# OVERCOMING THE BOUNDARY EFFECTS IN SURFACE SPLINE INTERPOLATION

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## 1. Introduction

Let $m, d \in \mathbb{N} := \{1, 2, 3, \dots\}$ be such that $m > d/2$, and define $\phi : \mathbb{R}^d \to \mathbb{R}$ by

$$\phi := \begin{cases} |\cdot|^{2m-d} & \text{if } d \text{ is odd,} \\ |\cdot|^{2m-d} \log |\cdot| & \text{if } d \text{ is even.} \end{cases}$$

Let $\Xi$ be a finite subset of $\mathbb{R}^d$ satisfying

$$\forall q \in \Pi_{m-1} (q_{|_\Xi} = 0 \Rightarrow q = 0), \tag{1.1}$$

where $\Pi_{m-1} :=$ {polynomials of total degree $\le m-1$}, and assume that $f$ is a function defined at least on $\Xi$. The *surface spline interpolant to $f$ at $\Xi$*, denoted $T_\Xi f$, is the unique function $s \in S(\phi; \Xi)$ satisfying $s_{|_\Xi} = f_{|_\Xi}$; here, $S(\phi; \Xi)$ denotes the space of all functions of the form

$$q + \sum_{\xi \in \Xi} \lambda_\xi \phi(\cdot - \xi)$$

where $q \in \Pi_{m-1}$ and the $\lambda_\xi$'s satisfy

$$\sum_{\xi \in \Xi} \lambda_\xi r(\xi) = 0, \quad \forall r \in \Pi_{m-1}. \tag{1.2}$$

The approximation power of surface spline interpolation is usually described via 'approximation orders'. For this we assume that we have a bounded open $\Omega \subset \mathbb{R}^d$ for which $\overline{\Omega}$ :=closure$(\Omega) \supset \Xi$, and we define the 'density of $\Xi$ in $\Omega$' to be the number

$$\delta := \delta(\Xi; \Omega) := \sup_{x \in \Omega} \inf_{\xi \in \Xi} |x - \xi| .$$

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Surface spline interpolation in $\Omega$ is said to provide *$L_p$-approximation of order $\gamma$* if

$$\|f - T_\Xi f\|_{L_p(\Omega)} = O(\delta^\gamma) \quad \text{as } \delta \to 0$$

for all sufficiently smooth functions $f$. The $L_p$-approximation order of surface spline interpolation is only partially understood at present (see [D2], [B1], [WS], [P2], [J1], [LW], [S2], [J2], [Bej], [J3] and the surveys [P1], [B2], [FH]). One aspect which has arisen is the definite presence of boundary effects which affect not only the rate at which $T_\Xi f$ converges to $f$ but also the rate at which the coefficients $\{\lambda_\xi\}_{\xi\in\Xi}$ grow/decay as $\delta \to 0$. We illustrate these boundary effects by comparing results in the special case $\Omega = \mathbb{R}^d$, $\Xi = h\mathbb{Z}^d$ with results when $\Omega = B$, $\Xi = \Xi_h := h\mathbb{Z}^d \cap (1-h)B$.

Although the case $\Omega = \mathbb{R}^d$, $\Xi = h\mathbb{Z}^d$ violates our initial assumptions, Buhmann [B1] has shown that $T_\Xi$ can be defined even when $\Xi$ is the infinite set $h\mathbb{Z}^d$ (more on this in section 5). Regarding approximation orders, it is known ([B1],[JL]) that $T_{h\mathbb{Z}^d}$ provides $L_p$-approximation of order $2m$ for $1 \le p \le \infty$, and that the order $2m$ is sharp. In case the function $f$ decays sufficiently fast, it can be shown that there exists $\lambda \in \ell_2 := \ell_2(\mathbb{Z}^d)$ such that $T_{h\mathbb{Z}^d} f = \sum_{j\in\mathbb{Z}^d} \lambda_j \phi(\cdot - hj)$. We will show, in this case, that if $f \ne 0$ is sufficiently smooth, then $\|\lambda\|_{\ell_2} = O(h^{d/2})$ and $\|\lambda\|_{\ell_2} \ne o(h^{d/2})$.

We look now at the special case $\Omega = B$, $\Xi = \Xi_h$. Regarding approximation, it is known [J1] that there exists an $f \in C^\infty(\mathbb{R}^d)$ such that $\|f - T_{\Xi_h} f\|_{L_p(B)} \ne o(h^{m+1/p})$; consequently, $T_{\Xi_h}$ does not provide $L_p$-approximation in $B$ of any order exceeding $m + 1/p$ for $1 \le p \le \infty$. Note that $m + 1/p < 2m$ unless $m = d = p = 1$. Regarding the size of $\{\lambda_\xi\}_{\xi\in\Xi_h}$, we show in Proposition 4.5 that for the same $f$, $\|\lambda\|_{\ell_2(\Xi_h)} \ne o(h^{(d+1)/2-m})$ as $h \to 0$. Note that $(d+1)/2 - m < d/2$.

The purpose of the present work is to present a modified form of surface spline interpolation which, to some extent, overcomes the above described boundary effects. Regarding approximation, our modified method provides $L_p$-approximation of order $\gamma_p + m$, where $\gamma_p := \min\{m, m + d/p - d/2\}$. Note that $\gamma_p + m = 2m$ if $1 \le p \le 2$; while $\gamma_p + m$ lies strictly between $m + 1/p$ and $2m$ when $2 < p \le \infty$. The stated order of approximation is obtained provided that $\Omega$ is bounded, open, and has the cone property (see Definition 4.1). Regarding the size of $\lambda$, our method enjoys an estimate which, roughly speaking, reduces to $\|\lambda\|_{\ell_2} = O(h^{d/2})$ when the interpolation points are on a grid. Before describing our interpolation method we introduce a family of seminorms defined on $S(\phi;\Xi)$.

Let $\eta \in C([0\,..\,\infty))$ be given by

$$\eta(t) = b t^{m-d/2} K_{m-d/2}(t),$$

where $K_{m-d/2}$ is the modified Bessel function of order $m - d/2$ (see [AS]) and the constant $b = b(m,d)$ is chosen so that $\eta(0) = 1$. For $h > 0$, we define the seminorm $|||\cdot|||_h$ on $S(\phi;\Xi)$ by

$$|||q + \sum_{\xi\in\Xi} \lambda_\xi \phi(\cdot - \xi)|||_h := \sqrt{\sum_{\xi,\xi'\in\Xi} \lambda_\xi \overline{\lambda_{\xi'}}\, \eta(|\xi - \xi'|/h)}.$$

**Interpolation Method 1.3.** We assume that we are given a bounded, open $\Omega \subset \mathbb{R}^d$ which has the cone property, a finite set $\Xi \subset \overline{\Omega}$ satisfying (1.1), and data $f_{|\Xi}$. Let $\Omega_2 \subset \mathbb{R}^d$ (depending only on $\Omega$) be a bounded, open set which contains $\overline{\Omega}$, and let $\Xi_2 \subset \overline{\Omega}_2$ be a finite set such that $\Xi_2 \supset \Xi$ and $\delta(\Xi_2; \Omega_2) \leq \text{const}(d,m)\delta(\Xi;\Omega)$. Let $s = q + \sum_{\xi\in\Xi_2} \lambda_\xi \phi(\cdot - \xi) \in S(\phi; \Xi_2)$ be chosen such that

$$s_{|\Xi} = f_{|\Xi} \quad \text{and} \tag{1.4}$$

$$|||s|||_\delta \leq \text{const}(d,m) \min\{|||\widetilde{s}|||_\delta : \widetilde{s} \in S(\phi;\Xi_2) \text{ and } \widetilde{s}_{|\Xi} = f_{|\Xi}\}, \tag{1.5}$$

where $\delta := \delta(\Xi;\Omega)$.

Two remarks are in order here. First, the method requires only the information $f_{|\Xi}$; in particular, it does not require that $f$ be known on any points in $\Xi_2\backslash\Xi$. Second, the fact that the method does not specify a unique choice of the function $s \in S(\phi;\Xi_2)$ should not be viewed as a negative feature. Since $\eta(|\cdot|)$ is a (strictly) poitive definite function (cf. [S1]), it follows that there exists a unique $s \in S(\phi;\Xi_2)$ which minimizes $|||s|||_\delta$ subject to the constraints (1.4). The point of (1.5) is that it is not necessary to completely minimize $|||s|||_\delta$; rather, it suffices to reduce $|||s|||_\delta$ to within a constant of its minimum value. This means that one can replace $|||\cdot|||_\delta$ in (1.5) with any equivalent seminorm so long as the equivalency constants are independent of $\delta$. For example, if $c > 0$ is a constant (independent of $\delta$), then $|||\cdot|||_\delta$ and $|||\cdot|||_{c\delta}$ are equivalent (see Proposition 2.9). Another example of an equivalent seminorm arises when a certain 'mesh ratio' remains bounded. For finite $\mathcal{N} \subset \mathbb{R}^d$, we define the minimum separation distance in $\mathcal{N}$ to be

$$sep(\mathcal{N}) := \min_{\substack{\xi,\xi'\in\mathcal{N}\\ \xi\neq\xi'}} |\xi - \xi'|\,.$$

If the mesh ratio $\delta/sep(\Xi_2)$ is bounded independently of $\delta$, then it turns out that $|||s|||_\delta$ is equivalent to $\|\lambda\|_{\ell_2(\Xi_2)}$ (see Proposition 2.3), and hence (1.5) can be replaced with

$$\|\lambda\|_{\ell_2(\Xi_2)} \leq \text{const}(d,m)\min\{\left\|\widetilde{\lambda}\right\|_{\ell_2(\Xi_2)} : \widetilde{s} = \widetilde{q} + \sum_{\xi\in\Xi_2} \widetilde{\lambda}_\xi \phi(\cdot - \xi) \in S(\phi;\Xi_2) \text{ and } \widetilde{s}_{|\Xi} = f_{|\Xi}\}. \tag{1.6}$$

The following is a simplified version of Theorem 4.6.

