$$

Proof. For any fixed $s < 1$, we choose u so that $\max\{s, 0\} < u < 1$. From Proposition 2.5 (for $\theta := \eta$, $\zeta := \xi$ and $\beta := u$), we get, with $\varepsilon_1 := u - s > 0$,

$$\begin{aligned} |\mathbf{M}_{j,l}(k,m)| &\lesssim 2^{-(j-l)(u+\frac{n}{2})} \left(1 + \frac{|2^{j-l}m - k|}{2^{j-l}}\right)^{-\gamma} \\ &= 2^{(l-j)(s+\frac{n}{2})}2^{-|l-j|\varepsilon_1} \left(1 + |2^{l-j}k - m|\right)^{-\gamma}. \end{aligned} \tag{2.4}$$

Thus by Proposition 2.3 (for $\beta := \varepsilon_1$), we obtain (2.3). □

In the opposite case, when $l > j$, the size of $|\mathbf{M}_{j,l}(k,m)| = |\langle \xi_{j,k}, \eta_{l,m}\rangle|$ is governed by the moment condition of η and the smoothness of ξ. Since ξ is not so smooth, we need to be a bit more careful in arguing this case. We use, to this end, the fact that ξ is a linear combination of a finite number of translates of $\chi(2\cdot)$, and that η decays rapidly and has one vanishing moment.

LEMMA 2.8. *Let $l > j$. Then*

$$\|\mathbf{M}_{j,l}\|_p \lesssim 2^{(l-j)(-\frac{n}{2})} \quad \textit{for} \quad 0 < p \le 1, \quad \textit{and} \quad \|\mathbf{M}_{j,l}\|_\infty \lesssim 2^{(l-j)(\frac{n}{2}-1)}.$$

Therefore, by interpolation, we get for $1 \le p \le \infty$

$$\|\mathbf{M}_{j,l}\|_p \lesssim 2^{(l-j)(\frac{n}{2}-\frac{n}{p})} 2^{-(l-j)(1-\frac{1}{p})}.$$

Proof. We first note that by (2.2) of Proposition 2.5 (for $\theta := \xi$ and $\zeta := \eta$)

$$\begin{aligned}|\mathbf{M}_{j,l}(k,m)| = |\langle \xi_{j,k}, \eta_{\,l,m}\rangle| &\lesssim 2^{-(l-j)\frac{n}{2}} \left(1 + \frac{|2^{l-j}k - m|}{2^{l-j}}\right)^{-\gamma} \\ &= \frac{2^{(l-j)\frac{n}{2}} 2^{-|l-j|(n-\mu)}}{2^{(l-j)\mu}} \left(1 + \frac{|2^{l-j}k - m|}{2^{l-j}}\right)^{-\gamma},\end{aligned}$$

where $\mu := \frac{n}{\min\{1,p\}}$. Thus by Proposition 2.3 (for $\beta := n - \mu$ and $s := 0$), we obtain, for all $0 < p \le \infty$,

$$\|\mathbf{M}_{j,l}\|_p \lesssim 2^{(l-j)(\frac{n}{2}-\frac{n}{p})} 2^{-|l-j|(n-\mu)}.$$

In particular, we have

$$\|\mathbf{M}_{j,l}\|_p \lesssim 2^{(l-j)(-\frac{n}{2})}, \quad \text{for} \quad 0 < p \le 1, \quad \text{and} \quad \|\mathbf{M}_{j,l}\|_\infty \lesssim 2^{(l-j)\frac{n}{2}}.$$

Note that this estimation is good enough for $\|\mathbf{M}_{j,l}\|_p$, $0 < p \le 1$, but not for $\|\mathbf{M}_{j,l}\|_\infty$. To improve the estimation of $\|\mathbf{M}_{j,l}\|_\infty$, we compute $|\langle \xi_{j,k}, \eta_{l,m}\rangle|$ directly. Without loss, we can replace $\xi_{j,k}$ by $\chi_{j,k}$. That is, we estimate $\sum_{m\in\mathbb{Z}^n} |\langle \chi_{j,k}, \eta_{l,m}\rangle|$. In fact, for later use, we look at the more general expression

$$\sum_{m\in\mathbb{Z}^n} |\langle \chi_{j,k}, \eta_{l,m}\rangle|^\alpha, \quad \alpha > 0.$$

We first note that, with $Q_0 := [0,1)^n$,

$$\begin{aligned}\langle \chi_{j,k}, \eta_{l,m}\rangle = \langle \chi_{j-l,0}, \eta_{0,m-2^{l-j}k}\rangle &= 2^{(j-l)\frac{n}{2}} \int_{2^{j-l}z\in Q_0} \overline{\eta(z + 2^{l-j}k - m)}\,dz \\ &= 2^{(j-l)\frac{n}{2}} \int_{z\in 2^{l-j}(k+Q_0)} \overline{\eta(z-m)}\,dz.\end{aligned}$$

Therefore, with $Q := 2^{l-j}(k + Q_0)$, we have

$$\sum_{m\in\mathbb{Z}^n} |\langle \chi_{j,k}, \eta_{l,m}\rangle|^\alpha = 2^{(j-l)\frac{n}{2}\alpha} \sum_{m\in\mathbb{Z}^n} \left|\int_{z\in Q} \overline{\eta(z-m)}\,dz\right|^\alpha.$$

Now, with ∂Q being the boundary of Q, we define Ω_N, $N \in \mathbb{N}_0 := \mathbb{N} \cup 0$, as follows:

$$\begin{aligned}\Omega_0 &:= \{m \in \mathbb{Z}^n : \operatorname{dist}(m, \partial Q) \le 1\}, \\ \Omega_N &:= \{m \in \mathbb{Z}^n : 2^{N-1} < \operatorname{dist}(m, \partial Q) \le 2^N\}, \quad N \in \mathbb{N}.\end{aligned}$$

Then, we have

$$\sum_{m\in\mathbb{Z}^n} \left|\int_{z\in Q} \overline{\eta(z-m)}\,dz\right|^\alpha = \sum_{N\ge 0} \sum_{m\in\Omega_N} \left|\int_{z\in Q} \overline{\eta(z-m)}\,dz\right|^\alpha.$$

Now we claim that if $m \in \Omega_N$, then

$$\left|\int_{z\in Q} \overline{\eta(z-m)}dz\right| \lesssim 2^{-N(\gamma-n)}.$$

When $N = 0$ the estimation is trivial from the fact that L_1-norm of η is finite, so we assume that $N \in \mathbb{N}$. If $m \in \Omega_N$ is outside Q, then for every $z \in Q$, $|z-m| \geq \operatorname{dist}(m, \partial Q) > 2^{N-1}$, thus we get

$$\left|\int_{z\in Q} \overline{\eta(z-m)}dz\right| \leq \int_{z\in Q-m} |\overline{\eta(z)}|dz \leq \int_{|z|\geq 2^{N-1}} \frac{c}{(1+|z|)^{\gamma}}dz \lesssim 2^{-N(\gamma-n)}.$$

