# Approximation orders of shift-invariant subspaces of $W_2^s(\mathbb{R}^d)$


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**Key words.** Approximation order, shift-invariant spaces, Sobolev spaces, polynomial reproduction, Strang-Fix conditions, refinement equation, sum rules, superfunction.

**AMS subject classification.** 41A25, 41A63, 41A30, 41A15, 42B99, 46E30.



**Abstract**

We extend the existing theory of approximation orders provided by shift-invariant subspaces of $L_2$ to the setting of Sobolev spaces, provide treatment of $L_2$ cases that have not been covered before, and apply our results to determine approximation order of solutions to a refinement equation with a higher-dimensional solution space.


## Contents




*Supported by the US National Science Foundation under Grant DMS-9872890, by Alexander von Humboldt Foundation and by the DFG Research Center "Mathematics for key technologies" in Berlin, Germany.

†Supported by the US National Science Foundation under Grants DMS-9872890, DBI-9983114. and ANI-0085984, and by the U.S. Army Research Office under Contract DAAG55-98-1-0443.

# 1 Introduction

## 1.1 General

We are interested in this paper in the approximation order of *shift-invariant (SI) spaces* of functions defined on the Euclidean space $\mathbb{R}^d$, $d \geq 1$. Such spaces play an important role in several areas of Real Analysis, including Spline Approximation, Wavelets, Subdivision Algorithms, Uniform Sampling and Gabor Systems. It is not surprising, thus, that the theory of approximation and representation from SI spaces received significant attention and enjoyed rapid development in the last 10–15 years. The determination and understanding of the *approximation orders* of these spaces is among the main pillars of this theory.

As the title of this article indicates, we restrict our attention to approximation in Sobolev spaces: given $s \in \mathbb{R}$, we denote by $W_2^s(\mathbb{R}^d)$ the **Sobolev space of smoothness** $s$, i.e., the space of all tempered distributions $f$ whose Fourier transform is locally in $L_2(\mathbb{R}^d)$ and satisfies

$$\|f\|^2_{W_2^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\cdot|)^{2s} |\widehat{f}|^2 < \infty.$$

(Here and elsewhere, $|\cdot|$ is the Euclidean distance in $\mathbb{R}^d$.) A closed subspace $S \subset W_2^s(\mathbb{R}^d)$ is **shift-invariant** if it is invariant under all **shifts**, i.e., integer translations, or more generally, scaled integer translations: given a fixed $h > 0$,

$$\text{for every } \alpha \in h\mathbb{Z}^d \text{ and every } f \in W_2^s(\mathbb{R}^d), \quad f \in S \Longrightarrow f(\cdot + \alpha) \in S.$$

When necessary, one identifies the underlying parameter $h$ by referring to $S$ as $h$-shift-invariant, and/or by denoting the SI space as $S^h$. Also, sometimes, in order to emphasize the ambient space $W_2^s(\mathbb{R}^d)$ we write $S(W_2^s)$, instead of simply $S$. The smallest SI space that contains a given $\Phi \subset W_2^s(\mathbb{R}^d)$ is denoted by

$$S_\Phi := S_\Phi(W_2^s),$$

or, in complete detail, $S_\Phi^h(W_2^s)$, and we refer then to $\Phi$ as a **generating set** of $S_\Phi$. The basic objective of shift-invariant space theory is to understand the properties of SI spaces in terms of properties of their generating sets. In this regard we recall that an SI space generated by a *singleton* $\Phi = \{\phi\}$ is known as **Principal Shift-Invariant** (PSI), while the one generated by a *finite* $\Phi$ is referred to as **Finitely-generated Shift-Invariant** (FSI).

Now, assume that we are given a *ladder* $\mathcal{S} := (S^h := S^h(W_2^s))_{h>0}$ of SI spaces. Let $k > s$. We say that $\mathcal{S}$ **provides approximation order** $k$ (in $W_2^s(\mathbb{R}^d)$), if, for every $f \in W_2^k(\mathbb{R}^d)$,

$$\mathrm{dist}_s(f, S^h) := \inf_{g \in S^h} \|f - g\|_{W_2^s(\mathbb{R}^d)} \leq C h^{k-s} \|f\|_{W_2^k(\mathbb{R}^d)},$$

with the constant $C$ independent of $f$ and $h$. As is essentially known [16] (and developed fully in this paper), the above notion of approximation order depends strongly on $k$ but only mildly on $s$. The ladder $\mathcal{S}$ is PSI or FSI if each of its components $S^h$ is a PSI, or, respectively, FSI space.

The literature on approximation orders of SI spaces is vast, and it is not within the scope of this paper to provide a comprehensive review of it. We refer to the introduction and the references of [3] as well as to the exposition and the references in the survey article [15]. Many specific results on the topic are reviewed in the body of this article. In particular, a complete characterization of the $L_2$-approximation orders (i.e., the case $s = 0$) of PSI ladders is obtained in [3], while the analogous results for FSI ladders are obtained in [3], [4] and [5]. There are also numerous results on approximations in other norms, for example, in $L_p$. Results and references in this direction can be found in [25] and [24]. In addition, we refer the reader to [14], [22], [29], [31] for information on approximation properties of refinable SI spaces, and to [27], [26], [18], [19], [21], [28] for results on wavelet constructions based on SI spaces.

## 1.2 Motivation

While the current level of mathematical understanding of the issue of approximation orders of SI spaces is quite advanced, there are numerous gaps and inconsistencies in it. This is exactly the motivation behind the present endeavor: obtaining seamless, cohesive (and, so we hope, final) theory. We provide a few examples for the "gaps" and "inconsistencies" in the state-of-the-art theory. Let us first define two important classes of SI ladders: stationary ladders, and local ones.

**Definition.** Let $\mathcal{S}$ be an SI ladder. We say that $\mathcal{S}$ is **stationary** if, for every $h > 0$, $S^h = S^1(\cdot/h) := \{f(\cdot/h) : f \in S^1\}$. Given a stationary ladder, we say that $\mathcal{S}$ is also **local** if $S^1$ is FSI and is generated by a *compactly supported* $\Phi$.

**Discussion 1 (1)** Let us assume that $\mathcal{S}$ is PSI, stationary and local. Then the entire ladder is determined by the (compactly supported) generator $\phi$ of $S^1$ (since the other spaces in the ladder are dilations of $S^1$). In this case, one usually refers to $\phi$ as **the generator of the ladder**. The current theory covers the case $s \geq 0$, and shows that the approximation order in $L_2$ (as well as in $W_2^s$, $s > 0$) provided by such ladders is intimately related to the order of the zeros $\widehat{\phi}$, the Fourier transform of $\phi$, has at the punctured lattice $2\pi\mathbb{Z}^d \backslash 0$ (cf. §3.2). The smoothness of $\phi$, on the other hand, does not play any role, provided, of course, that $\phi \in L_2$ (which is required for the definition of $L_2$-orders to make sense). Thus, if we replace $\phi \in L_2$ by its convolution product with a smooth generic mollifier, the $L_2$-approximation order of the ladder, in general, will not change. In contrast, if $\phi \notin L_2$ while its Fourier transform does have the requisite zeros on $2\pi\mathbb{Z}^d \backslash 0$, the smoothing may simply result in an $L_2$-function, and the ladder may then provide high approximation order in

$L_2$ (despite the fact that the $L_2$-approximation order provided by the initial ladder is zero). One expects that the extension of the notion of approximation order to $W_2^s$, $s<0$ will remove the above artificial hump, and this is, indeed, the case.

**(2)** Retaining the same setup as in (1), it is also quite well-known that if $\phi_1$ and $\phi_2$ are two compactly supported $L_2(\mathbb{R})$-functions, and if the PSI stationary ladder generated by $\phi_j$, $j=1,2$, provides approximation order $k_j>0$, then the PSI stationary ladder generated by $\phi_1*\phi_2$ provides (at a minimum) approximation order $k_1+k_2$. One expects then that, if $k_2=0$, the approximation order provided by $\phi_1*\phi_2$ will be at least $k_1$. This, however, is not the case, and there are examples when the aforementioned approximation order is smaller than $k_1$. This nuisance is fixed (in §3.4) via the introduction of negative approximation orders.

**(3)** Let us consider now the case of local stationary FSI ladders in $L_2$. In this case $S^1 = S^1_\Phi(L_2)$, with $\Phi :=\{\phi_1,\ldots,\phi_r\}\subset L_2(\mathbb{R}^d)$ compactly supported, and with $S^h = S^1(\cdot/h)$, $0<h<1$. A cornerstone in the analysis of the approximation order of such ladders in the existence of a superfunction, i.e., a compactly supported function $\psi\in S^1$ whose associated local stationary PSI ladder already provides the same approximation order as the original FSI ladder (cf. §4.2). The existence of such a superfunction is proved in [4]. However, the Fourier transform of the superfunction $\psi$ may vanish at the origin, a property that denies us the existence of effective numerical approximation schemes from its associated ladder (we refer to such superfunctions as "bad"). In [5], this problem is overcome, but at the price of imposing an additional condition on the vector $\Phi$ (its Gramian should be invertible at the origin; cf. §4.5 for a complete discussion). At the outset of the current venture, we observed that the condition assumed in [5] is not necessary for the existence of a "good" superfunction (i.e., a superfunction $\psi$ for which $\widehat{\psi}(0)\neq 0$). Unfortunately, a good superfunction may not always exist: in §4.8 we construct an FSI vector (with $d=r=2$) for which all the compactly supported superfunctions are bad, dashing thereby our hope that a good superfunction may be proved to exist in general.

**(4)** Let $\mathcal{S}$ be a ladder as in **(3)**, but assume, in addition, that $S^1 := S^1_\Phi$ is refinable, i.e., that $S^2\subset S^1$. It is then known (see, e.g., [27], [12] and [31] for the PSI case and [31] for the FSI case) that the $L_2$-approximation orders provided by the ladder are bounded below by the smoothness of $\Phi$: if $\Phi\subset W_2^k(\mathbb{R}^d)$, then the ladder provides approximation order $k+1$ or higher. Moreover, [31] proves (for $d=1$, and under some mild conditions on $\Phi$ for $d>1$) that approximation order $k+1$ is implied by the mere existence of a nonzero function $f$ in $S^1\cap W_2^k(\mathbb{R}^d)$. However, all these results assume more than the smoothness of $f$ and the refinability of $S^1$: they require in addition the entire vector $\Phi$ to lie in $L_2$. The removal of this condition (§4.9) leads to a conclusion that says, essentially, that for $\mathcal{S}$ to provide some approximation order, it should contain one nonzero function of corresponding smoothness, and nothing else.

**(5)** Our final example still deals with refinable ladders. One way to obtain a refinable space $S_\Phi$ is to select an $r\times r$ matrix $P$ whose entries are trigonometric polynomials and to seek a compactly supported vector-valued function $\Phi$ that solves the refinement equation $\widehat{\Phi}(2\cdot) = P\widehat{\Phi}$. A major goal in this direction is to reveal the approximation order of the stationary ladder generated by $\Phi$ in terms of properties of $P$ (see [14], [17], [5]). The ultimate known result in this direction, [5], requires a regularity condition on $\Phi$ that necessarily fails once the above refinement equation has (in a nontrivial way) more than one solution. Thus, there is no theory at present that deals with the approximation orders of refinable vectors, once the refinement equation has multiple solutions. §5 deals with the approximation order of stationary refinable ladders and provides a novel theory for the case when multiple solutions to the same refinement equation exist.

### 1.3 Layout of this article

In the introductory §2, we define the notions of shift-invariance and approximation order and make several basic observations that will be extensively used in the sequel.

Section 3 is devoted to PSI ladders. The section begins with a summary of the known characterization of the $L_2$-approximation orders provided by PSI stationary and nonstationary ladders. These results are then extended to general $W_2^s$ (the end of §3.1) and connected with the notion of the Strang-Fix conditions (§3.2) and polynomial reproduction (§3.5). The results from §3.2 are in turn used in §3.3 to analyze the dependence of the approximation order notion on the value of $s$, i.e., on the space where the error is measured. The issue of negative approximation orders is discussed in §3.4.

FSI ladders are considered in §4. It begins, analogously to §3, with a summary on the $L_2$-approximation orders of FSI spaces and with the extension of these results to the setting of Sobolev spaces. This takes up §4.1. Section 4.2 focuses on the notion of a superfunction, which is instrumental in the reduction of the FSI case to the PSI case. This notion is further used in §4.4 to understand polynomial reproduction from FSI spaces and in §4.3 to establish the consistency of the notion of approximation order as we operate in different Sobolev spaces. However, not all superfunctions are equally useful, as is made clear in §4.5 and §4.8. Regardless of whether 'good' superfunctions exist in the underlying FSI space, there is an alternative method proposed in §4.6 that can always be used to estimate approximation orders. The usefulness of that alternative approach is demonstrated by an example in §4.7. Section 4.9 is devoted to refinable FSI spaces. It shows that the approximation order of stationary refinable FSI spaces is bounded below by (a weak variant of) the decay rate of the Fourier transform of any (nonzero) function in the space.