**Theorem 1.7.** *If $f$ belongs to the Sobolev space $W_2^{2m}$ and $s = q + \sum_{\xi\in\Xi_2} \lambda_\xi\phi(\cdot - \xi)$ is chosen according to Interpolation Method 1.3, then for $1 \leq p \leq \infty$,*

$$(i) \qquad \|f - s\|_{L_p(\Omega)} = O(\delta^{\gamma_p + m}) \qquad \text{as } \delta \to 0, \text{ and}$$

$$(ii) \qquad \|\lambda\|_{\ell_2(\Xi_2)} = O((\delta/\epsilon)^{m-d/2}\delta^{d/2}) \qquad \text{as } \delta, \epsilon \to 0,$$

*where $\gamma_p := \min\{m, m + d/p - d/2\}$, $\delta := \delta(\Xi,\Omega)$, and $\epsilon := sep(\Xi_2)$.*

Note that if the mesh ratio $\delta/\epsilon$ is bounded independently of $\delta$ (eg. if $\Xi = h\mathbb{Z}^d \cap \Omega$ and $\Xi_2 = h\mathbb{Z}^d \cap \Omega_2$), then (ii) reduces to $\|\lambda\|_{\ell_2(\Xi_2)} = O(\delta^{d/2})$.

Throughout this paper we use standard multi-index notation: $D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$. The natural numbers are denoted $\mathbb{N} := \{1, 2, 3, \dots\}$, and the non-negative integers are denoted $\mathbb{N}_0$. For multi-indices $\alpha \in \mathbb{N}_0^d$, we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$, while for $x \in \mathbb{R}^d$, we define $|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. For multi-indices $\alpha$, we employ the notation $()^\alpha$ to represent the monomial $x \mapsto x^\alpha$, $x \in \mathbb{R}^d$, and we define $\alpha! := (\alpha_1!)(\alpha_2!)\cdots(\alpha_d!)$. The space of bivariate polynomials of total degree $\le k$ can then be expressed as $\Pi_k := \operatorname{span}\{()^\alpha : |\alpha| \le k\}$. For $x \in \mathbb{R}^d$, we define the complex exponential $e_x$ by $e_x(t) := e^{ix\cdot t}$, $t \in \mathbb{R}^d$. The Fourier transform of a function $f$ can then be expressed as $\widehat{f}(w) := \int_{\mathbb{R}^d} e_{-w}(x) f(x)\,dx$. The space of compactly supported $C^\infty$ functions is denoted $C_c^\infty(\mathbb{R}^d)$. If $\mu$ is a distribution and $g$ is a test function, then the application of $\mu$ to $g$ is denoted $\langle g, \mu\rangle$. We employ the notation const to denote a generic constant in the range $(0\,..\,\infty)$ whose value may change with each occurence. An important aspect of this notation is that const depends only on its arguments if any, and otherwise depends on nothing. Without further mention, we assume that the parameters $m, d$ are positive integers with $m > d/2$. Two oft employed sets in $\mathbb{R}^d$ are the open unit ball $B := \{x \in \mathbb{R}^d : |x| < 1\}$ and the unit cube $C := [1/2..1/2)^d$.

## 2. Preliminaries

The conclusion of Theorem 1.7 asserts that $\|f - s\|_{L_p(\Omega)} = O(\delta^{\gamma_p + m})$ as $\delta \to 0$. We prefer our conclusion to estimate $\|f - s\|_{L_p(\Omega)}$ for all values of $\delta$, not just asymptotically as $\delta \to 0$. To do this we need to place an additional assumption on the interpolation points $\Xi$.

**Definition 2.1.** A set $\mathcal{N} \subset \mathbb{R}^d$ is said to be *correct* for interpolation in $\Pi_n$ if for all functions $f$, defined at least on $\mathcal{N}$, there exists a unique $q \in \Pi_n$ such that $q_{|\mathcal{N}} = f_{|\mathcal{N}}$. We denote by $\mathcal{I}_n$ the set of all pointsets in $\mathbb{R}^d$ which are correct for interpolation in $\Pi_n$. For $\mathcal{N} \in \mathcal{I}_n$, we define $|\mathcal{N}|_{\mathcal{I}_n}$ as follows: Let $y_\mathcal{N} := \frac{1}{\#\mathcal{N}} \sum_{\xi\in\mathcal{N}} \xi$ be the center of $\mathcal{N}$. For each $\alpha$ with $|\alpha| \le n$, there exist unique numbers $\{a_{\alpha,\xi}\}_{\xi\in\mathcal{N}}$ such that $D^\alpha q(y_\mathcal{N}) = \sum_{\xi\in\mathcal{N}} a_{\alpha,\xi} q(\xi)$ for all $q \in \Pi_n$. Then

$$|\mathcal{N}|_{\mathcal{I}_n} := \max_{|\alpha|\le n, \xi\in\mathcal{N}} |a_{\alpha,\xi}|\,.$$

The additional assumption which we need is that there exists $\mathcal{N} \subset \Xi$ such that $\mathcal{N} \in \mathcal{I}_{2m-1}$ and $|\mathcal{N}|_{\mathcal{I}_{2m-1}} \le \operatorname{const}(m,d)$. Note that this is necessarily satisfied if $\delta(\Xi;\Omega)$ is sufficiently small.

The surface spline interpolant is intimately connected to a space of functions $H^m$ defined as follows: For $n > d/2$, let $H^n$ be the set of all continuous functions $g$ such that $D^\alpha g \in L_2 := L_2(\mathbb{R}^d)$ for all $|\alpha| = n$, and define the seminorm $|||\cdot|||_{H^n}$ on $H^n$ by

$$|||g|||_{H^n} := \| |\cdot|^n \,\widehat{g}\|_{L_2}, \quad g \in H^n.$$

Duchon [D1] has shown (assuming (1.1)) that $s = T_\Xi f$ is the unique function in $H^m$ which minimizes $|||s|||_{H^m}$ subject to the constraints $s_{|\Xi} = f_{|\Xi}$. The seminorm $|||\cdot|||_h$ which we defined on $S(\phi;\Xi)$ actually has a natural extension to all of $H^m$. Let $|||\cdot|||_*$ be the seminorm defined on $H^m$ by

$$|||g|||_* := \left\| \frac{|\cdot|^{2m}}{(1+|\cdot|^2)^{m/2}} \widehat{g} \right\|_{L_2}, \quad g \in H^m.$$

**Proposition 2.2.** *If $s = q + \sum_{\xi\in\Xi} \lambda_\xi \phi(\cdot - \xi) \in S(\phi;\Xi)$ and $h > 0$, then*

$$|||s|||_h = \operatorname{const}(d,m) h^{-2m+d} |||s(h\cdot)|||_*.$$

*Proof.* According to [GS], $\widehat{\eta}(|\cdot|) = c_\eta (1+|\cdot|^2)^{-m}$ and $\widehat{\phi}$ can be identified on $\mathbb{R}^d \backslash 0$ with $c_\phi |\cdot|^{-2m}$, where $c_\eta$, $c_\phi$ are constants depending only on $d, m$.

$$|||s(h\cdot)|||_* = \left\| \frac{|\cdot|^{2m}}{(1+|\cdot|^2)^{m/2}} (s(h\cdot))^\wedge \right\|_{L_2} = h^{-d} \left\| \frac{|\cdot|^{2m}}{(1+|\cdot|^2)^{m/2}} \widehat{s}(\cdot/h) \right\|_{L_2}$$
$$= h^{-d} |c_\phi| \left\| \frac{|\cdot|^{2m}}{(1+|\cdot|^2)^{m/2}} |\cdot/h|^{-2m} \sum_{\xi\in\Xi} \lambda_\xi e_{-\xi}(\cdot/h) \right\|_{L_2} = h^{2m-d} |c_\phi| \left\| (1+|\cdot|^2)^{-m/2} \sum_{\xi\in\Xi} \lambda_\xi e_{-\xi/h} \right\|_{L_2}.$$

Now,

$$\left\| (1+|\cdot|^2)^{-m/2} \sum_{\xi\in\Xi} \lambda_\xi e_{-\xi/h} \right\|_{L_2}^2$$
$$= \int_{\mathbb{R}^d} (1+|\cdot|^2)^{-m} \left( \sum_{\xi\in\Xi} \lambda_\xi e_{-\xi/h} \right) \overline{\left( \sum_{\xi'\in\Xi} \lambda_{\xi'} e_{-\xi'/h} \right)} \, dm$$
$$= \sum_{\xi,\xi'\in\Xi} \lambda_\xi \overline{\lambda_{\xi'}} \int_{\mathbb{R}^d} (1+|\cdot|^2)^{-m} e_{(\xi'-\xi)/h} \, dm = \frac{(2\pi)^d}{c_\eta} \sum_{\xi,\xi'\in\Xi} \lambda_\xi \overline{\lambda_{\xi'}} \eta(|\xi'-\xi|/h).$$

□

The following result shows that $|||s|||_h$ is equivalent to $\|\lambda\|_{\ell_2(\Xi)}$ whenever $h$ is sufficiently small.