If $m \in \Omega_N$ is inside Q, we first recall that $\int \eta(z)dz = 0$, thus by the same argument as above, we get with $Q^c := \mathbb{R}^n \backslash Q$,

$$\begin{aligned}\left|\int_{z\in Q} \overline{\eta(z-m)}dz\right| = \left|\int_{z\in Q^c} \overline{\eta(z-m)}dz\right| &\leq \int_{z\in Q^c-m} |\overline{\eta(z)}|dz \\ &\leq \int_{|z|\geq 2^{N-1}} \frac{c}{(1+|z|)^{\gamma}}dz \lesssim 2^{-N(\gamma-n)}.\end{aligned}$$

Since $\#\Omega_N \lesssim (2^N)^n 2^{(l-j)(n-1)}$, we obtain, with $c := c(n,\gamma)$,

$$\sum_{m\in\mathbb{Z}^n} \left|\int_{z\in Q} \overline{\eta(z-m)}dz\right|^{\alpha} \leq c \sum_{N\geq 0} \left(2^{(l-j)(n-1)} 2^{-N(\alpha(\gamma-n)-n)}\right) \lesssim 2^{(l-j)(n-1)},$$

provided that $\gamma > n(1+\frac{1}{\alpha})$. Therefore we get

$$\sup_{k\in\mathbb{Z}^n} \sum_{m\in\mathbb{Z}^n} |\mathbf{M}_{j,l}(k,m)|^{\alpha} \lesssim 2^{(j-l)\frac{n\alpha}{2}} 2^{(l-j)(n-1)} = 2^{(l-j)(n(1-\frac{\alpha}{2})-1)}. \tag{2.5}$$

Using this estimate for $\alpha = 1$, we obtain the desired bound for $\|\mathbf{M}_{j,l}\|_\infty$:

$$\|\mathbf{M}_{j,l}\|_\infty \leq \sup_{k\in\mathbb{Z}^n} \sum_{m\in\mathbb{Z}^n} |\mathbf{M}_{j,l}(k,m)| \lesssim 2^{(l-j)(\frac{n}{2}-1)}. \qquad \square$$

From the above lemma, we get that for $l > j$, $0 < p \leq \infty$ and $s \in \mathbb{R}$, there exists $\varepsilon_2 > 0$ such that

$$\|\mathbf{M}_{j,l}\|_p \lesssim 2^{(l-j)(s+\frac{n}{2}-\frac{n}{p})} 2^{-|l-j|\varepsilon_2}, \qquad \text{for } \begin{cases} s > n(\frac{1}{p}-1), & \text{if } p \leq 1, \\ s > \frac{1}{p}-1, & \text{if } p \geq 1. \end{cases}$$

This bound, when combined with Proposition 2.4 and (2.3), leads to the following bound on $\mathbf{M}$:

(2.6) $\mathbf{M}$ is a bounded endomorphism of $\dot{b}^s_{pq}$ for $0 < p, q \leq \infty$ and s satisfying

$$\begin{cases} n(\frac{1}{p}-1) < s < 1, & \text{if } p \leq 1, \\ \frac{1}{p}-1 < s < 1, & \text{if } p \geq 1. \end{cases} \tag{2.7}$$

Our next objective is to establish complementary bounds on the operator $\mathbf{M}^*$. We start with the case $l \geq j$.

LEMMA 2.9. *Let $s \in \mathbb{R}$, $0 < p \le \infty$, and*

$$\lambda := n\left(\frac{1}{\min\{1,p\}} - 1\right) - s.$$

Suppose that $\lambda < 1$ and $l \ge j$. Then we have, with $\varepsilon_1 := \varepsilon_1(p,s) > 0$, for all $0 < p \le \infty$,

$$\|\mathbf{M}^*_{j,l}\|_p \lesssim 2^{(l-j)(s+\frac{n}{2}-\frac{n}{p})} 2^{-|l-j|\varepsilon_1}.$$

Proof. For any fixed $\lambda < 1$, we choose u so that $\max\{\lambda, 0\} < u < 1$. From Proposition 2.5 (for $\theta := \eta$, $\zeta := \xi$ and $\beta := u$), we get, with $\varepsilon_1 := u - \lambda > 0$,

$$\begin{aligned}|\mathbf{M}^*_{j,l}(k,m)| &\lesssim 2^{-(l-j)(u+\frac{n}{2})} \left(1 + \frac{|2^{l-j}k - m|}{2^{l-j}}\right)^{-\gamma} \\ &= \frac{2^{(l-j)(s+\frac{n}{2})} 2^{-|l-j|\varepsilon_1}}{2^{(l-j)(\lambda+n+s)}} \left(1 + \frac{|2^{l-j}k - m|}{2^{l-j}}\right)^{-\gamma}.\end{aligned}$$

Thus by Proposition 2.3 (for $\beta := \varepsilon_1$), we obtain the stated result. $\square$

From the above lemma, we get that for $l \ge j$, $0 < p \le \infty$ and $s \in \mathbb{R}$, there exists $\varepsilon_1 > 0$ such that

$$\|\mathbf{M}^*_{j,l}\|_p \lesssim 2^{(l-j)(s+\frac{n}{2}-\frac{n}{p})} 2^{-|l-j|\varepsilon_1}, \quad \text{for } \begin{cases} s > n(\frac{1}{p} - 1) - 1, & \text{if } p \le 1, \\ s > -1, & \text{if } p \ge 1. \end{cases} \tag{2.8}$$

We need also to estimate $\|\mathbf{M}^*_{j,l}\|_p$ when $j > l$. Using a similar argument to the one used in Lemma 2.8, we obtain the following result:

LEMMA 2.10. *For $j > l$ we have*

$$\|\mathbf{M}^*_{j,l}\|_p \lesssim 2^{(l-j)(\frac{n}{2}-\frac{n}{p}+\frac{1}{p})} \ \text{for } 0 < p \le 1, \quad \|\mathbf{M}^*_{j,l}\|_\infty \lesssim 2^{(l-j)\frac{n}{2}}.$$

Therefore, by interpolation, we get for $1 \le p \le \infty$

$$\|\mathbf{M}^*_{j,l}\|_p \lesssim 2^{(l-j)(\frac{n}{2}-\frac{n}{p})} 2^{(l-j)\frac{1}{p}}.$$

Proof. By (2.2) of Proposition 2.5 (for $\theta := \xi$ and $\zeta := \eta$), we have

$$\begin{aligned}|\mathbf{M}^*_{j,l}(k,m)| &\lesssim 2^{-(j-l)\frac{n}{2}} \left(1 + \frac{|2^{j-l}m - k|}{2^{j-l}}\right)^{-\gamma} \\ &= 2^{(l-j)\frac{n}{2}} \left(1 + |2^{l-j}k - m|\right)^{-\gamma}.\end{aligned}$$