In §5, applications of the theory from the preceding sections to multiple solutions to a refinement equation are developed. We start by discussing, in §5.1, the structure of the solutions space to a refinement question. In §5.2, we introduce the notion of coherent approximation orders, which bundles together different solutions to the same refinement equation. In §5.3, the notion of coherent approximation order is associated with a corresponding (novel) notion of universal supervectors; those lead to a uniform way of constructing superfunctions in all the FSI spaces that are generated by the various solutions to the given refinement equation.

For the convenience of the reader, some of the notations used in the paper are collected immediately before the bibliography.

## 2 SI ladders: the prelude

We start our analysis with a few elementary, yet very useful, observations concerning the interplay between approximation orders in $W_2^s(\mathbb{R}^d)$ on the one hand, and in $L_2(\mathbb{R}^d)$ on the other hand.

As mentioned before, the symbol $W_2^k(\mathbb{R}^d)$ denotes the **Sobolev space of smoothness** $k$. Note also the isometry

$$J_{-k} : L_2(\mathbb{R}^d) \to W_2^k(\mathbb{R}^d) : f \mapsto \left((1 + |\cdot|^2)^{-k/2}\widehat{f}\right)^{\vee}. \tag{1}$$

Recall that the Sobolev spaces are ordered by embedding: $W_2^s \hookrightarrow W_2^t$ whenever $s \geq t$.

## 2.1 Shift-invariance

The notion of shift-invariance is valid in any function space $F$ whose elements are defined on $\mathbb{R}^d$, and is certainly not specific to $W_2^s(\mathbb{R}^d)$. Given such a space $F$, we consider now SI spaces that are invariant under *integer* translations; thus, we refer to a closed subspace $S \subset F$ as *shift-invariant* if $S$ is invariant under multi-integer shifts

$$s \in S \Longrightarrow s(\cdot - \alpha) \in S, \qquad \alpha \in \mathbb{Z}^d.$$

In agreement with the definitions of PSI and FSI ladders, a **principal shift-invariant (PSI) space** $S_\phi$ is generated by a single function $\phi \in F$ as the closure of

$$\operatorname{span}[\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^d]$$

in the topology of $F$, while a **finitely generated shift-invariant (FSI) space** $S_\Phi$ is the closure of $\sum_{\phi \in \Phi} S_\phi$, with $\Phi$ a finite subset of $F$.

It is known that an FSI subspace of $L_2(\mathbb{R}^d)$ can be characterized on the Fourier domain as follows:

**Result 2 ([4])** *For* $\Phi \subset L_2(\mathbb{R}^d)$,

$$S_\Phi(L_2(\mathbb{R}^d)) = \{f \in L_2(\mathbb{R}^d) : \widehat{f} = \tau^* \widehat{\Phi},\ \tau \text{ measurable},\ \ \tau(\cdot + \alpha) = \tau,\ \ \text{all}\ \ \alpha \in 2\pi\mathbb{Z}^d\}. \qquad (2)$$

That is, the Fourier transform of an element of $S_\Phi(L_2)$ is the inner product of two vector-valued functions: the vector $\tau$ (whose entries are measurable and $2\pi$-periodic but otherwise arbitrary), and the vector $\widehat{\Phi}$. Note that we tacitly assume that the entries of $\tau$ are indexed by $\Phi$ (or by the same index set that is used to index $\Phi$).

Since the operators $J_s$ commute with translations, one easily checks that

$$S_\Phi(W_2^s) = J_{-s} S_{J_s \Phi}(L_2), \qquad (3)$$

which, together with Result 2, leads to the following

**Corollary 3** *For* $\Phi \subset W_2^s(\mathbb{R}^d)$,

$$S_\Phi(W_2^s) = \{f \in W_2^s(\mathbb{R}^d) : \widehat{f} = \tau^* \widehat{\Phi},\ \tau \text{ measurable},\ \ \tau(\cdot + \alpha) = \tau,\ \ \text{all}\ \ \alpha \in 2\pi\mathbb{Z}^d\}. \qquad (4)$$

## 2.2 Approximation orders

The basic idea leading to the notion of approximation order is very simple. It is the heuristic understanding that increasing the density of translations used to define an SI space may improve their approximation "power". At the same time, for numerical reasons (and also for deeper theoretical reasons), one would, almost always, change the generator(s) of the SI space when switching from $S^1$ to $S^h$, $h < 1$: the new generators should be more localized, and one way, sometime adequate sometime not, to modify the generators is by dilation (see the definition of a stationary ladder in §1.2).

The following, simple but important, result connects the approximation orders of SI ladders in $L_2$ to the analogous approximation orders in $W_2^s$:

**Proposition 4** *The ladder $\mathcal{S} = (S^h = S^h(W_2^s))_h$ provides approximation order $k$ in $W_2^s$ if and only if $(J_s S^h)_h$ provides approximation order $k{-}s$ in $L_2$, where $J_s$ is defined as in (1).*

*Proof.* $J_s$ is an isometry from $W_2^k$ to $W_2^{k-s}$ as well as from $W_2^s$ to $L_2$. Thus, if $J_s\mathcal{S}$ provides approximation $k-s$ in $L_2$ then, for every $f \in W_2^k$,

$$\mathrm{dist}_s(f, S^h) = \mathrm{dist}_0(J_s f, J_s S^h)_{L_2} \le C h^{k-s} \|J_s f\|_{W_2^{k-s}} = C h^{k-s} \|f\|_{W_2^k}.$$

Hence $\mathcal{S}$ provides approximation order $k$ in $W_2^s$. The converse is proved in the same manner. □

As already indicated before, the two most important cases of SI ladders are

- **PSI:** each $S^h$ is an $h$-dilate of some PSI space, i.e., $S^h = S_{\phi_h}(\cdot/h)$; a PSI ladder may be stationary or nonstationary depending on whether or not the generator $\phi_h$ of $S^h$ is independent of $h$.
- **FSI:** each $S^h$ is an $h$-dilate of some FSI space $S_{\Phi_h}$; an FSI ladder, just like a PSI ladder, may be stationary or nonstationary.

Nonstationary FSI ladders are broad enough to cover almost all situations of interest in applications. Thus it is of primary importance to be able to characterize the approximation orders provided by such ladders. It turns out that nonstationary ladders are useful not only on their own, but also as a tool for analyzing stationary ladders.

**Corollary 5** *An FSI ladder $(S^h := S_{\Phi_h}(\cdot/h))_h$ provides approximation order $k$ in $W_2^s$ if and only if the FSI ladder $(S_{\Psi_h}(\cdot/h))_h$, $\widehat{\Psi}_h := (1 + |\cdot/h|^2)^{s/2}\widehat{\Phi}_h$, provides approximation order $k{-}s$ in $L_2$.*

*Proof.* In view of Proposition 4, we only need to identify $(J_s S^h)(h\cdot)$ as $S_{\Psi_h}$, with $\Psi_h$ as in the corollary. Now, by Corollary 3, $f \in S_{\Phi_h}(W_2^s)$ iff $f \in W_2^s$ and $\widehat{f} = \tau^*\widehat{\Phi}_h$, $\tau$ $2\pi$-periodic. Thus, $f \in S^h$ iff $f \in W_2^s$ and $\widehat{f} = \tau^*\widehat{\Phi}_h(h\cdot)$, with $\tau$ $2\pi/h$-periodic. Thus $f \in J_s S^h$ iff $f \in L_2$, and

$$\widehat{f} = (1 + |\cdot|^2)^{s/2}\tau^*\widehat{\Phi}_h(h\cdot).$$

Dilating the last equation, we obtain that $f \in (J_s S^h)(h\cdot)$ iff $f \in L_2$ and

$$\widehat{f} = (1 + |\cdot/h|^2)^{s/2}\tau^*\widehat{\Phi}_h$$

for a $2\pi$-periodic $\tau$. By Result 2, this last condition is equivalent to $f$ being in $S_{\Psi_h}(L_2)$. □

Note that the ladder associated with $(\Psi_h)_h$ in the above result is nonstationary even when we assume the original one to be stationary, i.e., when we assume $\Phi_h$ to be independent of $h$.

# 3 PSI ladders

We start our study of PSI ladders by recalling the characterization of the $L_2$-approximation order of these spaces. We then extend the result to the Sobolev space $W_2^s$. The general result is then connected with the notions of the Strang-Fix conditions and polynomial reproduction. In turn, those latter notions allow us to understand the dependence of the approximation order notion on the value of $s$, i.e., on the space where the error is measured.

### 3.1 Approximation orders of PSI ladders

Note that the first part of the next result is not entirely a special case of second part (although it can be derived from it with ease).

**Result 6 ([3, Theorems 1.6 and 4.3])**

1. *The stationary PSI ladder* $\mathcal{S} = (S^h := S^h(L_2))$, *with* $S^h = S_\phi(\cdot/h)$, $\phi \in L_2(\mathrm{IR}^d)$, *provides approximation order* $k$ *if and only if there exists a neighborhood* $\Omega$ *of* $0$ *such that*
$$\frac{[\widehat{\phi},\widehat{\phi}]^0}{[\widehat{\phi},\widehat{\phi}]}\,\frac{1}{|\cdot|^{2k}} \in L_\infty(\Omega).$$
*Here* $[\widehat{\phi},\widehat{\phi}] := \sum_{\alpha\in 2\pi\mathbb{Z}^d} |\widehat{\phi}(\cdot+\alpha)|^2$, $[\widehat{\phi},\widehat{\phi}]^0 := \sum_{\alpha\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}(\cdot+\alpha)|^2$.

2. *The nonstationary PSI ladder* $\mathcal{S} = (S^h := S^h(L_2))$, *with* $S^h = S_{\phi_h}(\cdot/h)$, $\phi_h \in L_2(\mathrm{IR}^d)$, *provides approximation order* $k$ *if and only if, for some* $h_0 > 0$ *and some neighborhood* $\Omega$ *of* $0$, *the collection of functions*
$$\frac{[\widehat{\phi}_h,\widehat{\phi}_h]^0}{[\widehat{\phi}_h,\widehat{\phi}_h]}\,\frac{1}{(|\cdot|^2+h^2)^k}, \qquad 0 < h < h_0,$$
*lies in* $L_\infty(\Omega)$ *and is bounded there.*

Combining Proposition 4 and Result 6 we obtain the analogous result for Sobolev spaces.

**Theorem 7** *Let* $s \in \mathrm{IR}$ *and* $k > s$. *Assume also that* $k$ *is nonnegative.*

1. *The stationary PSI ladder* $\mathcal{S} = (S^h := S^h(W_2^s))$, *with* $S^h = S_\phi(\cdot/h)$, $\phi \in W_2^s$, *provides approximation order* $k$ *if and only if there exists a neighborhood* $\Omega$ *of* $0$ *such that*
$$\mathcal{M}_{\phi,s} := \frac{[\widehat{\phi},\widehat{\phi}]_s^0}{[\widehat{\phi},\widehat{\phi}]_s}\,\frac{1}{|\cdot|^{2k-2s}} \in L_\infty(\Omega). \tag{5}$$
*Here* $[\widehat{\phi},\widehat{\phi}]_s := \sum_{\alpha\in 2\pi\mathbb{Z}^d} |\widehat{\phi}(\cdot+\alpha)|^2|\cdot+\alpha|^{2s}$, $[\widehat{\phi},\widehat{\phi}]_s^0 := \sum_{\alpha\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}(\cdot+\alpha)|^2|\cdot+\alpha|^{2s}$.

2. *The nonstationary PSI ladder* $\mathcal{S} = (S^h := S^h(W_2^s))$, *with* $S^h = S_{\phi_h}(\cdot/h)$, $\phi_h \in W_2^s$, *provides approximation order* $k$ *if and only if, for some* $h_0 > 0$ *and some neighborhood* $\Omega$ *of* $0$, *the collection of functions*
$$\frac{[\widehat{\phi}_h,\widehat{\phi}_h]_s^0}{[\widehat{\phi}_h,\widehat{\phi}_h]_{s,h}}\,\frac{1}{(|\cdot|^2+h^2)^{k-s}}, \qquad 0 < h < h_0 \tag{6}$$
*lies in* $L_\infty(\Omega)$ *and is bounded there. Here,* $[\widehat{\phi}_h,\widehat{\phi}_h]_{s,h} := [\widehat{\phi}_h,\widehat{\phi}_h]_s^0 + |\widehat{\phi}_h|^2(|\cdot|^2+h^2)^s$.