**Proposition 2.3.** *Let $\Xi$ be a finite subset of $\mathbb{R}^d$, and let $0 < h \le \operatorname{const}(d,m) sep(\Xi)$. If $s = q + \sum_{\xi\in\Xi_2} \lambda_\xi \phi(\cdot - \xi) \in S(\phi;\Xi)$, then*

$$\operatorname{const}(d,m) \|\lambda\|_{\ell_2(\Xi)} \le |||s|||_h \le \operatorname{const}(d,m) \|\lambda\|_{\ell_2(\Xi)}. \tag{2.4}$$

*Proof.* It is known (cf. [S1]) that since $sep(\Xi/h) \ge \operatorname{const}(d,m)$,

$$|||s|||_h = \sqrt{\sum_{\xi,\xi'\in\Xi} \lambda_\xi \overline{\lambda_{\xi'}} \eta(|\xi-\xi'|/h)} \ge \operatorname{const}(d,m) \|\lambda\|_{\ell_2(\Xi)}.$$

Put $C := [-1/2\,..\,1/2)^d$ and recall from the proof of Proposition 2.2 that

$$\begin{aligned} |||s|||_h^2 &= \text{const}(d,m)\left\|(1+|\cdot|^2)^{-m/2}\sum_{\xi\in\Xi}\lambda_\xi e_{-\xi/h}\right\|_{L_2}^2 \\ &\le \text{const}(d,m)\sum_{j\in\mathbb{Z}^d}\left\|(1+|\cdot|^2)^{-m/2}\right\|_{L_\infty(j+C)}^2\left\|\sum_{\xi\in\Xi}\lambda_\xi e_{-\xi/h}\right\|_{L_2(j+C)}^2. \end{aligned}$$

Since $sep(\Xi/h) \ge \text{const}(d,m)$, it follows that $\left\|\sum_{\xi\in\Xi}\lambda_\xi e_{-\xi/h}\right\|_{L_2(j+C)} \le \text{const}(d,m)\,\|\lambda\|_{\ell_2(\Xi)}$.

Hence

$$|||s|||_h^2 \le \text{const}(d,m)\sum_{j\in\mathbb{Z}^d}\left\|(1+|\cdot|^2)^{-m/2}\right\|_{L_\infty(j+C)}^2\|\lambda\|_{\ell_2(\Xi)}^2 \le \text{const}(d,m)\,\|\lambda\|_{\ell_2(\Xi)}^2\,.$$

□

Theorem 1.7 describes the approximation power of Interpolation Method 1.3 when the data comes from a function $f\in W_2^{2m}$. The theorem does not address the case when $f$ is less smooth. The theory actually applies when $f$ belongs to a certain range of smoothness spaces where $W_2^m$ is the roughest space and $W_2^{2m}$ is the smoothest. We now describe these spaces.

**Definition 2.5.** The Sobolev space $W_2^\gamma$, $\gamma\ge 0$, is the set of all $f\in L_2$ such that

$$\|f\|_{W_2^\gamma} := \left\|(1+|\cdot|^2)^{\gamma/2}\widehat{f}\right\|_{L_2} < \infty.$$

Let $A_0 := \overline{B}$, and for $k\in\mathbb{N}$, let $A_k := 2^k\overline{B}\backslash 2^{k-1}B$. The Besov space $B_{2,q}^\gamma$, $\gamma\in\mathbb{R}$, $1\le q\le\infty$, is defined to be the set of all tempered distributions $f$ for which

$$\|f\|_{B_{2,q}^\gamma} := \left\|k\mapsto 2^{k\gamma}\left\|\widehat{f}\right\|_{L_2(A_k)}\right\|_{\ell_q(\mathbb{N}_0)} < \infty.$$

These Besov spaces are Banach spaces; the reader is refered to [Pe] for a general reference.

**Definition.** For $\gamma\in[0\,..\,m]$, let $\mathcal{F}_\gamma$ be the space given by

$$\mathcal{F}_\gamma := \begin{cases} B_{2,\infty}^{m+\gamma} & \text{if } 0<\gamma<m, \\ W_2^{m+\gamma} & \text{if } \gamma\in\{0,m\}. \end{cases}$$

Incidentally, the space $B_{2,\infty}^{m+\gamma}$ is strictly larger than $W_2^{m+\gamma}$. The following lemma shows some useful relations between $|||\cdot|||_*$ and $|||\cdot|||_{H^m}$, $\|\cdot\|_{\mathcal{F}_\gamma}$.

**Lemma 2.6.** *If* $f \in H^m$, $h > 0$ *and* $\gamma \in [0\,..\,m]$, *then*

$$
\begin{aligned}
&(i) && |||f(h\cdot)|||_* \le h^{m-d/2}|||f|||_{H^m},\\
&(ii) && |||f|||_* \le h^{-2m+d/2}(1+h^m)|||f(h\cdot)|||_*, \qquad \textit{and}\\
&(iii) && |||f(h\cdot)|||_* \le \mathrm{const}(m,\gamma)h^{m+\gamma-d/2}(1+h^m)\,\|f\|_{\mathcal{F}_\gamma}.
\end{aligned}
$$

*Proof.* First note that

$$
\begin{aligned}
|||f(h\cdot)|||_* &= \left\| \frac{|\cdot|^{2m}}{(1+|\cdot|^2)^{m/2}}\,(f(h\cdot))\widehat{\ }\right\|_{L_2} = h^{-d}\left\| \frac{|\cdot|^{2m}}{(1+|\cdot|^2)^{m/2}}\widehat{f}(\cdot/h)\right\|_{L_2}\\
&= h^{-d/2}\left\| \frac{|h\cdot|^{2m}}{(1+|h\cdot|^2)^{m/2}}\widehat{f}\right\|_{L_2} = h^{2m-d/2}\left\| \frac{|\cdot|^{2m}}{(1+|h\cdot|^2)^{m/2}}\widehat{f}\right\|_{L_2}.
\end{aligned}
\tag{2.7}
$$

Hence,

$$
|||f(h\cdot)|||_* \le h^{2m-d/2}\left\| \frac{|\cdot|^{2m}}{(0+|h\cdot|^2)^{m/2}}\widehat{f}\right\|_{L_2} = h^{m-d/2}|||f|||_{H^m}
$$

which proves (i). For (ii) we note that by (2.7),

$$
\begin{aligned}
|||f|||_* &= \left\| \frac{|\cdot|^{2m}}{(1+|\cdot|^2)^{m/2}}\widehat{f}\right\|_{L_2} \le \left\| \frac{(1+|h\cdot|^2)^{m/2}}{(1+|\cdot|^2)^{m/2}}\right\|_{L_\infty}\left\| \frac{|\cdot|^{2m}}{(1+|h\cdot|^2)^{m/2}}\widehat{f}\right\|_{L_2}\\
&= \max\{1,h^m\}\left\| \frac{|\cdot|^{2m}}{(1+|h\cdot|^2)^{m/2}}\widehat{f}\right\|_{L_2} \le (1+h^m)h^{-2m+d/2}|||f(h\cdot)|||_*, \quad \text{by (2.7)}.
\end{aligned}
$$

For (iii), we mention that the factor $(1+h^m)$ is only needed in case $h \ge 1$ and $0 < \gamma < m$. The case $\gamma = 0$ of (iii) follows from (i) since $|||f|||_{H^m} \le \|f\|_{W_2^m}$. The case $\gamma = m$ of (iii) follows easily from (2.7) since

$$
|||f(h\cdot)|||_* \le h^{2m-d/2}\left\| \frac{|\cdot|^{2m}}{(1+0)^{m/2}}\widehat{f}\right\|_{L_2} = h^{2m-d/2}\,\|f\|_{H^{2m}}. \le h^{2m-d/2}\,\|f\|_{W_2^{2m}}.
$$

Now assume that $0 < \gamma < m$. By (2.7),

$$
\begin{aligned}
|||f(h\cdot)|||_* &\le h^{2m-d/2}\sum_{k=0}^{\infty}\left\| \frac{|\cdot|^{2m}}{(1+|h\cdot|^2)^{m/2}}\widehat{f}\right\|_{L_2(A_k)}\\
&\le \mathrm{const}(m)h^{2m-d/2}\left(\left\|\widehat{f}\right\|_{L_2(A_0)} + \sum_{k=1}^{\infty}\frac{2^{2km}}{(1+h^2 2^{2k})^{m/2}}\left\|\widehat{f}\right\|_{L_2(A_k)}\right)\\
&\le \mathrm{const}(m)h^{2m-d/2}\,\|f\|_{B^{m+\gamma}_{2,\infty}}\left(1+\sum_{k=1}^{\infty}\frac{2^{2km}}{(1+h^2 2^{2k})^{m/2}}2^{-k(m+\gamma)}\right).
\end{aligned}
\tag{2.8}
$$