Thus by Proposition 2.3 (for $\beta := 0$ and $s := 0$), we obtain, for all $0 < p \le \infty$,

$$\|\mathbf{M}^*_{j,l}\|_p \lesssim 2^{(l-j)(\frac{n}{2}-\frac{n}{p})}.$$

Note that this time we need to improve the estimation of $\|\mathbf{M}^*_{j,l}\|_p$, $0 < p \le 1$, not of $\|\mathbf{M}^*_{j,l}\|_\infty$. By (2.5) (with $\alpha := p$), we get

$$\|\mathbf{M}^*_{j,l}\|_p^p \le \sup_{m\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} |\mathbf{M}^*_{j,l}(k,m)|^p \lesssim 2^{(j-l)(n(1-\frac{p}{2})-1)} = 2^{(l-j)(\frac{n}{2}-\frac{n}{p}+\frac{1}{p})p}. \qquad \square$$

Thus, by the above lemma, we get that for $j > l$, $0 < p \le \infty$, $s \in \mathbb{R}$, and with $\varepsilon_2 := \frac{1}{p} - s$,

$$\|\mathbf{M}^*_{j,l}\|_p \lesssim 2^{(l-j)(s+\frac{n}{2}-\frac{n}{p})} 2^{-|l-j|\varepsilon_2}. \tag{2.9}$$

By combining (2.8) and (2.9), and by applying Proposition 2.4, we get the following:

(2.10) $\mathbf{M}^*$ is bounded endomorphism on $\dot{b}^s_{pq}$ for $0 < p, q \le \infty$ and s satisfying

$$\begin{cases} n(\frac{1}{p}-1) - 1 < s < \frac{1}{p}, & \text{if } p \le 1, \\ -1 < s < \frac{1}{p}, & \text{if } p \ge 1. \end{cases} \tag{2.11}$$

We now recall (2.6), which, together with (2.10), yields:

$\mathbf{M}$ and $\mathbf{M}^*$ are bounded endomorphisms on $\dot{b}^s_{pq}$ for $0 < p, q \le \infty$ and s satisfying

$$\begin{cases} n(\frac{1}{p}-1) < s < 1, & \text{if } p \le 1, \\ \frac{1}{p} - 1 < s < \frac{1}{p}, & \text{if } p \ge 1. \end{cases} \tag{2.12}$$

We also need the following related corollary:

Corollary 2.11. *Let $0 < p, q \le \infty$. If $s \in \mathbb{R}$ satisfies (2.7), then for every $h := (h(l,m) : l \in \mathbb{Z}, m \in \mathbb{Z}^n) \in \dot{b}^s_{pq}$,*

$$\sum_{l,m} |h(l,m)||\langle \xi, \eta_{l,m}\rangle| < \infty. \tag{2.13}$$

Also, if $s \in \mathbb{R}$ satisfies (2.11), then for every $h := (h(l,m) : l \in \mathbb{Z}, m \in \mathbb{Z}^n) \in \dot{b}^s_{pq}$,

$$\sum_{l,m} |h(l,m)||\langle \eta, \xi_{l,m}\rangle| < \infty. \tag{2.14}$$

Proof. We note that the sequence $(|\langle \xi, \eta_{l,m}\rangle|)_{l,m}$ comprises the ($j = k = 0$)-row of the matrix $\mathbf{M}$. Since (2.6) gives the boundedness of $\mathbf{M}$, the number $\mathbf{M}(h')(0,0)$ must be finite for every $h' \in \dot{b}^s_{pq}$. However, for the choice $h' := |h|$, we have that $\mathbf{M}(h')(0,0) = \sum_{l,m} |h(l,m)||\langle \xi, \eta_{l,m}\rangle|$, hence (2.13) is true.

Similarly, (2.14) is obtained by inspecting $\mathbf{M}^*$ instead of $\mathbf{M}$, and using (2.10) instead of (2.6). ∎

2.3. Characterizations of Besov spaces using PCFs. In this subsection, we obtain Besov space characterizations in terms of the wavelet coefficients of a piecewise-constant system by using the results obtained in the previous subsection. Throughout this subsection, we let $\varphi \in \mathcal{S}$ be a function satisfying the conditions in (2.1). We derive the characterization in two steps. The first step involves a Jackson-type inequality:

Theorem 2.12. *Let $0 < p, q \le \infty$, and let $s \in \mathbb{R}$ satisfy (2.7). Suppose that ψ is a piecewise-constant wavelet with one vanishing moment. Then*

$$\|T^*_{X(\psi)} f\|_{\dot{b}^s_{pq}} \lesssim \|f\|_{\dot{B}^s_{pq}(\mathbb{R}^n)}.$$

Remark. Following the discussion in [FJ2] and [HR], the expression $\langle f,\psi\rangle$, for $f\in\dot{B}^s_{pq}$ and a piecewise-constant ψ, is defined by

$$\langle f,\psi\rangle := \sum_{l,m}\langle\varphi_{l,m},\psi\rangle\langle f,\varphi_{l,m}\rangle.$$

The definition makes sense, since the sum in the right-hand side converges absolutely, by virtue of (2.13) of Corollary 2.11 (for $\xi:=\psi$ and $\eta:=\varphi$) and Proposition 2.1.

Proof of Theorem 2.12. By the previous remark, the expression $\langle f,\psi_{j,k}\rangle$ is well defined. That means the (suggestive) identity

$$T^*_{X(\psi)}f = (T^*_{X(\psi)}T_{X(\varphi)})T^*_{X(\varphi)}f$$

is valid for every $f\in\dot{B}^s_{pq}$. Since $T^*_{X(\psi)}T_{X(\varphi)}$ is bounded on $\dot{b}^s_{pq}$ (by (2.6), for $\xi:=\psi$ and $\eta:=\varphi$) and $T^*_{X(\varphi)}$ is a bijection from $\dot{B}^s_{pq}$ to $\dot{b}^s_{pq}$ by Proposition 2.1, we conclude that $T^*_{X(\psi)}:\dot{B}^s_{pq}\to\dot{b}^s_{pq}$ is bounded :

$$\|T^*_{X(\psi)}f\|_{\dot{b}^s_{pq}} \lesssim \|T^*_{X(\varphi)}f\|_{\dot{b}^s_{pq}} \approx \|f\|_{\dot{B}^s_{pq}(\mathbb{R}^n)},\quad f\in\dot{B}^s_{pq}. \qquad \square$$

We now state and prove the full-fledged characterization:

THEOREM 2.13. *Let $0<p,q\le\infty$, and let $s\in\mathbb{R}$ satisfy (2.12). Suppose that $(X(\Psi),X(\Psi^{\mathrm{d}}))$ is a bi-framelet, and both $X(\Psi)$ and $X(\Psi^{\mathrm{d}})$ are PCFs. Then for every $f\in\dot{B}^s_{pq}$, we have*

$$\sum_{\psi\in\Psi}\|T^*_{X(\psi)}f\|_{\dot{b}^s_{pq}} \approx \|f\|_{\dot{B}^s_{pq}(\mathbb{R}^n)}.$$

Proof. It was already proved in Theorem 2.12 that the expression $\langle f,\psi_{j,k}\rangle$ is well-defined for every $\psi\in\Psi$, $j\in\mathbb{Z}, k\in\mathbb{Z}^n$, and that $\|T^*_{X(\psi)}f\|_{\dot{b}^s_{pq}}\lesssim\|f\|_{\dot{B}^s_{pq}(\mathbb{R}^n)}<\infty$, for every $\psi\in\Psi$.