*Proof.* The second part of the current theorem follows from the second part of Result 6 and the PSI case of Corollary 5. Together, these two results yield the requisite characterization, but with $[\widehat{\phi}_h,\widehat{\phi}_h]_s^0$ replaced by $\sum_{\alpha\in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}_h(\cdot+\alpha)|^2(|\cdot+\alpha|^2+h^2)^s$. However, for $\alpha \neq 0$, we can replace $(|\cdot+\alpha|^2+h^2)^s$ by its equivalent expression $|\cdot+\alpha|^{2s}$.

It remains to show that in the stationary case, i.e., when $\phi_h = \phi$ for all $h$, (6) is equivalent to (5). The fact that the former *implies* the latter is obvious (one simply should take $h \to 0$ in (6) and invoke the uniform boundedness of the collection of functions that appears there). For the converse, we observe that (when $\phi_h := \phi$ for all $h$) the uniform boundedness of the functions in (6) is equivalent to the validity of the inequalities

$$\frac{[\widehat{\phi}, \widehat{\phi}]_s^0}{|\widehat{\phi}|^2} \le \frac{(|\cdot|^2 + h^2)^k}{c - (|\cdot|^2 + h^2)^{k-s}} \qquad \text{a.e.},$$

for some positive $c > 0$. Moreover, since we assume $k - s > 0$, we can force $(|\cdot|^2 + h^2)^{k-s} < c$ by making $h$ small enough and changing $\Omega$ if necessary. This leaves us with

$$\frac{[\widehat{\phi}, \widehat{\phi}]_s^0}{|\widehat{\phi}|^2} \le C(|\cdot|^2 + h^2)^k \qquad \text{a.e.}$$

as the requisite boundedness. This is definitely implied by (5), as the left hand side in the display above is independent of $h$ and since $k \ge 0$. □

**Remark on notation.** For brevity, we will use in the sequel the expressions '$S_\phi(W_2^s)$ provides approximation order $k$' and '$\phi$ provides approximation order $k$ in $W_2^s$' to mean that the *stationary* ladder generated by $S_\phi(W_2^s)$ provides approximation order $k$ in $W_2^s$.

### 3.2 Strang-Fix conditions

Given $\phi \in W_2^s(\mathbb{R}^d)$, and $k > 0$, one says that $\phi$ **satisfies the Strang-Fix (SF) condition of order** $k$ ([36]), if $\widehat{\phi}$ has a zero of order $k$ at each point $\alpha \in 2\pi\mathbb{Z}^d\backslash 0$. It is well known that the $L_2$-approximation order of a stationary PSI ladder is closely related to the order of the SF condition satisfied by the generator $\phi$ of the ladder. To be precise, a full characterization requires a nondegeneracy condition on $\widehat{\phi}$ at the origin. First, let us cite the $L_2$-result.

**Result 8 ([3, Theorems 1.14, 5.14])** *Assume that $0 < \eta \le |\widehat{\phi}|$ a.e. on some neighborhood $\Omega$ of the origin. Let $A := \Omega + 2\pi\mathbb{Z}^d\backslash 0$. If $\widehat{\phi} \in W_2^\rho(A)$ for some $\rho > k + d/2$, then $S_\phi(L_2)$ provides approximation order $k$ (in $L_2$) if and only if $\phi$ satisfies the Strang-Fix conditions of order $k$, i.e., near the origin*

$$|\widehat{\phi}(\cdot + \alpha)| = O(|\cdot|^k) \qquad \text{for all} \quad \alpha \in 2\pi\mathbb{Z}^d\backslash 0. \tag{7}$$

Here $W_2^\rho(A)$ is the local version of $W_2^\rho(\mathbb{R}^d)$; cf. [1, Chapter 7]. For our purposes, it is only important that the norm on $W_2^\rho(A)$ has a subadditivity property, i.e.,

$$\sum_\alpha \|f\|^2_{W_2^\rho(\alpha+\Omega)} \le \text{const}\, \|f\|^2_{W_2^\rho(A)}, \tag{8}$$

and that the Sobolev embedding theorem for such spaces still holds, in particular, that the bounded (compact) embedding

$$W_2^\rho(\alpha + \Omega) \hookrightarrow W_\infty^k(\alpha + \Omega) \tag{9}$$

is valid. Note that the condition $\widehat{\phi} \in W_2^\rho(A)$ is weaker than the more traditional decay condition on $\phi$

$$|\phi| = O(|\cdot|^{-k-d-\epsilon}), \quad \epsilon > 0,$$

which implies global smoothness of $\widehat{\phi}$.

We now show that the Strang-Fix conditions also characterize approximation power in a Sobolev space.

**Theorem 9** *Let $k \ge 0$, $s < k$, $\phi \in W_2^s$. Suppose that, for some $\eta_1, \eta_2 > 0$ and for some ball $\Omega$ centered at the origin,*

$$\eta_1 \le |\widehat{\phi}| \le \eta_2 \qquad \text{a.e. on} \quad \Omega \tag{10}$$

$$\|\widehat{\phi}\|^2_{k,A} := \sum_{\beta \in 2\pi\mathbb{Z}^d\backslash 0} |\beta|^{2s} \max_{\gamma:|\gamma|\le k} \|D^\gamma \widehat{\phi}\|^2_{L_\infty(\beta+\Omega)} < \infty, \tag{11}$$

*where $A$ denotes the set $\Omega + 2\pi\mathbb{Z}^d\backslash 0$ and $D^\gamma$ denotes the monomial derivative of order $\gamma \in \mathbb{Z}_+^d$ normalized by $\gamma!$. Then $S_\phi(W_2^s)$ provides approximation order $k$ (in $W_2^s$) if and only if (7) holds.*

*Proof.* Set

$$R := |\cdot|^{-2s} \sum_{\beta \in 2\pi\mathbb{Z}^d\backslash 0} |\widehat{\phi}(\cdot + \beta)|^2 \; |\cdot + \beta|^{2s}. \tag{12}$$

Suppose $\phi$ provides approximation order $k$ in $W_2^s$. Then (5) holds by Theorem 7, or equivalently, a.e. on $\Omega$,

$$\frac{R}{|\widehat{\phi}|^2 + R} = O(|\cdot|^{2k-2s}).$$

Since $k > s$, and $\widehat{\phi}$ is bounded on $\Omega$, we conclude that, around the origin, $R = O(|\cdot|^{2k-2s})$. This readily implies (7).

Now suppose $\phi$ satisfies (7). With $R$ as above, we invoke (11) to conclude that

$$\|R\|_{L_\infty(\Omega)} \le C|\cdot|^{2k-2s}\|\widehat{\phi}\|^2_{k,A} = O(|\cdot|^{2k-2s}).$$

However, the left-hand side $\mathcal{M}_{\phi,s}$ of (5) equals

$$\frac{|\cdot|^{2s-2k}R}{|\widehat{\phi}|^2 + R}.$$

We have just argued that the numerator in this expression is bounded. The denominator of the expression is bounded away of zero thanks to (10). This implies (5). $\square$

Note that condition (11) was required only for the "if" implication in the above result.

**Corollary 10** *In the notation of Theorem 9, let $\rho > k + d/2$ and let $A := \Omega + 2\pi\mathbb{Z}^d\backslash 0$. Then the conclusions of Theorem 9 remain valid when we replace condition (11) by:*

*(i) for $s \le 0$, the condition that $\widehat{\phi} \in W_2^\rho(A)$.*

*(ii) for $s \ge 0$, the condition that $(1 + |\cdot|^2)^{\rho/2}\phi \in W_2^s(\mathbb{R}^d)$, or the stronger condition that $\phi \in W_2^s$ and $\phi = O(|\cdot|^{-k-d-\epsilon})$, $\epsilon > 0$, at $\infty$.*

Note that the first condition in (ii) above implies, whenever $s \geq 0$, that $\widehat{\phi} \in W_2^\rho(\mathbb{R}^d)$, hence is stronger than the condition assumed in (i).

*Proof.* It is clearly sufficient to prove that each of the conditions in (i) and (ii) implies (11).

(i): Using (9), together with the fact that that the sets $(\beta + \Omega)$ are all translates of $\Omega$, the Sobolev embedding theorem applies to yield that, for $s \leq 0$,

$$\|\widehat{\phi}\|_{k,A}^2 \leq C_1 \sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} |\beta|^{2s} \|\widehat{\phi}\|_{W_2^\rho(\Omega+\beta)}^2 \leq C_2 \sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} \|\widehat{\phi}\|_{W_2^\rho(\Omega+\beta)}^2.$$

The right-hand-side in the above is bounded, thanks to (8), by a constant multiple of $\|\widehat{\phi}\|_{W_2^\rho(A)}^2$. Hence condition (11) is satisfied.

(ii): The second condition in (ii) clearly implies the first one. Now assume the first condition in (ii), i.e., that $f := (1 + |\cdot|^2)^{\rho/2}\phi \in W_2^s$. Then $\widehat{f}$ is locally in $L_2$ and

$$\sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} |\beta|^{2s} \|\widehat{f}\|_{L_2(\beta+2\Omega)}^2 \leq C\|f\|_{W_2^s}^2 < \infty.$$

However, $\|\widehat{\phi}\|_{W_2^\rho(\beta+\Omega)} \leq C\|\widehat{f}\|_{L_2(\beta+2\Omega)}$, and the argument in the proof of (i) then applies to yield (11). □

### 3.3 Approximation orders are independent of the underlying $W_2^s$ space

We are now in a position to observe that the definitions of approximation order, if made with respect to different Sobolev spaces, are consistent in the following sense.

**Proposition 11** *If $S_\phi(W_2^s)$ provides approximation order $k \geq 0$, $k > s$, then $S_\phi(W_2^t)$ provides the same approximation order for any $t \leq s$.*

*Proof.* First note that $\phi$ is an element of $W_2^t$ whenever $t \leq s$, since $W_2^s$ is embedded in $W_2^t$.

Now note that, by Theorem 7, $\phi$ provides approximation order $k$ if and only if (5) holds. The left-hand side $\mathcal{M}_{\phi,s}$ of (5) satisfies

$$\Big(1 - \mathcal{M}_{\phi,s}|\cdot|^{2(k-s)}\Big) \sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} |\widehat{\phi}(\cdot + \beta)|^2 |\cdot + \beta|^{2s} = \mathcal{M}_{\phi,s}|\widehat{\phi}|^2 |\cdot|^{2k}. \tag{13}$$

Since $\|\mathcal{M}_{\phi,s}\|_{L_\infty(\Omega)} \leq \text{const}_{\phi,s}$, the set $\Omega$ can be assumed to be small enough so that, e.g.,

$$1 - \mathcal{M}_{\phi,s}|\cdot|^{2(k-s)} \geq 1/2 \qquad \text{a.e. on} \quad \Omega.$$

Then

$$\sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} |\widehat{\phi}(\cdot + \beta)|^2 |\cdot + \beta|^{2t} \leq \sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} |\widehat{\phi}(\cdot + \beta)|^2 |\cdot + \beta|^{2s} \leq 2\mathcal{M}_{\phi,s}|\widehat{\phi}|^2 |\cdot|^{2k}.$$

This implies

$$\mathcal{M}_{\phi,t} \leq \frac{\sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} |\widehat{\phi}(\cdot + \beta)|^2 |\cdot + \beta|^{2t}}{|\widehat{\phi}|^2 |\cdot|^{2k}} \leq 2\mathcal{M}_{\phi,s}.$$

Thus, $S_\phi$ provides approximation order $k$ also in $W_2^t$. □

Proposition 11 shows that $\phi$ provides approximation order on the whole interval $\{W_2^t : t \leq s\}$ of Sobolev spaces once it does so in the space $W_2^s$.

Let us now show that, under the regularity assumptions already used in Theorem 7, a converse also holds.

**Theorem 12** *Let $t < s < k$, $k \geq 0$. Suppose $\phi \in W_2^s$ and that it satisfies (10)–(11) (with respect to s). Then $S_\phi(W_2^s)$ provides approximation order $k$ iff $S_\phi(W_2^t)$ provides that same approximation order.*

*Proof.* The "only if" implication was proved in Proposition 11 without appealing to (10)–(11).

We prove the "if" assertion as follows. First, since $\phi$ provides approximation order $k$ in $W_2^t$, while satisfying (10), it must satisfy the SF conditions or order $k$ (we do not need (11) for that part), by virtue of Theorem 9. Then, once $\phi$ satisfies the SF conditions of order $k$, the facts that it belongs to $W_2^s$ and satisfies (10)–(11) imply, again by Theorem 9, that it provides approximation order $k$ in $W_2^s$. $\square$

### 3.4 Negative approximation orders

When would it make sense to have a shift-invariant space that provides *negative* approximation order?