If $h \geq 1$, then by (2.8)

$$
\begin{aligned}
|||f(h\cdot)|||_* &\leq \mathrm{const}(m) h^{2m-d/2} \|f\|_{B^{m+\gamma}_{2,\infty}} \left(1 + \sum_{k=1}^{\infty} \frac{2^{2km}}{(0 + h^2 2^{2k})^{m/2}} 2^{-k(m+\gamma)}\right) \\
&= \mathrm{const}(m) h^{2m-d/2} \|f\|_{B^{m+\gamma}_{2,\infty}} \left(1 + h^{-m} \sum_{k=1}^{\infty} 2^{-k\gamma}\right) \leq \mathrm{const}(m,\gamma) h^{m-d/2}(1 + h^m) \|f\|_{B^{m+\gamma}_{2,\infty}}.
\end{aligned}
$$

On the other hand, if $h < 1$, then by (2.8)

$$
\begin{aligned}
|||f(h\cdot)|||_* &\leq \mathrm{const}(m) h^{2m-d/2} \|f\|_{B^{m+\gamma}_{2,\infty}} \sum_{k=0}^{\infty} \frac{2^{2km}}{(1 + h^2 2^{2k})^{m/2}} 2^{-k(m+\gamma)} \\
&\leq \mathrm{const}(m) h^{2m-d/2} \|f\|_{B^{m+\gamma}_{2,\infty}} \left( \sum_{k=0}^{\lceil -\log_2 h \rceil} 2^{2km} 2^{-k(m+\gamma)} + \sum_{k=\lceil -\log_2 h \rceil}^{\infty} \frac{2^{2km}}{h^m 2^{km}} 2^{-k(m+\gamma)} \right) \\
&= \mathrm{const}(m) h^{2m-d/2} \|f\|_{B^{m+\gamma}_{2,\infty}} \left( \sum_{k=0}^{\lceil -\log_2 h \rceil} 2^{k(m-\gamma)} + h^{-m} \sum_{k=\lceil -\log_2 h \rceil}^{\infty} 2^{-k\gamma} \right) \\
&\leq \mathrm{const}(m,\gamma) h^{m+\gamma-d/2} \|f\|_{B^{m+\gamma}_{2,\infty}}.
\end{aligned}
$$

□

With Proposition 2.2 and Lemma 2.6 in hand, we can prove an assertion contained in the second remark following Interpolation Method 1.3.

**Proposition 2.9.** *Let $\Xi \subset \mathbb{R}^d$ be finite and let $s = q + \sum_{\xi \in \Xi} \lambda_\xi \phi(\cdot - \xi) \in S(\phi; \Xi)$. If $h, h' > 0$ are such that $h/h' + h'/h \leq \mathrm{const}$, then*

$$\mathrm{const}(m) |||s|||_{h'} \leq |||s|||_h \leq \mathrm{const}(m) |||s|||_{h'}.$$

*Proof.* By Lemma 2.6 (ii),

$$
\begin{aligned}
h^{-2m+d} |||s(h\cdot)|||_* &\leq h^{-2m+d}(1 + (h'/h)^m)(h'/h)^{-2m+d/2} |||s(h'\cdot)|||_* \\
&\leq \mathrm{const}(m) h'^{-2m+d} |||s(h'\cdot)|||_*, \qquad \text{since } m > d/2.
\end{aligned}
$$

The desired conclusion now follows from Proposition 2.2 (and symmetry). □

If $f \in H^n$, then $\widehat{f}$ can be identified on $\mathbb{R}^d \backslash 0$ with a locally integrable function. However, on any neighborhood of 0, the distribution $\widehat{f}$ may be of a higher order. The following lemma gives a sufficient condition on the test function $g$ for which the higher order component of $\widehat{f}$ can be ignored when computing $\langle g, \widehat{f} \rangle$.

**Lemma 2.10.** *Let $n > d/2$. If $g \in C_c^\infty(\mathbb{R}^d)$ satisfies $|g(w)| = O(|w|^n)$ as $|w| \to 0$, then*

$$\langle g, \widehat{f}\rangle = \int_{\mathbb{R}^d \setminus 0} g(w)\widehat{f}(w)\,dw \quad \forall f \in H^n.$$

*Proof.* Let $\sigma \in C_c(\mathbb{R}^d)$ be such that $\sigma = 1$ on $B$. Define the tempered distribution $\nu$ by

$$\langle \psi, \widehat{\nu}\rangle := \int_{\mathbb{R}^d \setminus 0} [\psi(w) - \sum_{|\alpha|<n} \frac{D^\alpha \psi(0)}{\alpha!} w^\alpha \sigma(w)]\widehat{f}(w)\,dw, \quad \psi \in C_c^\infty(\mathbb{R}^d).$$

If $|\alpha| = n$, then $\langle \psi, ()^\alpha \widehat{\nu}\rangle = \int_{\mathbb{R}^d \setminus 0} \psi(w) w^\alpha \widehat{f}(w)\,dw$, and hence $()^\alpha \widehat{\nu} \in L_2$ (as $()^\alpha \widehat{f} \in L_2$). It follows from this that $\nu \in H^n$. Since $\widehat{\nu} = \widehat{f}$ on $\mathbb{R}^d \setminus 0$, it follows that $f - \nu$ is a polynomial. Since $f - \nu \in H^n$, it follows that $f - \nu \in \Pi_{n-1}$. Consequently, $f = \nu + q$ for some $q \in \Pi_{n-1}$. Now, if $g \in C_c^\infty(\mathbb{R}^d)$ satisfies $|g(w)| = O(|w|^n)$ as $|w| \to 0$, then $\langle g, \widehat{f}\rangle = \langle g, \widehat{\nu}\rangle + \langle g, \widehat{q}\rangle = \int_{\mathbb{R}^d \setminus 0} g(w)\widehat{f}(w)\,dw + 0$. $\square$

## 3. A result on $|||\cdot|||_*$

The purpose of this section is to prove the following:

**Proposition 3.1.** *Let $r > 0$ and for each $j \in \mathbb{Z}^d$, let $\mathcal{N}_j$ be a finite subset of $j + rB$. If $\{b_{j,\xi}\}_{j \in \mathbb{Z}^d, \xi \in \mathcal{N}_j}$ is such that*

$$\sum_{\xi \in \mathcal{N}_j} b_{j,\xi} q(\xi) = 0 \quad \forall q \in \Pi_{2m-1}, j \in \mathbb{Z}^d \quad \textit{and}$$

$$M := \sup_{j \in \mathbb{Z}^d} \sum_{\xi \in \mathcal{N}_j} |b_{j,\xi}| < \infty,$$

*then*

$$\sum_{j \in \mathbb{Z}^d} \left| \sum_{\xi \in \mathcal{N}_j} b_{j,\xi} f(\xi) \right|^2 \leq \operatorname{const}(d,m,r) M^2 |||f|||_*^2 \quad \forall f \in H^m.$$

Our proof of this proposition employs local versions of $|||\cdot|||_{H^n}$ and $\|\cdot\|_{W_2^n}$.

**Definition.** For $n > d/2$ and $A \subset \mathbb{R}^d$ open, we define

$$|||f|||_{H^n(A)} := \sqrt{\sum_{|\alpha|=n} \|D^\alpha f\|_{L_2(A)}^2},$$

$$\|f\|_{W_2^n(A)} := \sqrt{\sum_{|\alpha|\leq n} \|D^\alpha f\|_{L_2(A)}^2}.$$

It is a straightforward matter to show, via the Plancheral Theorem, that

(3.2)
$$\operatorname{const}(d,n)|||f|||_{H^n(\mathbb{R}^d)} \leq |||f|||_{H^n} \leq \operatorname{const}(d,n)|||f|||_{H^n(\mathbb{R}^d)} \quad \forall f \in H^n \quad \text{and}$$

(3.3)
$$\operatorname{const}(d,n)\,\|f\|_{W_2^n(\mathbb{R}^d)} \leq \|f\|_{W_2^n} \leq \operatorname{const}(d,n)\,\|f\|_{W_2^n(\mathbb{R}^d)} \quad \forall f \in W_2^n.$$

The proof of the following lemma can be found in [D2, p. 328].