For the other direction, we prove that, for every $f\in\dot{B}^s_{pq}$,

$$\sum_{\psi\in\Psi}\sum_{l,m}\langle f,\psi_{l,m}\rangle\,\psi^{\mathrm{d}}_{l,m} = f, \tag{2.15}$$

in the sense of $\mathcal{S}'/\mathcal{P}$. To this end, let $\mathcal{S}_\infty := \{\eta\in\mathcal{S} : \int_{\mathbb{R}^n}t^\alpha\eta(t)dt=0, \forall\alpha\in\mathbb{N}^n_0\}$. We need to show that

$$\langle\sum_{\psi\in\Psi}\sum_{l,m}\langle f,\psi_{l,m}\rangle\,\psi^{\mathrm{d}}_{l,m},\eta\rangle = \langle f,\eta\rangle,\quad \forall\eta\in\mathcal{S}_\infty.$$

By (2.14) of Corollary 2.11 (for $\xi:=\psi^{\mathrm{d}}$), $T^*_{X(\psi)}f\in\dot{b}^s_{pq}$, and the definition of $\langle f,\psi_{l,m}\rangle$, we get

$$\begin{aligned}\langle\sum_{\psi\in\Psi}\sum_{l,m}\langle f,\psi_{l,m}\rangle\,\psi^{\mathrm{d}}_{l,m},\eta\rangle &= \sum_{\psi\in\Psi}\sum_{l,m}\langle f,\psi_{l,m}\rangle\langle\psi^{\mathrm{d}}_{l,m},\eta\rangle\\ &= \sum_{\psi\in\Psi}\sum_{l,m}\sum_{j,k}\langle f,\varphi_{j,k}\rangle\langle\varphi_{j,k},\psi_{l,m}\rangle\langle\psi^{\mathrm{d}}_{l,m},\eta\rangle.\end{aligned}$$

Next, we note that

$$\sum_{\psi\in\Psi}\sum_{l,m}\left(\sum_{j,k}|\langle f,\varphi_{j,k}\rangle||\langle\varphi_{j,k},\psi_{l,m}\rangle|\right)|\langle\psi^{\mathrm{d}}_{l,m},\eta\rangle|<\infty,$$

which follows from the facts that $(|\langle\psi_{j,k},\,\varphi_{l,m}\rangle|:j,l\in\mathbb{Z},k,m\in\mathbb{Z}^n)$ is bounded on $\dot b^s_{pq}$ by (2.6) (for $\xi:=\psi$, $\eta:=\varphi$) and that $(|\langle f,\varphi_{j,k}\rangle|)_{j,k}\in\dot b^s_{pq}$, and by Corollary 2.11 (for $\xi:=\psi^{\mathrm{d}}$). Thus we have

$$\begin{aligned}\langle\sum_{\psi\in\Psi}\sum_{l,m}\langle f,\psi_{l,m}\rangle\,\psi^{\mathrm{d}}_{l,m},\eta\rangle&=\sum_{j,k}\langle f,\varphi_{j,k}\rangle\sum_{\psi\in\Psi}\sum_{l,m}\langle\varphi_{j,k},\psi_{l,m}\rangle\langle\psi^{\mathrm{d}}_{l,m},\eta\rangle\\&=\sum_{j,k}\langle f,\varphi_{j,k}\rangle\langle\varphi_{j,k},\eta\rangle\\&=\langle f,\eta\rangle.\end{aligned}$$

For the second equality, we used that $\sum_{\psi\in\Psi}T_{X(\psi^{\mathrm{d}})}T^*_{X(\psi)}\varphi_{j,k}=\varphi_{j,k}$ in the sense of L_2, and thus in the sense of $\mathcal{S}'$. Finally, the last equality is due to the fact that $\sum_{j,k}\langle f,\varphi_{j,k}\rangle\varphi_{j,k}=f$ in the sense of $\mathcal{S}'/\mathcal{P}$ (cf. Proposition 2.1).

Now that (2.15) is verified, we use it to conclude that, for every $f\in\dot B^s_{pq}$,

$$\langle f,\varphi_{j,k}\rangle=\sum_{\psi\in\Psi}\sum_{l,m}\langle\psi^{\mathrm{d}}_{l,m},\varphi_{j,k}\rangle\langle f,\psi_{l,m}\rangle,\quad\forall j,k.$$

That is,

$$T^*_{X(\varphi)}f=\sum_{\psi\in\Psi}(T^*_{X(\varphi)}T_{X(\psi^{\mathrm{d}})})T^*_{X(\psi)}f.$$

Since, for each ψ^{d}, $T^*_{X(\varphi)}T_{X(\psi^{\mathrm{d}})}$ is bounded $\dot b^s_{pq}$ by (2.10) (for $\xi:=\psi^{\mathrm{d}}$ and $\eta:=\varphi$), we obtain

$$\|T^*_{X(\varphi)}f\|_{\dot b^s_{pq}}\lesssim\sum_{\psi\in\Psi}\|(T^*_{X(\varphi)}T_{X(\psi^{\mathrm{d}})})T^*_{X(\psi)}f\|_{\dot b^s_{pq}}\lesssim\sum_{\psi\in\Psi}\|T^*_{X(\psi)}f\|_{\dot b^s_{pq}},\quad f\in\dot B^s_{pq}.$$

Invoking Proposition 2.1, we obtain the stated result. ∎

The range of parameters (p,s) for which Theorem 2.12 and Theorem 2.13 hold is depicted in Fig. 1.

3. Extremely local PCFs. The classical Haar wavelet system is commonly considered to be very local in space. In this section, we construct two PCFs that, in high spatial dimensions, are either far more local than Haar (the first construction) or are as local as Haar while delivering better performance (the second construction). Both representations are computed and inverted very fast, as we now explain.