Suppose we form a convolution of two compactly supported distributions $\phi_i$, $\widehat{\phi}_i(0) \neq 0$, $i = 1, 2$. If $\phi_i$, $i = 1, 2$, provides a positive approximation order $k_i > 0$, then their convolution $\psi$ provides approximation order $k_1 + k_2$. This is to be expected if one assumes that the Strang-Fix conditions are equivalent to approximation power (which is almost true): then $\widehat{\psi} = \widehat{\phi_1}\widehat{\phi_2}$ satisfies the Strang-Fix conditions of order $k_1 + k_2$ whenever each $\widehat{\phi}_i$ satisfies the Strang-Fix conditions of order $k_i$. A sample of a rigorous statement in this direction is as follows:

**Proposition 13** *Let $\phi_i \in W_2^{s_i}$, $i = 1, 2$ be nondegenerate compactly supported distributions, i.e., $\widehat{\phi}_i(0) \neq 0$, and provide approximation order $k_i > 0$, $k_i > s_i$, $i = 1, 2$ in their respective spaces. Then $\psi := \phi_1 * \phi_2$ provides approximation order $k_1 + k_2$ in $W_2^{s_1+s_2}$.*

*Proof.* First observe that $\psi \in W_2^{s_1+s_2}$. Indeed,

$$\|\psi\|^2_{W_2^{s_1+s_2}} \leq \||\widehat{\phi}_1|(1 + |\cdot|^2)^{s_1/2}\|_\infty \||\widehat{\phi}_2|(1 + |\cdot|^2)^{s_2/2}\|_\infty \|\phi_1\|_{W_2^{s_1}} \|\phi_2\|_{W_2^{s_2}}. \tag{14}$$

Since the $\phi_i$'s are compactly supported, their Fourier transforms are entire functions. Moreover, the products $\widehat{\phi}_i(1 + |\cdot|^2)^{s_i/2}$, $i = 1, 2$, are in $L_2$, and, therefore, their inverse transforms are in $L_2$, too. Those inverse transforms are the result of applying a singular convolution operator to $\phi$. Since the convolutor decays rapidly at $\infty$, and since $\phi$ is compactly supported, the result decays rapidly at $\infty$. Altogether, we conclude that $(\widehat{\phi}_i(1 + |\cdot|^2)^{s_i/2})^\vee$ is in $L_1$, and consequently each $\widehat{\phi}_i(1 + |\cdot|^2)^{s_i/2}$ must tend to zero at infinity. Therefore their $L_\infty$-norms must be finite. So, the right-hand side of (14) is finite, hence $\psi$ is in $W_2^{s_1+s_2}$.

Now, since we assume $\phi_i$ to provide approximation order $k_i$, and since $\widehat{\phi}_i$ is bounded around the origin, then (cf. the first part in the proof of Theorem 9)

$$[\widehat{\phi}_i, \widehat{\phi}_i]^0_{s_i} = O(|\cdot|^{2k_i}), \qquad i = 1, 2,$$

where we recall the notation $[g,g]_s^0 := [g,g]_s - |g|^2|\cdot|^{2s}$ from Theorem 7. But

$$[\widehat{\psi},\widehat{\psi}]_{s_1+s_2}^0 \le [\widehat{\phi}_1,\widehat{\phi}_1]_{s_1}^0[\widehat{\phi}_2,\widehat{\phi}_2]_{s_2}^0, \tag{15}$$

hence

$$[\widehat{\psi},\widehat{\psi}]_{s_1+s_2}^0 = O(|\cdot|^{2k_1+2k_2}).$$

Invoking the fact that $\widehat{\psi}(0) \ne 0$, we finally conclude that

$$\frac{[\widehat{\psi},\widehat{\psi}]_{s_1+s_2}^0}{[\widehat{\psi},\widehat{\psi}]_{s_1+s_2}} = O(|\cdot|^{2(k_1+k_2)-2(s_1+s_2)}).$$

This, in view of Theorem 7, finishes the proof. $\square$

Now, what if, upon convolving a given distribution $\phi_1$ with another distribution $\phi_2$ one discovers that the approximation order of $\phi_1 * \phi_2$ is smaller than that of $\phi_1$? Then it is natural to assign a negative approximation order to the distribution $\phi_2$. It makes little sense to define the notion of negative approximation order in terms of the ability to approximate functions. We choose, instead, the following technical definition, which is consistent with the discussion so far, as well as with the argument used in the proof of our last result.

**Definition.** Let $s < k \le 0$, and let $\phi \in W_2^s$. We say that the (stationary ladder generated by) $\phi$ provides approximation order $k$ (in $W_2^s$), if, for some neighborhood $\Omega$ of the origin,

$$\mathcal{M}_{\phi,s} := \frac{[\widehat{\phi},\widehat{\phi}]_s^0}{[\widehat{\phi},\widehat{\phi}]_s}\,\frac{1}{|\cdot|^{2k-2s}} \in L_\infty(\Omega).$$

Note that the above definition is consistent with the case $k > 0$. In this case, the *definition* of approximation order is different, but the characterization provided in (5) of Theorem 7 is exactly in the same terms.

Equipped with this last definition, we can extend Proposition 13 as follows:

**Proposition 14** *Let $\phi_i \in W_2^{s_i}(\mathbb{R}^d)$, $i = 1,2$ provide approximation order $k_i > s_i$ in $W_2^{s_i}(\mathbb{R}^d)$, $i = 1,2$. If the convolution product $\psi := \phi_1 * \phi_2$ lies in $W_2^{s_1+s_2}(\mathbb{R}^d)$, then it provides approximation order $k_1 + k_2$ there.*

*Proof.* Since $k_i > s_i$ for $i = 1,2$, it follows directly from the extended definition of approximation order that

$$\frac{[\widehat{\phi}_i,\widehat{\phi}_i]_{s_i}^0}{|\widehat{\phi}_i|^2} = O(|\cdot|^{2k_i}), \qquad i = 1,2.$$

These two estimates, together with the inequality (15), imply that $\frac{[\widehat{\psi},\widehat{\psi}]_{s_1+s_2}^0}{|\widehat{\psi}|^2} = O(|\cdot|^{2k_1+2k_2})$, hence

$$\frac{[\widehat{\psi},\widehat{\psi}]_{s_1+s_2}^0}{|\widehat{\psi}|^2|\cdot|^{2s_1+2s_2} + [\widehat{\psi},\widehat{\psi}]_{s_1+s_2}^0} \le \frac{[\widehat{\psi},\widehat{\psi}]_{s_1+s_2}^0}{|\widehat{\psi}|^2|\cdot|^{2s_1+2s_2}} = O(|\cdot|^{2k_1+2k_2-2s_1-2s_2}).$$

This completes the proof. $\square$

**Corollary 15** *Let $\phi_i \in W_2^{s_i}(\mathbb{R}^d)$, $i = 1,2$ be compactly supported distributions and provide approximation order $k_i > s_i$ in $W_2^{s_i}(\mathbb{R}^d)$, $i = 1,2$. Then $\phi_1 * \phi_2$ provides approximation order $k_1 + k_2$ in $W_2^{s_1+s_2}(\mathbb{R}^d)$.*

*Proof.* This fact follows from Proposition 14, since we know already from the proof of Proposition 13 that the convolution of two compactly supported distributions in $W_2^{s_i}(\mathbb{R}^d)$, $i = 1,2$, lies in $W_2^{s_1+s_2}(\mathbb{R}^d)$. □

**Remark.** In the rest of the paper, we only consider, by default, generators $\phi$ of stationary ladders that provide approximation order no smaller than 0. Note that this is the case when $\phi$ is of compact support and satisfies $\widehat{\phi}(0) \neq 0$.

### 3.5 Polynomial reproduction

We restrict our attention in this subsection to local stationary PSI ladders, and focus on the properties of the compactly supported generator $\phi$ of the underlying ladder. To be sure, all the results here extend, almost verbatim, to generators $\phi$ with sufficient decay at $\infty$, for example, $|\phi| = O(|\cdot|^{-k-d-\epsilon})$ at $\infty$, with $k$ the investigated approximation order and $\epsilon > 0$.

The theory of approximation orders of local stationary PSI ladders focuses, and rightly so, on the satisfaction of the SF conditions (cf. §3.2, and also the application of those conditions in §3.3). Under the compact support assumption on $\phi$, the SF conditions are known to be equivalent to the polynomial reproduction property, the latter being the subject of the current subsection. [1]

The connection between the SF conditions and polynomial reproduction is classically known, and can be dated back to Schoenberg ($d = 1$, [35]), and Strang and Fix, [36]. See also [2]. Our approach here follows [7]. Altogether, the results of this subsection are included for completeness, especially since the polynomial reproduction property in the PSI case is key to the understanding of the more complicated polynomial reproduction property of FSI spaces (§4.4), as well as the sum rules of refinable FSI spaces (§5.3).

Suppose that $\phi$ is compactly supported. Let us first attempt to connect the approximation orders provided by its stationary PSI ladder to the SF conditions. To this end, we would like to invoke Theorem 9. This theorem requires the satisfaction of (10) and (11). Condition (11) is satisfied once $\phi \in W_2^s$, as (ii) of Corollary 10 shows. The fact that a compactly supported distribution belongs to some $W_2^s$ is well-known, and follows from the fact that it is necessarily of finite order (as a distribution). As to (10), since $\widehat{\phi}$ is continuous, this condition is presently equivalent to the nondegeneracy requirement

$$\widehat{\phi}(0) \neq 0.$$

Thus we obtain the following result:

**Corollary 16** *Let $\phi$ be a compactly supported distribution, and assume that $\widehat{\phi}(0) \neq 0$. Then there exists $s \in \mathbb{R}$ such that $\phi \in W_2^s$. Moreover, the following conditions are then equivalent, for any given $k > 0$:*

[1] Prior to the publication of [8] and [3], approximation orders of stationary PSI ladders were usually derived directly from the polynomial reproduction property, while the SF conditions were considered to be a technical way for the verification of polynomial reproduction. However, as the discussion in this article clearly shows, the SF conditions characterize the approximation orders of the ladder even when a slow decay of the generator $\phi$ renders the polynomial reproduction property meaningless.

*(i) $\phi$ satisfies the SF conditions of order $k$.*

*(ii) The stationary PSI ladder generated by $\phi$ provides approximation order $k$ (in $W_2^s$).*

Now, recall that reproducing polynomials of total degree less than $k$ means that

$$\phi *' \Pi_{<k} \subseteq \Pi_{<k}.$$

The symbol $*'$ denotes the **semi-discrete convolution**

$$g*' : f \mapsto \sum_{j \in \mathbb{Z}^d} g(\cdot - j) f(j), \tag{16}$$

$\Pi := \Pi(\mathbb{R}^d)$ is the space of all $d$-variate polynomials, and $\Pi_{<k} := \{p \in \Pi : \ \deg p < k\}$.

One way to connect the polynomial reproduction to the SF condition is via the following variant of Poisson's summation formula, [33],

$$\phi *' f = \sum_{\alpha \in \mathbb{Z}^d} \phi * (e_\alpha f), \quad e_\alpha : x \mapsto e^{2\pi i \alpha \cdot x}, \tag{17}$$

which is valid for every compactly supported $\phi$ and every $C^\infty$-function $f$ (the convergence of the right-hand-side series is in the topology of tempered distributions). Now, for a polynomial $f$, one easily verifies that $\phi * (e_{-\alpha} f) = 0$ iff $\widehat{\phi}$ has a zero of order $\deg f + 1$ at $\alpha$. Thus, once $\phi$ satisfies the SF conditions of order $k$, we have that $\phi *' f = \phi * f$ for all $f \in \Pi_{<k}$. This establishes the sufficiency of the SF conditions, since $\phi *$ always maps $\Pi_{<k}$ into itself. On the other hand, if $\phi *' f$ is a polynomial of degree $< k$, then (17) shows that

$$\sum_{\alpha \in \mathbb{Z}^d \backslash \{0\}} \phi * (e_\alpha f) \tag{18}$$

is also a polynomial of degree $< k$. This is possible [7, Proof of (2.10) Lemma] only if all the summands in (18) vanish. In conclusion, $\phi$ satisfies the SF conditions of order $k$ if and only if

$$\phi *' = \phi *, \quad \text{on } \Pi_{<k}.$$

Since, as we already said, $\Pi_{<k}$ is an invariant subspace of $\phi *$ (with or without the SF conditions), we finally need only to guarantee that $\phi *$ be injective on polynomials, or equivalently, we need to assume that $\widehat{\phi}(0) \neq 0$. Indeed, the condition $\widehat{\phi}(0) \neq 0$ is necessary and sufficient for $\phi *$ to be an automorphism on $\Pi_{<k}$ (for any positive integer $k$), and we arrived at:

**Theorem 17** *Let $\phi$ be any compactly supported distribution with $\widehat{\phi}(0) \neq 0$, and let $k$ be a positive integer. Then $\phi$ provides approximation order $k$ in some Sobolev space $W_2^s$, $s < k$, if and only if it reproduces polynomials of total degree less than $k$.*

*Remark.* As alluded to before in a footnote, Theorem 17 could also be proved directly, avoiding the use of the Strang-Fix conditions and constructing instead a quasiinterpolant $Q : W_2^s \to S(\phi)$ such that $Sp = p$ for any $p \in \Pi_{<k}$; for a detailed discussion of this method see [7, Section 4].