**Lemma 3.4.** *Let $y \in \mathbb{R}^d$, $r > 0$, $n > d/2$, and let $\mathcal{N} \subset y + rB$ be such that $\mathcal{N} \in \mathcal{I}_{n-1}$. If $f \in W_2^n$ and $q \in \Pi_{n-1}$ are such that $f_{|\mathcal{N}} = q_{|\mathcal{N}}$, then*

$$\|f - q\|_{L_2(y+rB)} \le \mathrm{const}(r, n, \mathcal{N}) |||f|||_{H^n(y+rB)}.$$

**Lemma 3.5.** *Let $y \in \mathbb{R}^d$, $r > 0$, $n > d/2$, and let $\mathcal{N} \subset y + rB$ be such that $\mathcal{N} \in \mathcal{I}_{n-1}$. If $f \in W_2^n$, then*

$$\|f\|_{W_2^n(y+rB)} \le \mathrm{const}(r, n, \mathcal{N}) \left( \|f\|_{\ell_2(\mathcal{N})} + |||f|||_{H^n(y+rB)} \right).$$

*Proof.* Since all norms and seminorms under discussion are translation invariant, we may assume without loss of generality that $y = 0$. It is known [A, p. 79] that $\|\cdot\|_{W_2^n(rB)}$ is equivalent to $\|\cdot\|_{L_2(rB)} + |||\cdot|||_{H^n(rB)}$. Let $q \in \Pi_{n-1}$ be such that $q_{|\mathcal{N}} = f_{|\mathcal{N}}$. Then

$$\begin{aligned}
&\|f\|_{W_2^n(rB)} \le \|f - q\|_{W_2^n(rB)} + \|q\|_{W_2^n(rB)} \\
&\le \mathrm{const}(r, n, d) \left( \|f - q\|_{L_2(rB)} + |||f - q|||_{H^n(rB)} + \|q\|_{L_2(rB)} \right) \\
&\le \mathrm{const}(r, n, \mathcal{N}) \left( |||f|||_{H^n(rB)} + \|q\|_{\ell_2(\mathcal{N})} \right), \quad \text{by Lemma 3.4 and since } q \in \Pi_{n-1}, \\
&= \mathrm{const}(r, n, \mathcal{N}) \left( |||f|||_{H^n(rB)} + \|f\|_{\ell_2(\mathcal{N})} \right).
\end{aligned}$$

□

**Lemma 3.6.** *Let $y \in \mathbb{R}^d$, $r > 0$, and $n > d/2$. If $f \in H^n$, then there exists $\widetilde{f} \in H^n$ such that*

$$\begin{aligned}
&(i) \qquad \widetilde{f}_{|y+rB} = f_{|y+rB} \qquad \textit{and} \\
&(ii) \qquad |||\widetilde{f}|||_{H^n} \le \mathrm{const}(d, n, r) |||f|||_{H^n(y+rB)}.
\end{aligned}$$

*Proof.* Since the seminorms under discussion are translation invariant, we may assume without loss of generality that $y = 0$. Let $\mathcal{N} \subset rB$ be such that $\mathcal{N} \in \mathcal{I}_{n-1}$. Let $f \in H^n$. Let $q \in \Pi_{n-1}$ be such that $q_{|\mathcal{N}} = f_{|\mathcal{N}}$ and put $g := f - q$. By the Calderón Extension Theorem [A, p. 84], there exists $\widetilde{g} \in W_2^n$ such that $\widetilde{g}_{|rB} = g_{|rB}$ and $\|\widetilde{g}\|_{W_2^n} \le \mathrm{const}(d, n, r) \|g\|_{W_2^n(rB)}$. Since $\widetilde{g} \in W_2^n$ and $q \in \Pi_{n-1}$, it follows that $\widetilde{f} := \widetilde{g} + q \in H^n$. Note that $\widetilde{f}_{|rB} = f_{|rB}$ and

$$\begin{aligned}
&|||\widetilde{f}|||_{H^n} \le \|\widetilde{g}\|_{W_2^n} \le \mathrm{const}(d, n, r) \|g\|_{W_2^n(rB)} \\
&\le \mathrm{const}(\mathcal{N}, n, r) \left( \|g\|_{\ell_2(\mathcal{N})} + |||g|||_{H^n(rB)} \right), \quad \text{by Lemma 3.5,} \\
&= \mathrm{const}(\mathcal{N}, n, r) |||f|||_{H^n(rB)}
\end{aligned}$$

which (after a suitable choice of $\mathcal{N}$) proves the lemma. □

**Lemma 3.7.** *Let $n > d/2$, $r > 0$, $y \in \mathbb{R}^d$, and let $\mathcal{N}$ be a finite subset of $y + rB$. If $\{b_\xi\}_{\xi \in \mathcal{N}}$ is such that*

$$\sum_{\xi \in \mathcal{N}} b_\xi q(\xi) = 0 \quad \forall q \in \Pi_{n-1}, \tag{3.8}$$

*then*

$$\left| \sum_{\xi \in \mathcal{N}} b_\xi f(\xi) \right| \leq \mathrm{const}(d, n, r) |||f|||_{H^n(y+rB)} \sum_{\xi \in \mathcal{N}} |b_\xi| , \quad \forall f \in H^n.$$

*Proof.* Without loss of generality assume $y = 0$. Let $f \in H^n$ and let $\widetilde{f} \in H^n$ be as described in Lemma 3.6. Put $\tau := \sum_{\xi \in \mathcal{N}} b_\xi e_\xi$. Since $\widehat{\widetilde{f}}$ is integrable on $\mathbb{R}^d \backslash B$ and by Lemma 3.6 (i), it follows that $\sum_{\xi \in \mathcal{N}} b_\xi f(\xi) = \sum_{\xi \in \mathcal{N}} b_\xi \widetilde{f}(\xi) = (2\pi)^{-d} \langle \tau, \widehat{\widetilde{f}} \rangle$. Since $D^\alpha e_\xi(0) = (i\xi)^\alpha$, it follows from (3.8) that $D^\alpha \tau(0) = 0$ for all $|\alpha| < n$. Hence, $|\tau(w)| = O(|w|^n)$ as $|w| \to 0$. Therefore, by Lemma 2.10,

$$\left| \sum_{\xi \in \mathcal{N}} b_\xi f(\xi) \right| = (2\pi)^{-d} \left| \int_{\mathbb{R}^d \backslash 0} \tau(w) \widehat{\widetilde{f}}(w) \, dw \right| \leq (2\pi)^{-d} \left\| |\cdot|^{-n} \tau \right\|_{L_2} |||\widetilde{f}|||_{H^n}, \tag{3.9}$$

by Cauchy-Schwarz inequality. In order to estimate the factor containing $\tau$, we note that $\|\tau\|_{L_\infty} \leq \sum_{\xi \in \mathcal{N}} |b_\xi| =: M$. It follows by Taylor's Theorem that for $w \in B$,

$$\begin{aligned} &|\tau(w)| \leq \mathrm{const}(d, n) \max_{|\alpha|=n} \|D^\alpha \tau\|_{L_\infty(B)} |w|^n \\ &\leq \mathrm{const}(d, n) M \max_{|\alpha|=n, \xi \in \mathcal{N}} \|D^\alpha e_\xi\|_{L_\infty(B)} |w|^n \leq \mathrm{const}(d, n, r) M |w|^n . \end{aligned}$$

Hence,

$$\begin{aligned} &\left\| |\cdot|^{-n} \tau \right\|_{L_2} \leq \left\| |\cdot|^{-n} \tau \right\|_{L_2(\mathbb{R}^d \backslash B)} + \left\| |\cdot|^{-n} \tau \right\|_{L_2(B)} \\ &\leq M \left\| |\cdot|^{-n} \right\|_{L_2(\mathbb{R}^d \backslash B)} + \mathrm{const}(d, n, r) M \leq \mathrm{const}(d, n, r) M. \end{aligned}$$

which, in view of (3.9) and Lemma 3.6 (ii), completes the proof. □

**Lemma 3.10.** *Let $n > d/2$ and $r > 0$. For each $j \in \mathbb{Z}^d$, let $\mathcal{N}_j$ be a finite subset of $j + rB$. If $\{b_{j,\xi}\}_{j \in \mathbb{Z}^d, \xi \in \mathcal{N}_j}$ is such that*

$$\begin{aligned} &\sum_{\xi \in \mathcal{N}_j} b_{j,\xi} q(\xi) = 0 \quad \forall q \in \Pi_{n-1}, j \in \mathbb{Z}^d \quad \textit{and} \\ &M := \sup_{j \in \mathbb{Z}^d} \sum_{\xi \in \mathcal{N}_j} |b_{j,\xi}| < \infty, \end{aligned}$$

*then*

$$\sum_{j\in\mathbb{Z}^d}\left|\sum_{\xi\in\mathcal{N}_j} b_{j,\xi}f(\xi)\right|^2 \le \mathrm{const}(d,n,r)M^2\,|||f|||_{H^n}^2 \quad \forall f\in H^n.$$

*Proof.* By Lemma 3.7,

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^d}\left|\sum_{\xi\in\mathcal{N}_j} b_{j,\xi}f(\xi)\right|^2 \le \mathrm{const}(d,n,r)\sum_{j\in\mathbb{Z}^d} M^2\,|||f|||_{H^n(j+rB)}^2\\
&= \mathrm{const}(d,n,r)M^2\sum_{|\alpha|=n}\sum_{j\in\mathbb{Z}^d}\|D^\alpha f\|_{L_2(j+rB)}^2 \le \mathrm{const}(d,n,r)M^2\sum_{|\alpha|=n}\|D^\alpha f\|_{L_2}^2\\
&= \mathrm{const}(d,n,r)M^2\,|||f|||_{H^n(\mathbb{R}^d)}^2 \le \mathrm{const}(d,n,r)M^2\,|||f|||_{H^n}^2, \qquad \text{by (3.2).}
\end{aligned}$$

□

*Proof of Proposition 3.1.* Let $f\in H^m$ and define $f_1$ by $\widehat{f_1} := \chi_B\widehat{f}$ and put $f_2 := f - f_1$. Note that $f_1\in H^m\cap H^{2m}$, $f_2\in H^m$, $|||f|||_*^2 = |||f_1|||_*^2 + |||f_2|||_*^2$, $|||f_1|||_{H^{2m}} \le 2^{m/2}|||f_1|||_*$, and $|||f_2|||_{H^m}\le 2^{m/2}|||f_2|||_*$. Thus

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^d}\left|\sum_{\xi\in\mathcal{N}_j} b_{j,\xi}f(\xi)\right|^2 = \sum_{j\in\mathbb{Z}^d}\left|\sum_{\xi\in\mathcal{N}_j} b_{j,\xi}(f_1(\xi)+f_2(\xi))\right|^2\\
&\le 2\sum_{j\in\mathbb{Z}^d}\left|\sum_{\xi\in\mathcal{N}_j} b_{j,\xi}f_1(\xi)\right|^2 + 2\sum_{j\in\mathbb{Z}^d}\left|\sum_{\xi\in\mathcal{N}_j} b_{j,\xi}f_2(\xi)\right|^2\\
&\le \mathrm{const}(d,m,r)M^2\,|||f_1|||_{H^{2m}}^2 + \mathrm{const}(d,m,r)M^2\,|||f_2|||_{H^m}^2, \qquad \text{by Lemma 3.10,}\\
&\le \mathrm{const}(d,m,r)M^2\,|||f_1|||_*^2 + \mathrm{const}(d,m,r)M^2\,|||f_2|||_*^2 \le \mathrm{const}(d,m,r)M^2\,|||f|||_*^2.
\end{aligned}$$

□

## 4. The Main Result

The following is equivalent to the standard definition of the cone property. This form has been chosen simply to facilitate the proof of the lemma which follows.