3.1. Extremely local PCFs: the algorithms. Let $\mathrm{E}:=\{0,1\}^n$. We begin with a sequence $y_0:\mathbb{Z}^n\to\mathbb{C}$, which is considered to be "the data at full resolution". We derive the MRA representation of the data iteratively:

$$y_{j-1}(k):=2^{-n}\sum_{\upsilon\in\mathrm{E}}y_j(2k+\upsilon),\quad j=0,-1,\dots,\quad k\in\mathbb{Z}^n.$$

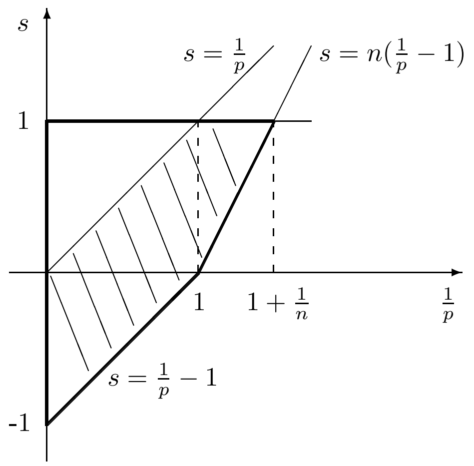

FIG. 1. *For (p, s) inside the polygon with thick boundary, we have the Jackson-type performance of Theorem 2.12. For (p, s) in smaller lined region, we have the Bernstein-type performance of Theorem 2.13.*

This resulted MRA $(y_j)_{j\le 0}$ is the MRA associated with χ, i.e., assuming $y_0(k) = \langle f, \chi_{0,k}\rangle$, $k \in \mathbb{Z}^n$, for some function f, it follows that

$$y_j(k) = 2^{j\frac{n}{2}}\langle f, \chi_{j,k}\rangle, \quad j < 0,\ k \in \mathbb{Z}^n.$$

(I) Bi-orthogonal construction.
Each y_j is associated with a sequence $d_j : \mathbb{Z}^n \to \mathbb{C}$ of detail coefficients, that are defined as follows:

Bi-orthogonal construction, decomposition:

$$d_j(2k + \upsilon) := y_j(2k + \upsilon) - y_j(2k), \quad k \in \mathbb{Z}^n,\ \upsilon \in \mathrm{E}.$$

The inversion (reconstruction) is iterative. At each iteration, it accepts y_{j-1} and d_j as its input and recovers y_j:

Bi-orthogonal construction, reconstruction: First, we compute $y_j(2k)$, $k \in \mathbb{Z}^n$ by:

$$y_j(2k) = y_{j-1}(k) - 2^{-n} \sum_{\upsilon\in \mathrm{E}\backslash 0} d_j(2k + \upsilon).$$

It is easy to see that this recovers correctly y_j at the even integers. The rest is trivial, since

$$y_j(2k + \upsilon) = d_j(2k + \upsilon) + y_j(2k), \quad k \in \mathbb{Z}^n,\ \upsilon \in \mathrm{E}\backslash 0. \qquad \square$$

Complexity. We measure complexity by counting the number of "operations" needed in order to fully derive y_{-1} and d_0 from y_0, and add the number of operations needed for the inversion. Here, we define "an operation" as the need to fetch an entry from some of our arrays/vectors. Thus, for example, computing one entry in y_{-1}

requires 2^n operations. Note that d_j vanishes on $2\mathbb{Z}^n$, hence that those coefficients can be ignored.

With that definition in mind, it is quite trivial to observe that the number of operation required to process the portion of y_0 that lies in a cube of lengthsize N is about $6 \times N^n$. This means that the cost of performing one complete cycle of decomposition/inversion is about 6 operations per one detail coefficient; in particular, this cost is independent of the spatial dimension n.

(II) Frame construction.
With $\mathbf{1} := (1 \ \cdots \ 1) \in \mathrm{E}$, the detail coefficients d_j are defined as follows:

Frame construction, decomposition:

$$d_j(2k) := y_j(2k-\mathbf{1}) - 2^{-n} \sum_{\upsilon \in \mathrm{E}\backslash 0} y_{j-1}(k+\upsilon-\mathbf{1}), \quad k \in \mathbb{Z}^n,$$
$$d_j(2k+\upsilon) := y_j(2k+\upsilon-\mathbf{1}) - y_j(2k-\mathbf{1}), \quad k \in \mathbb{Z}^n,\ \upsilon \in \mathrm{E}\backslash 0.$$

The inversion (reconstruction) is iterative. At each iteration, it accepts y_{j-1} and d_j as its input and recovers y_j:

Frame construction, reconstruction: First, we recover $y_j(2k-\mathbf{1})$, $k \in \mathbb{Z}^n$ by

$$y_j(2k-\mathbf{1}) = d_j(2k) + 2^{-n} \sum_{\upsilon \in \mathrm{E}} y_{j-1}(k+\upsilon-\mathbf{1}).$$

The rest is trivial, since

$$y_j(2k+\upsilon-\mathbf{1}) = d_j(2k+\upsilon) + y_j(2k-\mathbf{1}), \quad k \in \mathbb{Z}^n,\ \upsilon \in \mathrm{E}\backslash 0. \qquad \square$$

Complexity. With complexity defined as before, the only difference here is the need to compute d_j at the even integers. This adds about one operation per each detail coefficient. Switching between the two reconstruction algorithms does not affect complexity. Altogether, the cost here is about 7 operations per one detail coefficient.

3.2. Extremely local bi-orthogonal systems. We now provide the details of the bi-orthogonal wavelet system that underlies the first algorithm from the previous subsection.

We note that $\widehat{\chi}(2\cdot) = \tau\widehat{\chi}$ with

$$\tau(\omega) := \prod_{j=1}^{n} \left(\frac{1+e^{-i\omega_j}}{2}\right) = 2^{-n} \sum_{\upsilon \in \mathrm{E}} e_{-\upsilon}(\omega), \quad e_\upsilon : \omega \to e^{i\upsilon\cdot\omega}. \tag{3.1}$$

Let $\alpha_n := 2^{-n/2}$. For each $\upsilon \in \mathrm{E}\backslash 0$, we define τ_υ and $\tau_\upsilon^{\mathrm{d}}$ as

$$\tau_\upsilon := \alpha_n(e_{-\upsilon} - 1), \quad \tau_\upsilon^{\mathrm{d}} := \alpha_n(e_{-\upsilon} - \tau),$$

and

$$\widehat{\psi}_\upsilon(2\cdot) = \tau_\upsilon \widehat{\chi}, \quad \widehat{\psi}_\upsilon^{\mathrm{d}}(2\cdot) = \tau_\upsilon^{\mathrm{d}} \widehat{\chi}.$$

That is,

$$\psi_\upsilon = \alpha_n 2^n \left(\chi(2\cdot-\upsilon) - \chi(2\cdot)\right), \quad \psi_\upsilon^{\mathrm{d}} = \alpha_n 2^n \left(\chi(2\cdot-\upsilon) - 2^{-n}\chi\right).$$

Let $X(\Psi)$ and $X(\Psi^{\mathrm{d}})$ be the wavelet systems with mother wavelets $\Psi := \{\psi_\upsilon : \upsilon \in \mathrm{E}\backslash 0\}$ and dual mother wavelets $\Psi^{\mathrm{d}} := \{\psi^{\mathrm{d}}_\upsilon : \upsilon \in \mathrm{E}\backslash 0\}$. We note that each ψ_υ is supported on two cubes each of volume 2^{-n}. Considering the fact that each of the mother wavelets in the n-dimensional (and reasonably local) Haar wavelet system has the unit cube as its support, our Ψ is extremely local in high dimensions. In fact, even the convex hull of the support of ψ_υ is small: the sum of the volumes of the convex hulls of the supports of Ψ does not exceed $\frac{n+2}{2}$.