## 4 FSI ladders

We start this section, just like in the PSI case, by recalling the characterization of the $L_2$-approximation orders of FSI spaces and extending the result to the setting of Sobolev spaces. We then focus in §4.2 on the notion of a superfunction, which leads to the reduction of the FSI case to the PSI case. Besides, this notion proves to be very helpful in understanding polynomial reproduction from FSI spaces (see §4.4). In our setting of $W_2^s$, it also helps to establish the consistency of the notion of approximation order as we vary $s$ (see §4.3). However, not every superfunction can be used for these and/or for other purposes, and this brings one to the notions of 'good' and 'bad' superfunctions that are discussed in §4.5. We show, in §4.8, that there exist FSI spaces that do not contain any good superfunctions. Regardless of whether or not good superfunctions are around, an alternative approach, that is presented in §4.6, can always be used to bound the approximation order from *above*. The efficacy of this method is demonstrated in §4.7, where we recover the well-known example of $C^1$-cubics on a three-directional mesh, [6]: this is an example of a bivariate stationary local FSI ladder that, while reproducing all polynomials in $\Pi_3$, fails to provide the "expected" approximation order 4. Finally, §4.9 applies the results obtained in this section to the case when the vector $\Phi$ is refinable: in establishes a lower bound on the approximation order provided by $\Phi$ in terms of the decay of the Fourier transform of any non-zero function in $S_\Phi$.

### 4.1 Characterization of approximation power

The first three results of this section form a summary of the known characterization of approximation power valid in $L_2$, while the rest constitutes the characterization in the more general setting of $W_2^s$.

**Result 18 ([5, Theorem 2.2])** *The stationary FSI ladder $\mathcal{S} = (S^h := S^h(L_2))$, with $S^h = S_\Phi(\cdot/h)$, $\Phi \subset L_2(\mathbb{R}^d)$, provides approximation order $k$ if and only if there exists a neighborhood $\Omega$ of $0$ such that*

$$\left(1 - \widehat{\Phi}^* G_\Phi^{-1} \widehat{\Phi}\right) \frac{1}{|\cdot|^{2k}} \in L_\infty(\Omega).$$

*Here*

$$G_\Phi := \sum_{\alpha \in 2\pi\mathbb{Z}^d} \widehat{\Phi}(\cdot + \alpha)\widehat{\Phi}^*(\cdot + \alpha) = \left([\widehat{\phi}, \widehat{\varphi}]\right)_{\phi,\varphi\in\Phi}. \tag{19}$$

*Also, the expression $G_\Phi^{-1}\widehat{\Phi}$ is taken to mean any solution to the equation $G_\Phi \tau = \widehat{\Phi}$. A simple linear-algebraic argument shows that the latter equation is always solvable whether or not $G_\Phi$ is invertible, since one of the rank-one terms in (19) is $\widehat{\Phi}\widehat{\Phi}^*$.*

**Result 19 ([5, Theorem 2.7])** *An FSI nonstationary ladder $\mathcal{S} = (S^h := S^h(L_2))$, with $S^h = S_{\Phi_h}(\cdot/h)$, $\Phi_h \subset L_2(\mathbb{R}^d)$, provides approximation order $k$ if and only if, for some $h_0 > 0$ and some neighborhood $\Omega$ of the origin, the collection of functions*

$$\left(1 - \widehat{\Phi_h}^* G_{\Phi_h}^{-1} \widehat{\Phi_h}\right) \frac{1}{(|\cdot|^2 + h^2)^k}, \qquad h < h_0$$

*lies in $L_\infty(\Omega)$ and is bounded there.*

We also require the following equivalent formulation of the last characterization, in which we use the notation:

$$G_\Phi^0 := \sum_{\alpha \in 2\pi \mathbb{Z}^d \setminus 0} \widehat{\Phi}(\cdot + \alpha)\widehat{\Phi}^*(\cdot + \alpha) = G_\Phi - \widehat{\Phi}\widehat{\Phi}^*. \tag{20}$$

**Result 20** *(another version of Result 19) The FSI nonstationary ladder $\mathcal{S} = (S^h := S^h(L_2))$, with $S^h = S_{\Phi_h}(\cdot/h)$, $\Phi_h \subset L_2(\mathbb{R}^d)$, provides approximation order $k$ if and only if the collection of functions $(\mathcal{M}_{\Phi_h,s,h} : 0 < h < h_0)$, where*

$$\mathcal{M}_{\Phi,s,h} : \omega \mapsto \frac{1}{(|\omega|^2 + h^2)^k} \inf_{v \in \mathbb{C}^\Phi} \frac{v^* G_\Phi^0(\omega) v}{v^* G_\Phi(\omega) v}$$

*is bounded in $L_\infty(\Omega)$ for some neighborhood $\Omega$ of the origin and some $h_0 > 0$.*

Using these results, one obtains the following characterization of approximation power in $W_2^s$.

**Theorem 21**

1. *An FSI stationary ladder $\mathcal{S} = (S^h := S^h(W_2^s))$, with $S^h = S_\Phi(\cdot/h)$, $\Phi \subset W_2^s$, provides approximation order $k > 0$ if and only if there exists a neighborhood $\Omega$ of $0$ such that the function*

$$\mathcal{M}_{\Phi,s} : \omega \mapsto \frac{1}{|\omega|^{2k-2s}} \inf_{v \in \mathbb{C}^\Phi} \frac{v^* G_{\Phi,s}^0(\omega) v}{v^* G_{\Phi,s}(\omega) v} \quad \text{belongs to} \quad L_\infty(\Omega). \tag{21}$$

   *Here*

$$\begin{aligned} G_{\Phi,s} &:= \sum_{\alpha \in 2\pi \mathbb{Z}^d} \widehat{\Phi}(\cdot + \alpha)\widehat{\Phi}^*(\cdot + \alpha)|\cdot + \alpha|^{2s}, \\ G_{\Phi,s}^0 &:= \sum_{\alpha \in 2\pi \mathbb{Z}^d \setminus 0} \widehat{\Phi}(\cdot + \alpha)\widehat{\Phi}^*(\cdot + \alpha)|\cdot + \alpha|^{2s}. \end{aligned}$$

2. *An FSI nonstationary ladder $\mathcal{S} = (S^h := S^h(W_2^s))$, with $S^h = S_{\Phi_h}(\cdot/h)$, $\Phi_h \subset W_2^s$, provides approximation order $k \geq 0$ if and only if there exists a neighborhood $\Omega$ of $0$ and $h_0 > 0$ such that the collection of functions $(\mathcal{M}_{\Phi,s,h} : 0 < h < h_0)$, with*

$$\mathcal{M}_{\Phi_h,s,h} : \omega \mapsto \frac{1}{(|\omega|^2 + h^2)^{k-s}} \inf_v \frac{v^* G_{\Phi_h,s,h}^0(\omega) v}{v^* G_{\Phi_h,s,h}(\omega) v}, \qquad \text{is bounded in} \quad L_\infty(\Omega). \tag{22}$$

   *Here,*

$$\begin{aligned} G_{\Phi_h,s,h}(\omega) &:= \sum_{\alpha \in 2\pi \mathbb{Z}^d} \widehat{\Phi_h}(\omega + \alpha)\widehat{\Phi_h}^*(\omega + \alpha)(|\omega + \alpha|^2 + h^2)^s, \\ G_{\Phi_h,s,h}^0(\omega) &:= \sum_{\alpha \in 2\pi \mathbb{Z}^d \setminus 0} \widehat{\Phi_h}(\omega + \alpha)\widehat{\Phi_h}^*(\omega + \alpha)(|\omega + \alpha|^2 + h^2)^s. \end{aligned}$$

*Proof.* The proof is analogous to that of Theorem 7. In particular, part 2 of the current theorem is a direct consequence of Result 20 and Proposition 4.

Now we use the result of part 2 to derive part 1. In the stationary case, $\Phi_h = \Phi$ for all $h$, so the left-hand side of (22) becomes

$$\mathcal{M}_{\Phi,s,h}(\omega) = \frac{1}{(|\omega|^2+h^2)^{k-s}} \inf_v \frac{v^* \sum_{\alpha\in 2\pi\mathbb{Z}^d\backslash 0} \widehat{\Phi}(\omega+\alpha)\widehat{\Phi}^*(\omega+\alpha)(|\omega+\alpha|^2+h^2)^s\, v}{v^* \sum_{\alpha\in 2\pi\mathbb{Z}^d} \widehat{\Phi}(\omega+\alpha)\widehat{\Phi}^*(\omega+\alpha)(|\omega+\alpha|^2+h^2)^s\, v}.$$

Since the numerator of the infimum expression is bounded above and below by positive multiples of $v^* G^0_{\Phi,s} v$, the collection $(\mathcal{M}_{\Phi,s,h})$ is bounded in $L_\infty(\Omega)$ if and only if the collection of functions

$$\omega \mapsto \frac{1}{(|\omega|^2+h^2)^{k-s}} \inf_v \frac{v^* G^0_{\Phi,s}(\omega) v}{v^* G^0_{\Phi,s}(\omega) v + v^*\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)(|\omega|^2+h^2)^s v} \tag{23}$$

is bounded in $L_\infty(\Omega)$. Since $k$ and $k-s$ are nonnegative, for a fixed $v$ and a fixed $\omega \in \Omega$ (assuming $\Omega$ is sufficiently small), the expression

$$\frac{1}{(|\omega|^2+h^2)^{k-s}} \frac{v^* G^0_{\Phi,s}(\omega) v}{v^* G^0_{\Phi,s}(\omega) v + v^*(\widehat{\Phi}\widehat{\Phi}^*)(\omega)(|\omega|^2+h^2)^s v}$$

monotonically increases (as $h \to 0$) to

$$\frac{1}{|\omega|^{2(k-s)}} \frac{v^* G^0_{\Phi,s}(\omega) v}{v^* G_{\Phi,s}(\omega) v},$$

hence (23) monotonically increases to the function in (21). Therefore, the collection $(\mathcal{M}_{\phi,s,h})$ is bounded if and only if (21) holds. $\square$

**Remark on notation.** As in the PSI case, we shall use in the sequel language such as 'an FSI space $S_\Phi(W_2^s)$ provides approximation order $k$' and even '$\Phi \subset W_2^s$ provides approximation order $k$' and will mean by that that the FSI stationary ladder generated by $S_\Phi(W_2^s)$ provides approximation order $k$ in $W_2^s$.

The $L_2$-characterizations above (Results 18–20) are connected to the notion of superfunctions. We will now discuss this notion, and extend it to the setting of Sobolev spaces.

### 4.2 Superfunctions

Let $S$ be an SI space, and let $\mathcal{S} = (S^h := S(\cdot/h))$ be the associated stationary ladder. A function $g \in S \subset L_2$ is a **superfunction** in $S$ if the PSI stationary ladder it generates provides the same approximation order as that of $S$ (or, more precisely, of $\mathcal{S}$). For sure, $L_2$ in this definition can be replaced by any Sobolev space $W_2^s$.

The question of existence of superfunctions in FSI spaces can be answered in the affirmative using Theorem 21.

**Theorem 22** *Any FSI space $S_\Phi \subset W_2^s(\mathbb{R}^d)$ contains a superfunction.*

*Proof.* Let $\Omega$ be as in Theorem 21. For each fixed $\omega \in \Omega$, there exists a vector $v_0(\omega) \in \mathbb{C}^\Phi$ of (e.g., Euclidean) norm 1 that minimizes the ratio $v^* G^0_{\Phi,s}(\omega)v / v^* G_{\Phi,s}(\omega)v$. Now extend $v_0$ to the cube $[-\pi, \pi)^d$ in an arbitrary way, provided the norm $v_0$ is everywhere equal to 1. Finally, extend $v_0$ so defined to a $2\pi$-periodic vector-valued function.