**Definition 4.1.** A set $\Omega\subset\mathbb{R}^d$ is said to have the *cone property* if there exists $\epsilon_\Omega, r_\Omega\in(0\,..\,\infty)$ such that for all $x\in\Omega$ there exists $y\in\Omega$ such that $|x-y|=\epsilon_\Omega$ and

$$x + t(y - x + r_\Omega B)\subset\Omega \quad \forall t\in[0\,..\,1].$$

**Lemma 4.2.** *Let $n \geq 0$. If $\Omega \subset \mathbb{R}^d$ is bounded, open, and has the cone property, then there exists $\delta_0, r_0 \in (0\,..\,\infty)$ (depending only on $n$ and $\Omega$) such that if $\Xi$ is a finite subset of $\overline{\Omega}$ with $\delta := \delta(\Xi;\Omega) \leq \delta_0$, then for all $x \in \Omega/\delta$ there exists a finite $\mathcal{N} \subset (\Xi/\delta) \cap (x + r_0 B)$ and $\{b_\xi\}_{\xi \in \mathcal{N}}$ such that*

$$q(x) + \sum_{\xi \in \mathcal{N}} b_\xi q(\xi) = 0 \quad \forall q \in \Pi_n \quad \textit{and}$$
$$\sum_{\xi \in \mathcal{N}} |b_\xi| \leq \mathrm{const}(n, \Omega).$$

*Proof.* There exists $r_1 \in (0\,..\,\infty)$ (depending only on $d$ and $n$) such that if $z \in \mathbb{R}^d$ and $\widetilde{\Xi} \subset \mathbb{R}^d$ are such that $\delta(\widetilde{\Xi}; z + r_1 B) \leq 1$, then there exists $\mathcal{N} \subset \widetilde{\Xi} \cap (z + r_1 B)$ such that $\mathcal{N} \in \mathcal{I}_n$ and $|\mathcal{N}|_{\mathcal{I}_n} \leq \mathrm{const}(d, n)$. Let $\epsilon_\Omega, r_\Omega$ be as in Definition 4.1, and put $\delta_0 := r_\Omega / r_1$, $r_0 := r_1(1 + \epsilon_\Omega / r_\Omega)$. Assume $\delta \leq \delta_0$ and $x \in \Omega/\delta$. By Definition 4.1, there exists $y \in \Omega$ such that $|\delta x - y| = \epsilon_\Omega$ and $\delta x + t(y - \delta x + r_\Omega B) \subset \Omega$ for all $t \in [0\,..\,1]$. By substituting $t = \delta r_1 / r_\Omega$ and putting $z := x + (r_1/r_\Omega)(y - \delta x)$ it follows that $|x - z| = r_1 \epsilon_\Omega / r_\Omega$ and $z + r_1 B \subset (\Omega/\delta) \cap (x + r_0 B)$. Since $\delta(\Xi/\delta; z + r_1 B) \leq \delta(\Xi/\delta; \Omega/\delta) = 1$, there exists $\mathcal{N} \subset (\Xi/\delta) \cap (z + r_1 B)$ such that $\mathcal{N} \in \mathcal{I}_n$ and $|\mathcal{N}|_{\mathcal{I}_n} \leq \mathrm{const}(d, n)$. Let $y_{\mathcal{N}}$ and $\{a_{\alpha,\xi}\}_{|\alpha| \leq n, \xi \in \mathcal{N}}$ be as in Definition 2.1. If $q \in \Pi_n$, then

$$q(x) = \sum_{|\alpha| \leq n} \frac{1}{\alpha!} D^\alpha q(y_{\mathcal{N}})(x - y_{\mathcal{N}})^\alpha = \sum_{|\alpha| \leq n} \frac{1}{\alpha!} \sum_{\xi \in \mathcal{N}} a_{\alpha,\xi} q(\xi)(x - y_{\mathcal{N}})^\alpha$$
$$= \sum_{\xi \in \mathcal{N}} \left[ \sum_{|\alpha| \leq n} \frac{1}{\alpha!} a_{\alpha,\xi} (x - y_{\mathcal{N}})^\alpha \right] q(\xi).$$

Hence, if $b_\xi := -\sum_{|\alpha| \leq n} \frac{1}{\alpha!} a_{\alpha,\xi}(x - y_{\mathcal{N}})^\alpha$, then $q(x) + \sum_{\xi \in \mathcal{N}} b_\xi q(\xi) = 0 \; \forall q \in \Pi_n$ and

$$\sum_{\xi \in \mathcal{N}} |b_\xi| \leq \sum_{\xi \in \mathcal{N}} \sum_{|\alpha| \leq n} \frac{1}{\alpha!} |\mathcal{N}|_{\mathcal{I}_n} |x - y_{\mathcal{N}}|^{|\alpha|} \leq \mathrm{const}(d, n, r_0) = \mathrm{const}(n, \Omega).$$

□

The following result shows that if $s$ is any surface spline which happens to interpolate $f_{|\Xi}$, then $\|f - s\|_{L_p(\Omega)}$ can be estimated in terms of the smoothness of $f$ and $|||s|||_\delta$.

**Theorem 4.3.** *Let $\gamma \in [0\,..\,m]$ and $f \in \mathcal{F}_\gamma$. Let $\Omega$ be an open, bounded subset of $\mathbb{R}^d$ having the cone property and let $\Xi$ be a finite subset of $\overline{\Omega}$ for which there exists $\mathcal{N} \subset \Xi$ such that $\mathcal{N} \in \mathcal{I}_{2m-1}$ and $|\mathcal{N}|_{\mathcal{I}_{2m-1}} \leq \mathrm{const}(d, m)$. Let $\Xi_3$ be any finite subset of $\mathbb{R}^d$. If $s \in S(\phi; \Xi_3)$ satisfies $s_{|\Xi} = f_{|\Xi}$, then*

$$\|f - s\|_{L_p(\Omega)} \leq \mathrm{const}(\Omega, m, \gamma)(\delta^{\gamma_p + \gamma} \|f\|_{\mathcal{F}_\gamma} + \delta^{\gamma_p + m - d/2} |||s|||_\delta)$$

*where* $\delta := \delta(\Xi;\Omega)$ *and* $\gamma_p := \min\{m, m + d/p - d/2\}$, $1 \le p \le \infty$.

*Proof.* First note that

$$\begin{aligned} |||f(\delta\cdot) - s(\delta\cdot)|||_* &\le |||f(\delta\cdot)|||_* + |||s(\delta\cdot)|||_* \\ &\le \text{const}(\Omega, m, \gamma)\delta^{m+\gamma-d/2} \|f\|_{\mathcal{F}_\gamma} + \text{const}(d, m)\delta^{2m-d}|||s|||_\delta, \end{aligned} \tag{4.4}$$

by Lemma 2.6 (iii) and Proposition 2.2. Let $\delta_0$ and $r_0$ be as in Lemma 4.2 with $n = 2m-1$.

*Case 1.* $\delta \in (0\,..\,\delta_0]$.