Next, we show that the two PCF systems $X(\Psi)$ and $X(\Psi^{\mathrm{d}})$ are in fact bi-orthogonal.

THEOREM 3.1. *Suppose that $X(\Psi)$ and $X(\Psi^{\mathrm{d}})$ are defined as above. Then $X(\Psi)$ and $X(\Psi^{\mathrm{d}})$ are Bessel systems, and $T^*_{X(\Psi)}T_{X(\Psi^{\mathrm{d}})} = \mathrm{Id}_{\ell_2(X(\Psi^{\mathrm{d}}))}$.*

Proof. Note that, for $X := X(\Psi)$ or $X := X(\Psi^{\mathrm{d}})$, $\|T^*_X\|^2_{L_2(\mathbb{R}^n)\to\ell_2(X)} = \|T^*_X T_X\|_2$, the norm of $T^*_X T_X$ as an endomorphism of $\ell_2(X)$. Thus, to see that $X(\Psi)$ is a Bessel system, we estimate $\|T^*_{X(\Psi)}T_{X(\Psi)}\|_2$. It is easy to see that $T^*_{X(\Psi)}T_{X(\Psi)}$ is block-diagonal, with each block being $\langle\psi_\upsilon, \psi_{\upsilon'}\rangle$, for $\upsilon, \upsilon' \in \mathrm{E}\backslash 0$. Since direct computation gives

$$\langle\psi_\upsilon, \psi_{\upsilon'}\rangle = \begin{cases} 2, & \upsilon = \upsilon', \\ 1, & \upsilon \neq \upsilon', \end{cases}$$

we need to find the spectrum of the matrix

$$\mathbf{I} + \mathbf{v}^t\mathbf{v},$$

where $\mathbf{I}$ is the identity matrix of size $2^n - 1$ and $\mathbf{v}$ is the row vector $[1 \ \cdots \ 1]$ of length $2^n - 1$. Thus we get $\|T^*_{X(\Psi)}T_{X(\Psi)}\|_2 = 2^n$, hence we see that $X(\Psi)$ is Bessel.

Similarly, since

$$\langle\psi^{\mathrm{d}}_\upsilon, \psi^{\mathrm{d}}_{\upsilon'}\rangle = \begin{cases} 1 - 2^{-n}, & \upsilon = \upsilon', \\ -2^{-n}, & \upsilon \neq \upsilon', \end{cases}$$

the value $\|T^*_{X(\Psi^{\mathrm{d}})}T_{X(\Psi^{\mathrm{d}})}\|_2$ is the spectral radius of the matrix

$$\mathbf{I} - 2^{-n}\mathbf{v}^t\mathbf{v},$$

which is 1. Thus $X(\Psi^{\mathrm{d}})$ is Bessel.

To verify that $T^*_{X(\Psi)}T_{X(\Psi^{\mathrm{d}})} = \mathrm{Id}_{\ell_2(X(\Psi^{\mathrm{d}}))}$, we use the fact that $T^*_{X(\Psi)}T_{X(\Psi^{\mathrm{d}})}$ is block-diagonal, too, with each block being $\langle\psi^{\mathrm{d}}_\upsilon, \psi_{\upsilon'}\rangle$, $\upsilon, \upsilon' \in \mathrm{E}\backslash 0$. Since direct computation gives

$$\langle\psi^{\mathrm{d}}_\upsilon, \psi_{\upsilon'}\rangle = \begin{cases} 1, & \upsilon = \upsilon', \\ 0, & \upsilon \neq \upsilon', \end{cases}$$

we obtain the bi-orthogonality. $\square$

Note that we computed in the proof the exact frame bounds for each system, viz., for every $f \in L_2(\mathbb{R}^n)$,

$$\|f\|^2_{L_2(\mathbb{R}^n)} \le \sum_{x\in X(\Psi)} |\langle f, x\rangle|^2 \le 2^n \, \|f\|^2_{L_2(\mathbb{R}^n)},$$

$$2^{-n} \, \|f\|^2_{L_2(\mathbb{R}^n)} \le \sum_{x^{\mathrm{d}}\in X(\Psi^{\mathrm{d}})} |\langle f, x^{\mathrm{d}}\rangle|^2 \le \|f\|^2_{L_2(\mathbb{R}^n)}.$$

The condition number of the basis $X(\Psi)$ is thus $2^{\frac{n}{2}}$. The dual basis, obviously, must have the same condition number.

Finally, the performance of the above bi-orthogonal system is according to Theorems 2.12 and 2.13.

3.3. Extremely local PCF : bi-frames. The bi-orthogonal piecewise-constant system that was constructed in the previous subsection performs as every system with piecewise-constant decomposition and reconstruction mother wavelets. We will now show that, by adding one additional mother wavelet to the construction, we can improve substantially the Bernstein-type performance (Theorem 2.13). The new system is no more bi-orthogonal, but a frame. The algorithms associated with this frame representation were detailed in §3.1. Here are the system details.

Let $\alpha_n := 2^{-n/2}$ and $\mathbf{1} := (1 \ \cdots \ 1) \in \mathrm{E}$. For each $\upsilon \in \mathrm{E}$, we define τ_υ as

$$\tau_0 := \alpha_n e_{\mathbf{1}}(1 - e_{\mathbf{1}}\tau(2\cdot)\tau), \quad \tau_\upsilon := \alpha_n e_{\mathbf{1}}(e_{-\upsilon} - 1), \ \ \upsilon \in \mathrm{E}\backslash 0, \tag{3.2}$$

and let $\Psi_0 := \{\psi_\upsilon\}_{\upsilon\in\mathrm{E}}$ where $\widehat{\psi}_\upsilon(2\cdot) = \tau_\upsilon\widehat{\chi}$. That is,

$$\psi_0 := \alpha_n 2^n \left(\chi(2\cdot+\mathbf{1}) - 2^{-2n}\chi((\cdot+\mathbf{1})/2)\right),$$
$$\psi_\upsilon := \alpha_n 2^n \left(\chi(2\cdot-\upsilon+\mathbf{1}) - \chi(2\cdot+\mathbf{1})\right), \quad \upsilon \in \mathrm{E}\backslash 0.$$

Note that $X(\Psi_0)$ is Bessel. Since each $\psi_\upsilon \in \Psi_0$ is a piecewise-constant with one vanishing moment, the fact that $X(\psi_\upsilon)$ is Bessel follows from Theorem 2.12 with $s := 0, p := q := 2$ (noting that $\dot{B}^0_{22} = L_2$, and $\dot{b}^0_{22} = \ell_2(\mathbb{Z}\times\mathbb{Z}^n)$.)