Now, suppose that $\Phi$ provides approximation order $k$. Then, in view of Theorem 21, the vector $v_0$ satisfies

$$v_0 G^0_{\Phi,s} v_0 \leq \text{const}\,|\cdot|^{2k-2s}(v_0 G^0_{\Phi,s} v_0 + v_0^* \widehat{\Phi}\widehat{\Phi}^* v_0 |\cdot|^{2s}) \qquad \text{a.e. in } \Omega$$

or, equivalently

$$(1 - \text{const}\,|\cdot|^{2(k-s)}) v_0 G^0_{\Phi,s} v_0 \leq \text{const}\,|\cdot|^{2k} v_0^* \widehat{\Phi}\widehat{\Phi}^* v_0 \qquad \text{a.e. in } \Omega.$$

By changing $\Omega$ if needed (and using the fact that $k > s$), we obtain that

$$v_0 G^0_{\Phi,s} v_0 \leq C|\cdot|^{2k} v_0^* \widehat{\Phi}\widehat{\Phi}^* v_0 \qquad \text{a.e. in } \Omega$$

for some constant $C$. This implies that, for almost every fixed $\omega \in \Omega$, the smallest eigenvalue of the measurable Hermitian matrix $H(\omega) := (G^0_{\Phi,s} - C|\cdot|^{2k}\widehat{\Phi}\widehat{\Phi}^*)(\omega)$ is nonpositive. By Lemma 2.3.5 from [32], we can define a map $w$ on $\Omega$ such that (i) for almost every $\omega \in \Omega$, $w(\omega)$ is a normalized eigenvector of $H(\omega)$ that corresponds to the minimal eigenvalue, and (ii) $w$ is measurable on $\Omega$. Without loss, we assume that our original $v_0$ coincides with $w$ on $\Omega$. In particular, $v_0$ is now known to be measurable.

Let $\phi$ be the (scalar) distribution whose Fourier transform satisfies $\widehat{\phi} = v_0^* \widehat{\Phi}$. To show that $\phi$ is a superfunction for $S_\Phi$, we only need to verify that it belongs to $S_\Phi$, since it follows directly from the construction of $\phi$ that the space $S_\phi$ provides approximation order $k$. Since $\widehat{\phi} = v_0^* \widehat{\Phi}$, we only need, in view of Corollary 3, to show that $\phi \in W_2^s$. This final result is a simple consequence of the representation $\widehat{\phi} = v_0^* \widehat{\Phi}$, using the facts that $\Phi \subset W_2^s$ and that the entries of $v_0$ are bounded. ∎

This theorem extends the known $L_2$ result. However, in the $L_2$-case, a superfunction was originally constructed as the orthogonal projection $P_\Phi : L_2 \to S_\Phi$ of the sinc-function

$$\operatorname{sinc}(x) := \prod_{i=1}^{d} \frac{\sin(\pi x(i))}{\pi x(i)}.$$

The fact that $P_\Phi(\operatorname{sinc})$ is a superfunction follows from the general principle:

**Result 23 ([3])** *Let $S_\Phi(L_2)$ be an FSI space that provides approximation order $k_1 \geq 0$, and let $S_g(L_2)$ be a PSI space that provides approximation order $k_2 \geq 0$. Then the PSI space generated by the orthogonal projection $P_\Phi g$ on $S_\Phi$ in $L_2$ provides approximation order* $\min\{k_1, k_2\}$.

If, in Result 23 above, we choose $g$ such that $k_2 \geq k_1$, and $S_\Phi(L_2)$ does not provide an approximation order greater than $k_1$, then $P_\Phi g$ is a superfunction. It is easily checked that the approximation order provided by the space $S_{\text{sinc}}$ is infinite (i.e., exceeds any finite $k$), hence the following corollary.

**Result 24 ([3])** *An FSI space $S_\Phi(L_2)$ provides the same approximation order as the PSI space generated by* $P_\Phi(\operatorname{sinc})$ *in $L_2$.*

To find the $L_2$-projection on a FSI space of a function $f$, one solves equation (24).

**Result 25 ([3])** *The $L_2$-projection $P_\Phi(f)$ of $f \in L_2$ on an FSI space $S_\Phi$ satisfies*

$$(P_\Phi(f))^\wedge = \tau_f^* \widehat{\Phi} \tag{24}$$

*with $\tau_f$ any solution of*

$$G_\Phi \tau_f = [\widehat{\Phi}, \widehat{f}].$$

*Here, $\tau_f$ is a vector-valued function (indexed by $\Phi$) whose entries are measurable and $2\pi$-periodic, and the symbol $[\widehat{\Phi}, \widehat{f}]$ stands for $([\widehat{\phi}, \widehat{f}])_{\phi\in\Phi}$, where $[\widehat{f}, \widehat{g}] := \sum_{\alpha\in\mathbb{Z}^d} \widehat{f}(\cdot + 2\pi\alpha)\overline{\widehat{g}}(\cdot + 2\pi\alpha)$.*

**Remark:** Results 23, 24 and 25 are all corollaries to Theorem 3.3 of [3].

The above results extend easily to Sobolev spaces. Indeed, Results 23 and 24 require only one assumption, viz.

$$P_A P_g = P_{P_A g} P_g, \tag{25}$$

where $A$ denotes an arbitrary SI subspace of $W_2^s$, and $P_A, P_g, P_{P_A g}$ are the orthogonal projectors from $W_2^s$ onto $A, S_g(W_2^s), S_{P_A g}(W_2^s)$, respectively.

Under this condition, the analysis from [3, Section 3] leading to Results 23 and 24 goes through verbatim. Since the $W_2^s$-version of (25) is a simple consequence of the $L_2$-version when combined with the identity $P_{J_t A} J_t = J_t P_A$ (where $A$ an SI subspace of $W_2^s$ and $P_{J_t A}$ the orthogonal projector onto the space $J_t A$ in $L_2$), we obtain the following extension.

**Theorem 26** *Let $S_\Phi(W_2^s)$ be an FSI space that provides approximation order $k_1 \geq 0$, and let $S_g(W_2^s)$ be a PSI space that provides approximation order $k_2 \geq 0$. Set $\psi := P_\Phi g$, with $P_\Phi$ the orthogonal projection of $W_2^s$ onto $S_\Phi(W_2^s)$. Then the stationary PSI ladder generated by $\psi$ provides approximation order* $\min\{k_1, k_2\}$. *Specifically, for every $f \in W_2^s$ and $h > 0$,*

$$\operatorname{dist}_s(f, S_\psi^h(W_2^s)) \leq \operatorname{dist}_s(f, S_\Phi^h(W_2^s)) + 2\operatorname{dist}_s(f, S_g^h(W_2^s)).$$

*In particular, $S_\Phi(W_2^s)$ provides the same approximation order as the PSI space generated by $P_\Phi(\text{sinc})$ in $W_2^s$.*

Superfunctions are obviously useful if one wishes to approximate functions from a given SI space S, for if a superfunction $\phi$ is known explicitly, one can instead approximate from the simpler space $S_\phi$. In addition, it is already well established in the $L_2$-theory that superfunctions give rise to quasi-interpolants, i.e., bounded linear maps into the underlying SI space $S$ that reproduce polynomials contained in $S$ (see, e.g., [2], [7]). Superfunctions were also used in [3], [5] as purely theoretical tools.

The natural expectation is that superfunctions play similar roles in the setting of Sobolev spaces. That turns out to be the case. In particular, the superfunction method allows us to lift painlessly various results from the PSI setup to the FSI one. This includes the discussion concerning the consistency of the definitions of approximation orders in different Sobolev spaces, which we embark on in the next subsection.

### 4.3 Approximation orders are independent of the underlying $W_2^s$ space

**Proposition 27** *If an FSI space $S_\Phi(W_2^s)$ provides approximation order $k \geq 0$, $k > s$, then $S_\Phi(W_2^t)$ does so for any $t \leq s$.*

*Proof.* Assuming that $S_\Phi(W_2^s)$ provides approximation order $k$, Theorem 22 ascertains that $S_\Phi(W_2^s)$ contains a PSI subspace $S_\phi(W_2^s)$ that already provides approximation order $k$. Since $S_\Phi(W_2^t) \supset S_\Phi(W_2^s)$ (as easily follows from Corollary 3), $\phi \in S_\Phi(W_2^t)$, too. By Proposition 11, $S_\phi(W_2^t)$ then provides approximation order $k$ in $W_2^t$, therefore $S_\Phi(W_2^t)$ provides approximation order (at least) $k$ in $W_2^t$. □

Similarly to the PSI case, a converse also holds under some regularity assumptions on the superfunction.

**Proposition 28** *Let $k > s > t$, $k \ge 0$. Suppose $\phi, \Phi \subset W_2^s$. Suppose that $S_\Phi(W_2^t)$ provides approximation order $k$ in $W_2^t$, and that $\phi \in S_\Phi(W_2^t)$ is a corresponding superfunction. If $\phi$ satisfies (10)–(11), then $S_\Phi(W_2^s)$ provides approximation order $k$ in $W_2^s$, too.*

*Proof.* We apply Theorem 12 to the function $\phi$ to show that $S_\phi(W_2^s)$ provides approximation order $k$, which implies that $S_\Phi(W_2^s)$ also provides approximation order (at least) $k$. □

The theorem highlights a central point: it is useful to know that an FSI space contains a "good" superfunction. In the current context "good" in interpreted as "satisfying (10)–(11)". We will come back to this issue later, but first we show how the superfunction method reduces the polynomial reproduction issue in the FSI setup back to the simpler PSI setup.

### 4.4 Polynomial reproduction

Let $\Phi$ be a vector of compactly supported elements in $W_2^s$, $s \in \mathbb{R}$. Suppose that $\Phi$ provides approximation order $k > 0$ in $W_2^s$. Let us assume, further, that $S_\Phi(W_2^s)$ contains a *good* superfunction $\psi$ is the sense that:

1. $\widehat{\psi}(0) \neq 0$, and
2. $\psi$ is a *finite* linear combination of the shifts of $\Phi$ (hence, in particular, is compactly supported).

We note that the current notion of "good" is stronger (i.e., implies) the one that was discussed at the end of the last subsection (as the argument in §3.5 shows).

By our assumptions here, $\widehat{\psi} = v^*\widehat{\Phi}$, with $v$ a vector of *trigonometric polynomials*. Therefore, with $(a_\phi)_{\phi\in\Phi}$ the Fourier coefficients of the entries of $v$, we have the representation

$$\psi = \sum_{\phi\in\Phi} \phi *' a_\phi,$$

and each $a_\phi : \mathbb{Z}^d \to \mathbb{C}$ is finitely supported. Here, $*'$ is the semi-discrete convolution, (16).

Next, since $\widehat{\psi}(0) \neq 0$, and $S_\psi(W_2^s)$ provides approximation order $k$, we conclude from Theorem 17 that $\psi *'$ maps $\Pi_{<k}$ onto itself. Writing $\psi *'$ in terms of $\Phi$ we obtains

$$\psi *' f = (\sum_{\phi\in\Phi} \phi *' a_\phi) *' f = \sum_{\phi\in\Phi} \phi *' (a_\phi *' f).$$

The above representation leads to several conclusions that we summarize in our next result:

**Corollary 29** *Let $\Phi$ be a compactly supported vector that provides approximation order $k > 0$ in $W_2^s$, and assume that $S_\Phi(W_2^s)$ contains a good superfunction in the above sense. Then there exist finitely supported sequences $a_\phi : \mathbb{Z}^d \to \mathbb{C}$, $\phi \in \Phi$ such that, for every $f \in \Pi_{<k}$,*

$$Tf := \sum_{\phi\in\Phi} \phi *' (a_\phi * f)$$

*is a polynomial. Here, $a_\phi * f$ denotes the* discrete *convolution of $a_\phi$ and $f_{|_{\mathbb{Z}^d}}$.*
*The polynomial $Tf$ is identical to the result of the following continuous convolution*

$$\sum_{\phi\in\Phi} \phi * (a_\phi * f) = \sum_{\phi\in\Phi} \sum_{j\in\mathbb{Z}^d} a_\phi(j)(\phi * f)(\cdot - j). \tag{26}$$

*Moreover, the map $T_k := T_{|_{\Pi_{<k}}}$ is an automorphism.*

There are several immediate conclusions that can be derived directly from the above corollary. For example, since $T_k$ can be extended to a convolution operator, it commutes with differentiation in the sense that $D^\gamma T = TD^\gamma$ for every $\gamma \in \mathbb{Z}_+^d$, and commutes also with translations.

A simpler consequence is as follows: since $T_k$ is an automorphism, then every monomial $()^\alpha$, $|\alpha| < k$, lies in its range. (Here, the symbol $()^\alpha$ stands for the normalized monomial

$$()^\alpha : x \mapsto (x)^\alpha := x^\alpha/\alpha!.$$

We also use in the sequel $D^\alpha$ for the *normalized* monomial derivative.) Thus, the following is true:

**Corollary 30** *Let $\Phi$ be as in Corollary 29. Then there exist polynomials $(g_\alpha)_{\alpha\in\mathbb{Z}_+^d}$ such that, for $|\alpha| < k$,*

$$\sum_{\phi\in\Phi} \phi *' (a_\phi * g_\alpha) = ()^\alpha.$$

The result shows that every $()^\alpha$ is writable as $\sum_\phi \phi *' f_{\phi,\alpha}$, for suitable polynomials $(f_{\phi,\alpha})_\phi$. However, the result shows more: it decomposes each $f_{\phi,\alpha}$ into $a_\phi * g_\alpha$, with the first factor independent of $\alpha$ (and is finitely supported), and the second independent of $\phi$ (and is a polynomial).