Since, for $1 \le p \le 2$, $\gamma_p$ is constantly $m$ and $\|f - s\|_{L_p(\Omega)} \le \text{const}(\Omega)\|f - s\|_{L_2(\Omega)}$, we may assume without loss of generality that $2 \le p \le \infty$. Put $C := [-1/2\,..\,1/2)^d$ and $\mathcal{A} := \{j \in \mathbb{Z}^d : (j + C) \cap (\Omega/\delta) \ne \emptyset\}$. For each $j \in \mathcal{A}$, let $x_j \in j + C$ be such that $\|f(\delta\cdot) - s(\delta\cdot)\|_{L_\infty((j+C)\cap(\Omega/\delta))} \le 2\,|f(\delta x_j) - s(\delta x_j)|$. By Lemma 4.2, for each $j \in \mathcal{A}$, there exists $\mathcal{N}_j \subset (\Xi/\delta) \cap (x_j + r_0 B)$ and $\{b_{j,\xi}\}_{\xi\in\mathcal{N}_j}$ such that

$$\begin{aligned} &q(x_j) + \sum_{\xi\in\mathcal{N}_j} b_{j,\xi}\, q(\xi) = 0 \quad \forall q \in \Pi_{2m-1} \quad \text{and} \\ &\sum_{\xi\in\mathcal{N}_j} |b_{j,\xi}| \le \text{const}(m, \Omega). \end{aligned}$$

Put $r := r_0 + \sqrt{d/2}$ and note that $\{x_j\} \cup \mathcal{N}_j \subset j + rB$ for all $j \in \mathcal{A}$. Now,

$$\begin{aligned} \|f - s\|_{L_p(\Omega)} &= \delta^{d/p}\|f(\delta\cdot) - s(\delta\cdot)\|_{L_p(\Omega/\delta)} \le \delta^{d/p}\left\| j \mapsto \|f(\delta\cdot) - s(\delta\cdot)\|_{L_\infty((j+C)\cap(\Omega/\delta))}\right\|_{\ell_p(\mathcal{A})} \\ &\le 2\delta^{d/p}\,\|j \mapsto |f(\delta x_j) - s(\delta x_j)|\|_{\ell_p(\mathcal{A})} \le 2\delta^{d/p}\,\|j \mapsto |f(\delta x_j) - s(\delta x_j)|\|_{\ell_2(\mathcal{A})} \\ &= 2\delta^{d/p}\sqrt{\sum_{j\in\mathcal{A}} |f(\delta x_j) - s(\delta x_j)|^2}. \end{aligned}$$

Since $f(\delta\xi) - s(\delta\xi) = 0$ for all $\xi \in \Xi/\delta$, we have

$$|f(\delta x_j) - s(\delta x_j)| = \left| f(\delta x_j) - s(\delta x_j) + \sum_{\xi\in\mathcal{N}_j} (f(\delta\xi) - s(\delta\xi)) \right|, \quad \forall j \in \mathcal{A}.$$

It thus follows by Proposition 3.1 that

$$\sum_{j\in\mathcal{A}} |f(\delta x_j) - s(\delta x_j)|^2 \le \text{const}(m, \Omega)|||f(\delta\cdot) - s(\delta\cdot)|||_*^2.$$

Therefore,

$$\begin{aligned} \|f - s\|_{L_p(\Omega)} &\le \text{const}(m, \Omega)\delta^{d/p}|||f(\delta\cdot) - s(\delta\cdot)|||_* \\ &\le \text{const}(\Omega, m, \gamma)(\delta^{\gamma_p+\gamma}\|f\|_{\mathcal{F}_\gamma} + \delta^{\gamma_p+m-d/2}|||s|||_\delta) \end{aligned}$$

by (4.4).

*Case 2.* $\delta > \delta_0$.

It suffices to show that $\|f-s\|_{L_\infty(\Omega)} \le \text{const}(\Omega, m, \gamma)(\|f\|_{\mathcal{F}_\gamma} + |||s|||_\delta)$. Let $x \in \Omega$. As was shown in the proof of Lemma 4.2, if $y_{\mathcal{N}}$, $\{a_{\alpha,\xi}\}_{|\alpha|\le n, \xi\in\mathcal{N}}$ are as in Definition 2.1 and $b_\xi := -\sum_{|\alpha|\le 2m-1} \frac{1}{\alpha!} a_{\alpha,\xi}(x-y_{\mathcal{N}})^\alpha$, $\xi \in \mathcal{N}$, then $q(x) + \sum_{\xi\in\mathcal{N}} q(\xi) = 0$ $\forall q \in \Pi_{2m-1}$. Let $r$ be the smallest positive real number for which $\Omega \subset y_{\mathcal{N}} + rB$. Then

$$\begin{aligned}
|f(x)-s(x)| &= \left| f(x)-s(x) + \sum_{\xi\in\mathcal{N}} (f(\xi)-s(\xi)) \right| \\
&\le \text{const}(d,m,r)|||f-s|||_*, \qquad \text{by Proposition 3.1}, \\
&\le \text{const}(\Omega,m)(1+\delta^m)\delta^{-2m+d/2}|||f(\delta\cdot)-s(\delta\cdot)|||_*, \qquad \text{by Lemma 2.6 (ii)}, \\
&\le \text{const}(\Omega,m,\gamma)(\|f\|_{\mathcal{F}_\gamma} + |||s|||_\delta), \qquad \text{by (4.4)},
\end{aligned}$$

since $\delta_0 \le \delta \le \text{const}(\Omega)$. □

Our first application of Theorem 4.3 is to prove a result mentioned in the introduction regarding the size of $\lambda$ in the case when $\Omega = B$ and $\Xi = h\mathbb{Z}^d \cap (1-h)B$.

**Proposition 4.5.** *There exists $f \in C^\infty(\mathbb{R}^d)$ such that if $\Omega = B$, $\Xi = h\mathbb{Z}^d \cap (1-h)B$ and $T_\Xi f = q + \sum_{\xi\in\Xi} \lambda_\xi \phi(\cdot - \xi)$, then*

$$\begin{aligned}
&(i) \qquad \|f - T_\Xi f\|_{L_p(B)} \ne o(h^{m+1/p}), \quad 1 \le p \le \infty, \qquad \textit{and} \\
&(ii) \qquad \|\lambda\|_{\ell_2(\Xi)} \ne o(h^{(d+1)/2-m}) \qquad \textit{as } h \to 0.
\end{aligned}$$

*Proof.* It was shown in [J1] that there exists a compactly supported $f \in C^\infty(\mathbb{R}^d)$ such that (i) holds. In order to prove that (ii) holds for the same function $f$, suppose to the contrary that $\|\lambda\|_{\ell_2(\Xi)} = o(h^{(d+1)/2-m})$. Since $\|f\|_{W_2^{2m}} < \infty$, it follows by Theorem 4.3 (with $s = T_\Xi f$, $\gamma = m$, $\Xi_3 = \Xi$, $p = 2$) that for sufficiently small $h$

$$\begin{aligned}
\|f - T_\Xi f\|_{L_2(B)} &\le \text{const}(d,m)(h^{2m}\|f\|_{W_2^{2m}} + h^{2m-d/2}|||T_\Xi f|||_h) \\
&\le \text{const}(d,m)(h^{2m}\|f\|_{W_2^{2m}} + h^{2m-d/2}\|\lambda\|_{\ell_2(\Xi)}), \qquad \text{by Proposition 2.3}, \\
&= o(h^{m+1/2})
\end{aligned}$$

which contradicts (i). □

Our main result is now obtained by applying Theorem 4.3 in the case when $s$ is chosen according to Interpolation Method 1.3. We employ results from [J3] to estimate $|||s|||_\delta$.

**Theorem 4.6.** *Let $\gamma \in [0\,..\,m]$ and $f \in \mathcal{F}_\gamma$. Let $s = q + \sum_{\xi\in\Xi_2} \lambda_\xi \phi(\cdot - \xi)$ be chosen according to Interpolation Method 1.3 and assume that there exists $\mathcal{N} \subset \Xi$ such that $\mathcal{N} \in \mathcal{I}_{2m-1}$ and $|\mathcal{N}|_{\mathcal{I}_{2m-1}} \le \text{const}(d,m)$. Then*

$$\begin{aligned}
&(i) \qquad \|f-s\|_{L_p(\Omega)} \le \text{const}(\Omega, \Omega_2, m, \gamma)\delta^{\gamma_p+\gamma}\|f\|_{\mathcal{F}_\gamma}, \\
&(ii) \qquad |||s|||_\delta \le \text{const}(\Omega, \Omega_2, m, \gamma)\delta^{\gamma-m+d/2}\|f\|_{\mathcal{F}_\gamma}, \qquad \textit{and} \\
&(iii) \qquad \|\lambda\|_{\ell_2(\Xi_2)} \le \text{const}(\Omega, \Omega_2, m, \gamma)(\delta/\epsilon)^{m-d/2}\delta^{\gamma-m+d/2}\|f\|_{\mathcal{F}_\gamma},
\end{aligned}$$

*where* $\gamma_p := \min\{m, m+d/p-d/2\}$, $\delta := \delta(\Xi;\Omega)$, *and* $\epsilon := sep(\Xi_2)$.