We show next that the Bernstein-type inequality $\|f\|_{\dot{B}^s_{pq}} \lesssim \|T^*_{X(\Psi_0)}f\|_{\dot{b}^s_{pq}}$ is valid for a broader range of spaces. The improvement is particularly notable for large values of p (e.g., $p=\infty$).

Theorem 3.2. *Let $0 < p, q \le \infty$, and let $s \in \mathbb{R}$ satisfy (2.7). Then for every $f \in \dot{B}^s_{pq}$, we have*

$$\sum_{\psi\in\Psi_0} \|T^*_{X(\psi)}f\|_{\dot{b}^s_{pq}} \approx \|f\|_{\dot{B}^s_{pq}(\mathbb{R}^n)}.$$

Thus, remarkably, the Bernstein-type performance of the system $X(\Psi_0)$ is identical to its Jackson-type performance! The (p,s)-region for which Theorem 3.2 holds is depicted in Fig. 2.

Discussion. The addition of the mother wavelet ψ_0 is not only enhancing the performance of the representation, but also degrades its extreme locality: whereas the sum of the volumes of the supports of 2^n-1 mother wavelets $\{\psi_\upsilon\}_{\upsilon\in\mathrm{E}\backslash 0}$ never exceeds 2, ψ_0 alone is supported in a cube of volume 2^n. That said, the average volume of the supports of the wavelets in the current PCF is

$$(2^n+2)/2^n \approx 1,$$

which is on par with the tensor Haar system. However, the tensor Haar system performs only according to Theorem 2.13, hence is lagging in performance behind our current system. □

To prove Theorem 3.2, we first find a dual frame $X(\Psi_0^{\mathrm{d}})$ for $X(\Psi_0)$.

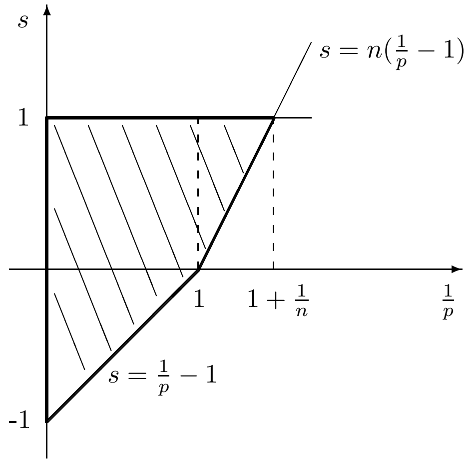

FIG. 2. *By Theorem 3.2, the Bernstein-type performance is now valid for the same range of parameters as the Jackson-type performance. Compare this graph with Fig. 1.*

LEMMA 3.3. *Let ξ be any trigonometric polynomial that satisfies $\xi(0) = 1$. We define the dual refinement mask τ^{d} and dual wavelet masks $(\tau_\upsilon^{\mathrm{d}})_{\upsilon\in\mathrm{E}}$ by*

$$\begin{aligned}\tau^{\mathrm{d}} &:= e_{-\mathbf{1}}\overline{\tau}(2\cdot)\tau\left(1+\xi\left[1-|\tau(2\cdot)|^2\right]\right),\\ \tau_0^{\mathrm{d}} &:= \alpha_n 2^n e_{\mathbf{1}}\tau(1-\xi|\tau(2\cdot)|^2),\\ \tau_\upsilon^{\mathrm{d}} &:= \alpha_n e_{\mathbf{1}}\left(e_{-\upsilon} - e_{-2\upsilon}\xi\overline{\tau}(2\cdot)\tau\right), \quad \upsilon\in\mathrm{E}\backslash 0.\end{aligned} \tag{3.3}$$

Then the masks $(\tau, (\tau_\upsilon)_{\upsilon\in\mathrm{E}})$ and $(\tau^{\mathrm{d}}, (\tau_\upsilon^{\mathrm{d}})_{\upsilon\in\mathrm{E}})$ satisfy the MUEP condition (1.5), i.e.

$$\overline{\tau}\tau^{\mathrm{d}}(\cdot+\mu) + \sum_{\upsilon\in\mathrm{E}}\overline{\tau}_\upsilon\tau_\upsilon^{\mathrm{d}}(\cdot+\mu) = \begin{cases}1, & \mu = 0,\\ 0, & \mu\in\{0,\pi\}^n\backslash 0.\end{cases}$$

Proof. For $\mu\in\{0,\pi\}^n$, we have

$$\begin{aligned}&\overline{\tau(\omega)}\tau^{\mathrm{d}}(\omega+\mu) + \overline{\tau_0(\omega)}\tau_0^{\mathrm{d}}(\omega+\mu) + \sum_{\upsilon\in\mathrm{E}\backslash 0}\overline{\tau_\upsilon(\omega)}\tau_\upsilon^{\mathrm{d}}(\omega+\mu)\\ &= \overline{\tau(\omega)}\left\{e^{-i\mathbf{1}\cdot(\omega+\mu)}\overline{\tau(2\omega)}\tau(\omega+\mu)\left(1+\xi(\omega+\mu)[1-|\tau(2\omega)|^2]\right)\right\}\\ &\quad +\alpha_n^2 2^n e^{-i\mathbf{1}\cdot\omega}(1-e^{-i\mathbf{1}\cdot\omega}\overline{\tau(2\omega)}\;\overline{\tau(\omega)})\left\{e^{i\mathbf{1}\cdot(\omega+\mu)}\tau(\omega+\mu)\left(1-\xi(\omega+\mu)|\tau(2\omega)|^2\right)\right\}\\ &\quad +\alpha_n^2 e^{-i\mathbf{1}\cdot\omega}\sum_{\upsilon\in\mathrm{E}\backslash 0}(e^{i\upsilon\cdot\omega}-1)\left\{e^{i\mathbf{1}\cdot(\omega+\mu)}\left(e^{-i\upsilon\cdot(\omega+\mu)} - e^{-2i\upsilon\cdot(\omega+\mu)}\xi(\omega+\mu)\overline{\tau(2\omega)}\tau(\omega+\mu)\right)\right\}\\ &= e^{i\mathbf{1}\cdot\mu}\tau(\omega+\mu) - e^{i\mathbf{1}\cdot\mu}|\tau(2\omega)|^2\tau(\omega+\mu)\xi(\omega+\mu) + e^{i\mathbf{1}\cdot\mu}e^{-i\mathbf{1}\cdot\omega}\overline{\tau(\omega)}\;\overline{\tau(2\omega)}\tau(\omega+\mu)\xi(\omega+\mu)\\ &\quad +e^{i\mathbf{1}\cdot\mu}2^{-n}\left(\sum_{\upsilon\in\mathrm{E}}e^{-i\upsilon\cdot\mu}-1\right) - e^{i\mathbf{1}\cdot\mu}2^{-n}\left(\sum_{\upsilon\in\mathrm{E}}e^{-i\upsilon\cdot(\omega+\mu)}-1\right)\\ &\quad -e^{i\mathbf{1}\cdot\mu}\overline{\tau(2\omega)}\tau(\omega+\mu)\xi(\omega+\mu)2^{-n}\left(\sum_{\upsilon\in\mathrm{E}}e^{-i\upsilon\cdot(\omega+2\mu)}-1\right)\end{aligned}$$