The reader might wonder how realistic the assumption about the existence of a good superfunction is. We discuss that issue in this section as well as in §5. A sufficient condition for the existence of a good superfunction as above is the invertibility, in a suitable sense, of the Gramian $G_{\Phi,s}$ around the origin. We also note that our results here recover the results of [11, Section 3] (cf. also [9], [10]). The underlying assumption in [11] is that the shifts of the distributions $\phi \in \Phi$ are linearly independent, a condition that is significantly stronger than the Gramian invertibility that we have alluded to above. At the same time, our derivation here is simpler due to the superfunction approach.

Next, one might also wonder how to invert the operator $T$, i.e., how to compute the above polynomials $(g_\alpha)_\alpha$. That inversion is the key for the so-called quasi-interpolation approach, and is discussed in detail in [2] and [7] (in the PSI context; our superfunction approach already reduced the problem to that setup). At base, we seek a simple linear functional $\mu$ such that $\mu*$ inverts on $\Pi_{<k}$ either the convolution $\psi*$ or the map $f \mapsto f *' \psi$.

Among the various methods, we describe a general recursive approach (see [13], [2], [11]). To this end, we need first to present this approach in the nondegenerate PSI case, i.e., when the (single) generator $\psi$ satisfies the condition $\widehat{\psi}(0) \neq 0$. The superfunction method will allow us then to lift the result to the FSI setup.

**Proposition 31** *Let $\psi$ be a compactly supported distribution with $\widehat{\psi}(0) = 1$ that provides approximation order $k$ in some $W_2^s$, $s \in \mathbb{R}$. Define the polynomials $g_\alpha$, $\alpha \in \mathbb{Z}_+^d$, $|\alpha| < k$, by the recurrence*

$$g_\alpha := ()^\alpha - \sum_{\beta<\alpha} c(\alpha-\beta)\, g_\beta, \tag{27}$$

*where*

$$c(\gamma) := (\psi *' ()^\gamma)(0) = (\psi * ()^\gamma)(0) = (()^\gamma *' \psi)(0), \quad \gamma \in \mathbb{Z}_+^d,\ |\gamma| < k.$$

*Then these polynomials satisfy*

$$()^\alpha = \psi * g_\alpha, \qquad |\alpha| < k. \tag{28}$$

Note that for the expression $()^\gamma *' \psi = \sum_{j\in\mathbb{Z}^d} (\cdot - j)^\gamma \psi(j)$ to make sense $\psi$ needs to be continuous. The other two representations of $c(\gamma)$ are valid for an arbitrary compactly supported distribution $\psi$.

*Proof.* By Theorem 17, $\psi *'$ reproduces all polynomials of degree $< k$, and hence (cf. e.g., [7])

$$\psi *' ()^\alpha = \psi * ()^\alpha = ()^\alpha *' \psi, \quad \forall \alpha \in \mathbb{Z}_+^d,\ |\alpha| < k.$$

Thus, $c(\alpha)$ is well-defined.

Now, given $\alpha$ as above, it is elementary that (since $\psi * ()^0 = \widehat{\psi}(0) = 1$)

$$\psi * ()^\alpha = \sum_{|\beta|\le|\alpha|} (\psi * ()^{\alpha-\beta})(0)\, ()^\beta = ()^\alpha + \sum_{|\beta|<|\alpha|} c(\alpha-\beta) ()^\beta.$$

However, with $g_\alpha$ as in (27), we obtain (by convolving $g_\alpha$ with $\psi$, assuming by induction that $\psi * g_\beta = ()^\beta$ for $|\beta| < \alpha$, and using the last identity) that

$$\psi * g_\alpha := \psi * ()^\alpha - \sum_{|\beta|<|\alpha|} c(\alpha-\beta)\, ()^\beta = ()^\alpha. \tag{29}$$

□

Using this proposition with respect to the superfunction $\psi := \sum_{\phi\in\Phi} \phi *' a_\phi$, we obtain the following:

**Theorem 32** *Under the assumptions of Corollary 29, the polynomials $(g_\alpha)$ from Corollary 30 satisfy the following recurrence relation:*

$$g_\alpha := ()^\alpha - \sum_{\beta<\alpha} c(\alpha-\beta)\, g_\beta, \tag{30}$$

*where*

$$c(\gamma) := \sum_{\phi\in\Phi} c(\gamma, \phi),$$

*while*

$$c(\gamma, \phi) := \sum_{j\in\mathbb{Z}^d} (\phi * ()^\gamma)(j) a_\phi(-j).$$

*Here, $\phi * ()^\gamma$ is continuous convolution, while $a_\phi$ is the finitely supported sequence that appears in Corollary 30. Moreover, if each $\phi \in \Phi$ is continuous, we have the alternative discrete convolution representation*

$$c(\gamma, \phi) := (\phi_{|_{\mathbb{Z}^d}} * a_\phi * ()^\gamma_{|_{\mathbb{Z}^d}})(0) = \sum_{j,k\in\mathbb{Z}^d} \phi(j)\, a_\phi(k-j)\, (-k)^\gamma.$$

*Remark.* Compare the last theorem with Theorem 1 of [11].

### 4.5 Good and bad superfunctions

Every FSI space contains a superfunction. This positive statement can be turned negative: the existence of a superfunction in a given FSI space tells us nothing about the structure of the space. In contrast, Proposition 28, Corollary 29 and Theorem 32 show that the existence of "good" superfunctions does lead us to useful conclusions about the space and about the given generating set. A particular useful condition is that the Fourier transform of the superfunction be bounded away from zero near the origin. This condition is important also from the numerical stability point of view. In view of the above, we say that a superfunction $\widehat{\psi}$ is **nondegenerate**, if $\widehat{\psi}$ near 0 is bounded away from zero.

We note that the nondegeneracy itself falls short of classifying "good" superfunctions. For example, it can be checked that the $L_2$-projection of the sinc-function on an FSI space $S \subset L_2$ is always nondegenerate as long as $S$ provides a positive approximation order (by [5, Corollary 2.6]). However, the superfunctions obtained in this way may prove to be of little use due to their slow decay at $\infty$. Thus, we require a complementary property of a superfunction: we say that the superfunction has the **finite span property** if it is in the *finite* span of the shifts of the generating set $\Phi$. Such superfunctions are compactly supported if $\Phi$ itself is. It is proved in [4] that every local FSI space in $L_2$ contains a superfunction that satisfies the finite span property (with $\Phi$ being any compactly supported generating set for the space).

We call a superfunction **good**, if it is nondegenerate and finitely spanned by the shifts of $\Phi$. Such superfunctions are needed for constructing quasi-interpolation schemes. Indeed, the requirement appearing in Theorem 32 is exactly that the superfunction $\psi$ be good.

Corollary 3 shows that any function in $S_\Phi(W_2^s)$ is of the form $(v^*\widehat{\Phi})^\vee$, for some $2\pi$-periodic vector-valued function $v$. Theorem 21 adds that the vector $v$ associated with a superfunction satisfies

$$\frac{v^* G^0_{\Phi,s} v}{v^* G_{\Phi,s} v} = O(|\cdot|^{2k-2s}),$$

with $k$ the approximation order of the FSI space $S_\Phi(W_2^s)$. The known $L_2$-theory of approximation orders of FSI spaces offers then a recipe for constructing good superfunctions: first solve the equation $Gv = \widehat{\Phi}$ around the origin (cf. Theorem 25) and then approximate $v$ by a trigonometric polynomial vector $u$ such that $v-u$ has a zero of order $k$ at the origin. This is possible whenever the Gramian $G$ is $k$ times continuously differentiable around the origin and $G(0)$ is invertible (see [5, Theorem 4.2]). Next, $v - u = O(|\cdot|^k)$ implies $v^*\widehat{\Phi} - u^*\widehat{\Phi} = O(|\cdot|^k)$, hence $(u^*\widehat{\Phi})^\vee$ is a good superfunction.

Once $G(0)$ is not invertible, the notion of a "good" superfunction becomes more subtle. Are we only interested in the *existence* of a superfunction $\psi \in S_\Phi$ such that $\psi$ is "reasonably local" and $\widehat{\psi}(0) \neq 0$, or do we also insist on simple ways to obtain that function from the given generating set $\Phi$? Our discussion and development focuses on the latter approach: after all, the SI space is *given* to us in terms of the generating set $\Phi$, and we would like then the analysis to stay as close as possible to this set. Once we agree on that principle, it should be clear that "very bad" generating sets $\Phi$ are not going to yield good superfunctions: for example if $\Phi$ is compactly supported and $\widehat{\Phi}(0) = 0$, there is no hope to get from $\Phi$ in a simple way a superfunction $\psi$ with $\widehat{\psi}(0) \neq 0$. The ultimate question is how to define "good" vectors $\Phi$. Our suggestion is simple: these are the vectors that yield good superfunctions!

Our next results (in the next subsection) offer analysis of vectors $\Phi$ whose Gramian is singular.

We show the utility of this analysis by providing a new proof to a famous example of de Boor and Höllig concerning the approximation order of $C^1$-cubics on three direction mesh. We then provide an example of a "seemingly good" vector $\Phi$ that cannot yield good superfunctions.

### 4.6 Estimating approximation orders when Gramians are singular

Theorem 21 enables us, at least in principle, to determine the order of approximation provided by a given (stationary or nonstationary) FSI ladder in any Sobolev space. But, as we already saw in the $L_2$-case, such analysis is hard to carry out if the Gramian $|\cdot|^{-2s}G_{\Phi,s}$ is singular at the origin. The problem is exacerbated by the fact that the entries of $G_{\Phi,s}$ may be hard to compute.

Let us examine closely the source of the difficulty. For $s = 0$, the computation of approximation orders depends on estimating ratios of the form

$$\frac{v^*G^0v}{v^*Gv}$$

around the origin. Without loss, one can assume that the vector $v$ is normalized pointwise. If $G$ is continuous at 0 and invertible there, we can then dismiss the denominator, since it does not affect the asymptotic behaviour of the above expression. In contrast, if $G$ is singular at the origin, the denominator might affect the approximation order. The use of the verb "might" is justified: roughly speaking, there is hope that the specific vectors $v$ that minimize the numerator are far enough from the kernel of $G(0)$. Whenever this is the case, the problem is reduced to examining the behaviour of the numerator only. The current subsection translates the above discussion into rigorous analysis.

We first provide below a theorem that establishes an upper bound on the approximation order of an FSI space. The upper bound does not require the invertibility of the associated Gramian. To this end, we denote by $\rho_{\min}(A)$ the smallest eigenvalue of a positive-definite Hermitian matrix $A$.

**Theorem 33** *Suppose $\widehat{\Phi} \subset L_\infty(\Omega)$ for some neighborhood $\Omega$ of the origin. Given any set $\mathcal{I} \subseteq 2\pi\mathbb{Z}^d\backslash 0$, denote by $k(\Phi, \mathcal{I}, s)$ the order of the zero that the scalar function*

$$\omega \mapsto \rho_{\min}(A(\omega)), \quad A(\omega) := \sum_{\alpha\in\mathcal{I}} \widehat{\Phi}(\omega+\alpha)\widehat{\Phi}(\omega+\alpha)^*|\omega+\alpha|^{2s}$$

*has at the origin. Then the approximation order provided by the FSI space $S_\Phi(W_2^s)$ is no larger than $k(\Phi, \mathcal{I}, s)/2$.*

*Proof.* Suppose that $S_\Phi$ provides approximation order $k$ in $W_2^s$. Then the characterization from Theorem 21 implies that, for $v_0 : \omega \mapsto v_0(\omega)$ that minimizes (21) pointwise, the expression

$$\frac{v_0^*G^0_{\Phi,s}v_0}{v_0^*G_{\Phi,s}v_0}\frac{1}{|\cdot|^{2k-2s}} =: \mathcal{M}_{\Phi,s}$$

is bounded in a neighborhood $\Omega$ of the origin. Using the identity

$$(1-\mathcal{M}_{\Phi,s}|\cdot|^{2k-2s})v_0^*G^0_{\Phi,s}v_0 = \mathcal{M}_{\Phi,s}|v_0^*\widehat{\Phi}|^2\,|\cdot|^{2k},$$

and the fact that $k > s$, we conclude that $v_0^*G_{\Phi,s}v_0$ is bounded above by a constant multiple of $|v_0^*\widehat{\Phi}|^2\,|\cdot|^{2s}$ (compare with the argument used in the proof of Proposition 11). Thus,

$$\inf_v \frac{v^*G^0_{\Phi,s}v}{v^*G_{\Phi,s}v} = \frac{v_0^*G^0_{\Phi,s}v_0}{v_0^*G_{\Phi,s}v_0} \geq \text{const}\,\frac{v_0^*G^0_{\Phi,s}v_0}{|v_0^*\widehat{\Phi}|^2|\cdot|^{2s}} \geq \text{const}\inf_{|v|=1}\frac{v^*G^0_{\Phi,s}v}{|v^*\widehat{\Phi}|^2|\cdot|^{2s}} \geq \text{const}\,|\cdot|^{-2s}\inf_{|v|=1} v^*G^0_{\Phi,s}v.$$

The last inequality uses the assumption that $\widehat{\Phi}$ is bounded around the origin. We conclude then that, for some $C>0$ and a.e. around the origin,

$$\inf_v \frac{v^* G^0_{\Phi,s} v}{v^* G_{\Phi,s} v} \ge C|\cdot|^{-2s}\rho_{\min}(G^0_{\Phi,s}).$$

But $G^0_{\Phi,s} - A$ is (pointwise) a nonnegative definite Hermitian matrix, hence, pointwise, $\rho_{\min}(G^0_{\Phi,s}) \ge \rho_{\min}(A)$. The desired result then follows from Theorem 21. $\square$

As alluded to before, we know quite precisely when the above upper bound matches the associated approximation order.