*Proof.* We first prove (ii). Since $\delta(\Xi_2;\Omega_2) \le \mathrm{const}(d,m)\delta(\Xi;\Omega)$ and with Proposition 2.9 in view, we may assume without loss of generality that $\delta(\Xi_2;\Omega_2) \le \delta$. Let $\sigma \in C_c^\infty(\mathbb{R}^d)$ be such that $\sigma = 1$ on $\Omega$ and $K := \operatorname{supp}\sigma \subset \Omega_2$. Put $\widetilde{f} := \sigma f$ and note that $\operatorname{supp}\widetilde{f} \subset K$ and $\left\|\widetilde{f}\right\|_{\mathcal{F}_\gamma} \le \mathrm{const}(d,m,\sigma)\left\|f\right\|_{\mathcal{F}_\gamma}$. The following is known [D1] for $\gamma = 0$ and is proved in [J3; th. 5.1] for $\gamma \in (0\,..\,m]$.

$$
\begin{aligned}
|||\widetilde{f} - T_{\Xi_2}\widetilde{f}|||_{H^m} &\le \mathrm{const}(K,\Omega_2,m,\gamma)\delta^\gamma \left\|\widetilde{f}\right\|_{\mathcal{F}_\gamma} \\
&\le \mathrm{const}(\Omega_2,m,\gamma,\sigma)\delta^\gamma \left\|f\right\|_{\mathcal{F}_\gamma}.
\end{aligned}
\tag{4.7}
$$

Since $T_{\Xi_2}\widetilde{f} \in S(\phi;\Xi_2)$ and satisfies $(T_{\Xi_2}\widetilde{f})_{|\Xi} = f_{|\Xi}$, it follows by Proposition 2.2 that

$$
\begin{aligned}
&|||s(\delta\cdot)|||_* \le \mathrm{const}(d,m)|||(T_{\Xi_2}\widetilde{f})(\delta\cdot)|||_* \\
&\le \mathrm{const}(d,m)|||\widetilde{f}(\delta\cdot)|||_* + \mathrm{const}(d,m)|||\widetilde{f}(\delta\cdot) - (T_{\Xi_2}\widetilde{f})(\delta\cdot)|||_* \\
&\le \mathrm{const}(d,m,\gamma)\delta^{m+\gamma-d/2}(1+\delta^m)\left\|\widetilde{f}\right\|_{\mathcal{F}_\gamma} + \mathrm{const}(d,m)\delta^{m-d/2}|||\widetilde{f} - T_{\Xi_2}\widetilde{f}|||_{H^m}, \quad \text{by Lemma 2.6}, \\
&\le \mathrm{const}(\Omega_2,m,\gamma,\sigma)\delta^{m+\gamma-d/2}\left\|f\right\|_{\mathcal{F}_\gamma}, \quad \text{by (4.7) and since } \delta \le \mathrm{const}(\Omega_2),
\end{aligned}
$$

which in view of Propostion 2.2 (and after a suitable choice of $\sigma$) proves (ii) . Note that (i) follows from (ii) via Theorem 4.3. In order to prove (iii), note that by Proposition 2.3 and Proposition 2.6,

$$
\begin{aligned}
&\|\lambda\|_{\ell_2(\Xi_2)} \le \mathrm{const}(d,m)|||s|||_\epsilon = \mathrm{const}(d,m)\epsilon^{-2m+d}|||s(\epsilon\cdot)|||_* \\
&\le \mathrm{const}(d,m)\epsilon^{-2m+d}(\delta/\epsilon)^{-2m+d/2}(1+(\delta/\epsilon)^m)|||s(\delta\cdot)|||_*, \quad \text{by Lemma 2.6 (ii)}, \\
&= \mathrm{const}(d,m)(\delta/\epsilon)^{-d/2}(1+(\delta/\epsilon)^m)|||s|||_\delta \le \mathrm{const}(\Omega,\Omega_2,m,\gamma)(\delta/\epsilon)^{m-d/2}\delta^{\gamma-m+d/2}\left\|f\right\|_{\mathcal{F}_\gamma},
\end{aligned}
$$

by (ii). $\square$

## 5. Some bounds on $\|\lambda\|_{\ell_2(\Xi)}$ in case $\Omega = \mathbb{R}^d$ and $\Xi = h\mathbb{Z}^d$

Buhmann's [B1] extension of the definition of $T_\Xi f$ to the case $\Xi = h\mathbb{Z}^d$ is well defined under very minimal restrictions on the growth of $f$ at infinity. However, $T_{h\mathbb{Z}^d} f$ cannot necessarily be written as a series of the form $\sum_{j\in\mathbb{Z}^d}\lambda_j\phi(\cdot - hj)$ which converges uniformly on compact sets unless we make some decay assumptions on $f$. The following can easily be derived from [B1]:

**Theorem 5.1.** *Let* $h > 0$ *and* $k > \max\{2m, m+d\}$. *If* $f \in C(\mathbb{R}^d)$ *satisfies* $\left\||\cdot|^k f\right\|_{L_\infty} < \infty$, *then there exists a unique* $\lambda \in \ell_2$ *such that*

$$
\begin{aligned}
&(i) && \left\||\cdot|^k \lambda\right\|_{\ell_\infty} < \infty, \\
&(ii) && \sum_{j\in\mathbb{Z}^d}\lambda_j q(hj) = 0 \quad \forall q \in \Pi_{m-1}, \quad \textit{and} \\
&(iii) && s := \sum_{j\in\mathbb{Z}^d}\lambda_j\phi(\cdot - hj) \quad \textit{satisfies} \quad s_{|h\mathbb{Z}^d} = f_{|h\mathbb{Z}^d}.
\end{aligned}
$$

The coefficients $\{\lambda_j\}_{j\in\mathbb{Z}^d}$ above are given by $\lambda_j := h^{-2m+d} \sum_{\ell\in\mathbb{Z}^d} f(h\ell)c_{\ell-j}$ where $\{c_j\}_{j\in\mathbb{Z}^d}$ is an exponentially decaying sequence defined by

$$\sum_{j\in\mathbb{Z}^d} c_j e_{-j} = \omega := \frac{1}{c_\phi \sum_{j\in\mathbb{Z}^d} |\cdot + 2\pi j|^{-2m}},$$

where $c_\phi$ is a nonzero constant depending only on $d$, $m$. Assuming $f \in W_2^m$, it is a direct application of Poisson's summation formula to show that $\sum_{j\in\mathbb{Z}^d} \lambda_j e_{-j} = h^{-2m}\omega \sum_{j\in\mathbb{Z}^d} \widehat{f}(\cdot/h + 2\pi j/h)$. Consequently,

$$\|\lambda\|_{\ell_2} = (2\pi)^{-d/2} h^{-2m} \left\| \omega \sum_{j\in\mathbb{Z}^d} \widehat{f}(\cdot/h + 2\pi j/h) \right\|_{L_2(2\pi C)}. \tag{5.2}$$

Much can be derived from (5.2). For example, it is possible to show that if $0 < \gamma < m$, then $\|\lambda\|_{\ell_2} \le \mathrm{const}(d,m,\gamma) h^{\gamma-m+d/2} \|f\|_{B^{m+\gamma}_{2,\infty}}$ and there exists an exponentially decaying $f \in B^{m+\gamma}_{2,\infty}$ such that $\|\lambda\|_{\ell_2} \ne o(h^{\gamma-m+d/2})$ as $h \to 0$. We refrain from proving this result, but instead prove the following:

**Proposition 5.3.** *If $f \in W_2^{2m}\backslash 0$ satisfies $\left\| |\cdot|^k f \right\|_{L_\infty} < \infty$ for some $k > \max\{2m, m+d\}$ and $\lambda$ is as in Theorem 5.1, then*

$$\begin{aligned} &(i) \qquad \|\lambda\|_{\ell_2} \le \mathrm{const}(d,m) h^{d/2} |||f|||_{H^{2m}}, \quad \forall h > 0, \quad \textit{and} \\ &(ii) \qquad \|\lambda\|_{\ell_2} \ne o(h^{d/2}) \quad \textit{as } h \to 0. \end{aligned}$$

*Proof.* Noting that $\omega$ satisfies $\mathrm{const}(d,m)\,|x|^{2m} \le |\omega(x)| \le \mathrm{const}(d,m)\,|x|^{2m}$, $x \in 2\pi C$, we obtain from (5.2) that

$$\begin{aligned}
\|\lambda\|_{\ell_2} &\le \mathrm{const}(d,m) h^{-2m} \left( \left\| |\cdot|^{2m} \widehat{f}(\cdot/h) \right\|_{L_2(2\pi C)} + \sum_{j\in\mathbb{Z}^d\backslash 0} \left\| \widehat{f}(\cdot/h + 2\pi j/h) \right\|_{L_2(2\pi C)} \right) \\
&= \mathrm{const}(d,m) h^{d/2} \left( \left\| |\cdot|^{2m} \widehat{f} \right\|_{L_2(2\pi C/h)} + h^{-2m} \sum_{j\in\mathbb{Z}^d\backslash 0} \left\| \widehat{f} \right\|_{L_2(2\pi(j+C)/h)} \right) \\
&\le \mathrm{const}(d,m) h^{d/2} \left( |||f|||_{H^{2m}} + h^{-2m} \sum_{j\in\mathbb{Z}^d\backslash 0} \left\| |\cdot|^{-2m} \right\|_{L_\infty(2\pi(j+C)/h)} \left\| |\cdot|^{2m} \widehat{f} \right\|_{L_2(2\pi(j+C)/h)} \right) \\
&\le \mathrm{const}(d,m) h^{d/2} \left( |||f|||_{H^{2m}} + \left\| |\cdot|^{2m} \widehat{f} \right\|_{L_2(\mathbb{R}^d\backslash 2\pi h^{-1} C)} \right), \quad \text{by Cauchy-Schwarz ineq.,} \\
&\le \mathrm{const}(d,m) h^{d/2} |||f|||_{H^{2m}}
\end{aligned}$$

which proves (i). The above argument can be restructured to yield

$$\|\lambda\|_{\ell_2} \geq \mathrm{const}(d,m)h^{d/2} \left\| |\cdot|^{2m} \widehat{f} \right\|_{L_2(2\pi C/h)} - \mathrm{const}(d,m)h^{d/2} \left\| |\cdot|^{2m} \widehat{f} \right\|_{L_2(\mathbb{R}^d \setminus 2\pi h^{-1} C)}$$
$$\neq o(h^{d/2})$$

since $\left\| |\cdot|^{2m} \widehat{f} \right\|_{L_2(\mathbb{R}^d \setminus 2\pi h^{-1} C)} = o(1)$. □

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