$$+e^{i\mathbf{1}\cdot\mu}\overline{\tau(2\omega)}\tau(\omega+\mu)\xi(\omega+\mu)2^{-n}\left(\sum_{\upsilon\in\mathrm{E}}e^{-i\upsilon\cdot(2\omega+2\mu)}-1\right)$$

$$=e^{i\mathbf{1}\cdot\mu}2^{-n}\sum_{\upsilon\in\mathrm{E}}e^{-i\upsilon\cdot\mu}=\begin{cases}1, & \text{if } \mu=0,\\ 0, & \text{if } \mu\in\{0,\pi\}^n\backslash 0,\end{cases}$$

where, for the second to last equality, we used (3.1) and the identity $e_{-\mathbf{1}}\overline{\tau}=\tau$. □

Now we show that the pair $(X(\Psi_0), X(\Psi_0^{\mathrm{d}}))$ is a bi-framelet.

LEMMA 3.4. *Let $0<\beta<1$. Let $\{\tau_\upsilon\}_{\upsilon\in\mathrm{E}}$ be as in (3.2). Then there exists a framelet system $X(\Psi_0^{\mathrm{d}})$ associated with a refinable function $\phi^{\mathrm{d}}\in\mathcal{R}^\beta$ and corresponding τ^{d} so that the pair $(X(\Psi_0), X(\Psi_0^{\mathrm{d}}))$ is a bi-framelet.*

To prove the above lemma, we first recall a result from [HR]. In fact, we state a simplified version of it.

PROPOSITION 3.5. *Suppose that ζ is some fixed trigonometric polynomial which has a zero of order 2 at the origin. Let $\widetilde{\phi}$ be some refinable function with a refinement mask $\widetilde{\tau}$ that satisfies $\widetilde{\phi}\in\mathcal{R}^\alpha$ for some $0<\alpha<1$. Then, for every $\varepsilon>0$, there exists a trigonometric polynomial ξ such that $\xi(0)=1$, and such that the refinable function ϕ with mask $\widetilde{\tau}(1+\xi\zeta)$ belongs to $\mathcal{R}^{\alpha-\varepsilon}$.*

We also need the following (again simplified) result from [Me] :

PROPOSITION 3.6. *Suppose that $f\in\mathcal{R}^\alpha$, for some $0<\alpha<1$. Then the system $(f_{j,k}:\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n)$ is Bessel if $\widehat{f}(0)=0$.*

Proof of Lemma 3.4. First we note that the refinable function $\widetilde{\phi}$ whose mask is $e_{-\mathbf{1}}\overline{\tau}(2\cdot)\tau$, with τ being as in (3.1), is a continuous piecewise-linear function, hence satisfies $\widetilde{\phi}\in\mathcal{R}^\alpha$ for any $0<\alpha<1$. For any given $0<\beta<1$, we choose α so that $\beta<\alpha<1$. Then we use Proposition 3.5 to conclude that there exists ξ which gives the refinable function $\phi^{\mathrm{d}}\in\mathcal{R}^\beta$. Here we used the fact that $\zeta:=1-|\tau(2\cdot)|^2$ has zero of order 2 at the origin.

Now we argue that the dual wavelet system $X(\Psi_0^{\mathrm{d}})$ determined by the above ξ is Bessel. For that, it suffices to show that $X(\psi_\upsilon^{\mathrm{d}})$ is Bessel, for each $\upsilon\in\mathrm{E}$, which will follow once we verify that $\psi_\upsilon^{\mathrm{d}}$ satisfies the assumptions needed in Proposition 3.6. Since all the dual masks (cf. (3.3)) are trigonometric polynomials, $\phi^{\mathrm{d}}\in\mathcal{R}^\beta$ implies that $\psi_\upsilon^{\mathrm{d}}\in\mathcal{R}^\beta$. The condition $\widehat{\psi}_\upsilon^{\mathrm{d}}(0)=0$ is equivalent to $\tau_\upsilon^{\mathrm{d}}(0)=0$. This latter condition trivially follows from the assumption $\tau(0)=\xi(0)=1$.

By combining the above results with Lemma 3.3 and the fact that $X(\Psi_0)$ is Bessel, we see that all the requirements for $(X(\Psi_0), X(\Psi_0^{\mathrm{d}}))$ to be a UEP bi-framelet are satisfied. □

Proof of Theorem 3.2. For any fixed $s<1$, we let u be such that $\max\{s,0\}<u<1$. Then by Lemma 3.4, we can construct a dual framelet system $X(\Psi_0^{\mathrm{d}})$ in a way that $\phi^{\mathrm{d}}\in\mathcal{R}^u$. For each $\psi^{\mathrm{d}}\in\Psi_0^{\mathrm{d}}$, we let

$$\mathbf{M}^*:=(\mathbf{M}^*_{j,l}(k,m):=\mathbf{M}^*(j,k;l,m):j,l\in\mathbb{Z},\ k,m\in\mathbb{Z}^n),$$

with $\mathbf{M}^*(j,k;l,m):=\delta_{j,k;l,m}\,\langle\eta_{j,k},\psi^{\mathrm{d}}_{l,m}\rangle$, $\delta_{j,k;l,m}\in\{\pm1\}$, where η is a function with one (or more) vanishing moment and satisfying $\eta\in\mathcal{R}^\alpha_\gamma$ for any $0\le\alpha<1$ and for any $\gamma\in\mathbb{N}$.

Since $\psi^{\mathrm{d}}\in\mathcal{R}^u$, by Proposition 2.5 (for $\theta:=\psi^{\mathrm{d}}$, $\zeta:=\eta$ and $\beta:=u$), we get, for a suitably large γ and for $j>l$, and with $\varepsilon_3:=u-s$,

$$|\mathbf{M}^*_{j,l}(k,m)|\lesssim 2^{-(j-l)(u+\frac{n}{2})}\left(1+\frac{|2^{j-l}m-k|}{2^{j-l}}\right)^{-\gamma}$$

$$= 2^{(l-j)(s+\frac{n}{2})}2^{-|l-j|\varepsilon_3}\left(1+|2^{l-j}k-m|\right)^{-\gamma}.$$

Thus by Proposition 2.3 (for $\beta := \varepsilon_3$), we obtain (2.9) with ε_2 there replaced by $\varepsilon_3 > 0$, for any $s < 1$. The rest of the proof is identical to the proof of Theorem 2.13. Therefore we get the improved Bernstein result with s satisfying (2.7) instead of s satisfying (2.12). $\square$

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