**Theorem 34** *Suppose $\widehat{\Phi} \subset L_\infty(\Omega)$ for some neighborhood $\Omega$ of the origin. Let $v_0$ be a normalized eigenvector of $G^0_{\Phi,s}$ associated with the minimal eigenvalue (i.e., for a.e. $\omega \in \Omega$, the pair $(\rho_{\min}(G^0_{\Phi,s}(\omega)), v_0(\omega))$ is an eigenpair of $G^0_{\Phi,s}(\omega)$). If $|v_0^*\widehat{\Phi}|$ is bounded away from zero almost everywhere in $\Omega$, then the approximation order of $S_\Phi(W_2^s)$ is exactly $k(\Phi, 2\pi\mathbb{Z}^d\backslash 0, s)/2$.*

*Proof.* One only needs to show that the approximation order of $S_\Phi$ is bounded below by $k(\Phi, 2\pi\mathbb{Z}^d\backslash 0, s)/2$. But

$$\inf_v \frac{v^* G^0_{\Phi,s} v}{v^* G_{\Phi,s} v} \le \frac{v_0^* G^0_{\Phi,s} v_0}{v_0^* G_{\Phi,s} v_0} = \frac{\rho_{\min}(G^0_{\Phi,s})}{|v_0^*\widehat{\Phi}|^2|\cdot|^{2s} + \rho_{\min}(G^0_{\Phi,s})} \le \frac{\rho_{\min}(G^0_{\Phi,s})}{|v_0^*\widehat{\Phi}|^2|\cdot|^{2s}} \le \text{const}\,|\cdot|^{-2s}\rho_{\min}(G^0_{\Phi,s}).$$

Theorem 21 then yields the requisite lower bound. $\square$

### 4.7 Example: bivariate $C^1$-cubics

The results of the last section raise two questions. The first is whether the upper bounds provided in Theorem 33 are useful, i.e., whether they can be applied to solve a nontrivial problem. We provide in the current subsection an affirmative answer to this question.

The other, more fundamental question, is whether the setup of Theorem 34 is universal, i.e., whether we can *always* dispense with the denominator in the characterization provided in Theorem 21. This question is intimately related to the existence of good superfunctions. In the next subsection we provide a (-n unfortunate) negative answer to that second question.

As said, we describe now an example where Theorem 33 applies in a situation when direct evaluation of the approximation order is quite complicated. We choose the notorious example of an FSI space that reproduces all polynomials of order $\le 3$, but provides only approximation order 3. The example first appears in [6]. A second, completely different, proof of this result appears in [5]. Our proof is thus the third one for this result.

Consider the following two bivariate compactly supported piecewise polynomial functions whose Fourier transforms are given by

$$\widehat{\phi}_1(u,v) = \mathrm{i}\tfrac{(v(1-e^{-\mathrm{i}w}) - w(1-e^{-\mathrm{i}v}))(1-e^{-\mathrm{i}u})(1-e^{-\mathrm{i}v})(1-e^{-\mathrm{i}w})}{(uvw)^2},$$
$$\widehat{\phi}_2(u,v) = \widehat{\phi}_1(v,u),$$

where $w = u+v$. These functions are known as the Fredrickson elements. With $\Phi \subset L_2(\mathbb{R}^2)$ the 2-vector consisting of the above functions, the Gramian $G_\Phi$ is invertible in a punctured neighborhood of the origin. Hence it is still possible to compute enough coefficients in the Taylor expansion of $1-\widehat{\Phi}^* G_\Phi^{-1}\widehat{\Phi}$ to find the first nonvanishing nonconstant term. This complicated analysis was carried out in [5]. It shows that the first nonzero term in that expansion if of order 6, so $S_\Phi$ provides approximation order 3.

We use here, instead, Theorem 33 to arrive at the same conclusion more easily. Let $\mathcal{I} = \{(0,2\pi),(2\pi,0)\}$. Then

$$\sum_{\alpha\in\mathcal{I}} \widehat{\Phi}((u,v)+\alpha)\widehat{\Phi}((u,v)+\alpha)^* = \Psi_1(u,v)\Psi_1^*(u,v) + \Psi_2(u,v)\Psi_2^*(u,v) + o((|u|^2+|v|^2)^3) \tag{31}$$

where

$$\Psi_1(u,v) = \frac{(1-e^{-\mathrm{i}u})(1-e^{-\mathrm{i}v})(1-e^{-\mathrm{i}(u+v)})}{(u+v+2\pi)^2}\begin{bmatrix} \frac{-2\pi+\pi\mathrm{i}u+2\pi\mathrm{i}v+\mathrm{i}uv/2+\mathrm{i}v^2/2}{u(v+2\pi)^2} \\ \frac{2\pi-\pi\mathrm{i}u+\mathrm{i}uv/2+\mathrm{i}v^2/2}{u(v+2\pi)^2}\end{bmatrix},$$

$$\Psi_2(u,v) = \frac{(1-e^{-\mathrm{i}u})(1-e^{-\mathrm{i}v})(1-e^{-\mathrm{i}(u+v)})}{(u+v+2\pi)^2}\begin{bmatrix} \frac{2\pi-\pi\mathrm{i}v+\mathrm{i}uv/2+\mathrm{i}u^2/2}{v(u+2\pi)^2} \\ \frac{-2\pi+\pi\mathrm{i}v+2\pi\mathrm{i}u+\mathrm{i}uv/2+\mathrm{i}u^2/2}{v(u+2\pi)^2}\end{bmatrix}.$$

The trace of the matrix $\Psi_1\Psi_1^*+\Psi_2\Psi_2^*$ is of order 4, whereas its determinant is of order 10, so its minimal eigenvalue vanishes to order 6 and its maximal eigenvalue to order 4. Since the (matrix) terms that were left out of the computation are all of order $o(|\cdot|^6)$, the eigenvalues of the left hand side of (31) are also of order 6 and 4, respectively. Now Theorem 33 implies that the approximation order of $S_\Phi$ is at most 3.

The fact that the approximation order is at least 3 is trivial: the sum $\phi_1+\phi_2$ yields a superfunction which is nothing but the box spline $M_{2,2,1}$ (whose approximation order is indeed 3). The vector $\Phi$ is thus an example where the singularity of the Gramian does not preclude the existence of a good superfunction.

¿From our standpoint, the $C^1$-cubic vector $\Phi$ is "good", since the finite span of its shifts contains a good superfunction. The notoriety of this case is due to the difficulty in asserting that this $\Phi$ provides approximation order no higher than 3. The fact that the space reproduces all cubic polynomials is a sad, misleading, accident. The reader may claim that we ignore the fact that $\Phi$ here provides an approximation order which is "disappointing". While that might be the case, it goes beyond the realm of this article: we are only interested in ways to capture the approximation order of the given space, and not in the construction of SI spaces that provide "satisfactory" approximation order.

## 4.8 Good and bad superfunctions, continued

We will now show an example of a vector $\Phi$ whose entries seem to be "reasonable" but nonetheless it does not admit a good superfunction. This example of a bad $\Phi$, together with the example from the last section of a good $\Phi$, illustrates the depth of the difficulty in pinning down the notion of a good generating set for an FSI space.

Let $g$ be a compactly supported bivariate function whose Fourier transform $\widehat{g}$ has a zero of order $k>2$ at each of the $2\pi\mathbb{Z}^d\backslash 0$ points. Moreover, we assume that $1-\widehat{g}$ has a zero of order $k$ at the origin. There are many ways to construct such a function. For example, one can take the univariate B-spline of order $k$, apply a suitable differential operator $p(D)$ or order $k-1$ to it, and then use its tensor product in 2 dimensions. In this case, $g$ is piecewise polynomial of local degree

$k-1$ in each of its variables, and with support $[0,k]^2$. It provides approximation order $k$ (in $L_2$, for example).

Now, let $e$ be the bivariate exponential with frequency $(2\pi,0)$, i.e., $e : x \mapsto e^{2\pi i x(1)}$. We define a vector $\Phi$ with two components

$$\phi_1 := g + eD^{(0,2)}g, \quad \phi_2 := g - eD^{(2,0)}g.$$

Here, to recall, $D^\alpha$ is the normalized monomial differentiation, viz., $2D^{(2,0)}g$ is the second derivative of $g$ in the first argument. Despite the fact that each $\phi_i$ provides only approximation order 2, we contend that the FSI space $S_\Phi(L_2)$ provides approximation order $k$. We construct, to this end, a compactly supported superfunction as follows.

We choose a vector-valued function $v$ with two components that are trigonometric polynomials such that

$$v - \begin{pmatrix} ()^{2,0} \\ ()^{0,2} \end{pmatrix} = O(|\cdot|^{k+2})$$

around the origin. We then note that the Taylor expansion of order $k$ of $\widehat{\Phi}$ around the point $(2\pi, 0)$ is

$$\frac{1}{2}\begin{pmatrix} -()^{0,2} \\ ()^{2,0} \end{pmatrix}.$$

At the same time, the Taylor expansion of order $k$ of $\widehat{\Phi}$ around any point of $2\pi\mathbb{Z}^d\backslash 0$ other than $(2\pi, 0)$ is zero. From this, we conclude that the compactly supported function $\psi$ that is defined by

$$\widehat{\psi} := v^*\widehat{\Phi}$$

has a zero of order $k+2$ at *every* point of $2\pi\mathbb{Z}^d\backslash 0$. Finally, at the origin, $\widehat{\psi} - |\cdot|^2\widehat{g}$ has a zero of order $k+2$.

In order to determine the approximation order provided by $\psi$, we consider the expression

$$\frac{[\widehat{\psi},\widehat{\psi}] - |\widehat{\psi}|^2}{|\widehat{\psi}|^2} = \frac{[\widehat{\psi},\widehat{\psi}] - |\widehat{\psi}|^2}{|\cdot|^4|\widehat{g}|^2}\frac{|\cdot|^4|\widehat{g}|^2}{|\widehat{\psi}|^2}.$$

The term

$$\frac{|\cdot|^4|\widehat{g}|^2}{|\widehat{\psi}|^2}$$

is bounded around the origin. The other term,

$$\frac{[\widehat{\psi},\widehat{\psi}] - |\widehat{\psi}|^2}{|\cdot|^4|\widehat{g}|^2}$$

remains bounded (around the origin) even when multiplied by $|\cdot|^{-2k}$. Thus, $\psi$ provides approximation order $k$ (in $L_2$), *a fortiori* $\Phi$ provides that approximation order.

Note that the superfunction $\psi$ does not satisfy the desired condition $\widehat{\psi}(0) \neq 0$. In fact, this is necessary in a certain sense. Indeed, let $\tau$ be a $2\pi$-periodic vector-valued function that is continuous at the origin and does not vanish there. Let us further define a function $f$ by

$$\widehat{f} = \tau^*\widehat{\Phi}.$$

Then, up to a non zero multiplicative constant, the low order derivatives of $\widehat{f}$ at $(2\pi, 0)$ coincide with the derivatives at the origin of the function

$$(\tau_1()^{(0,2)} - \tau_2()^{(2,0)})\widehat{g}.$$

Since we assume $\tau$ not to vanish at the origin, it is clear that some second order derivative of the above expression does not vanish at the origin. As such, $f$ cannot provide approximation order larger than 2.

While the vector $\Phi$ in this example does not yield a good superfunction, it satisfies the following positive property: we could use the truncated Gramian $G^0_{\Phi,0}$ is order to determine the approximation order of $S_\Phi$. Indeed, if we normalize the given vector $v$ (i.e., redefine it pointwise as $v/|v|$), we obtain a vector for which $v^* G^0_{\Phi,0} v$ yields the correct decay rate $(k)$ at the origin. This means, in turn, that the smallest eigenvalue of $G^0_{\Phi,0}$ still determines the approximation order of the space $S_\Phi$. The superfunction that we obtain in this way (i.e., by using the normalized $v$) is still not good: it decays painfully slowly at $\infty$.

We close this section with two comments:

- We do not know at present of an example where the smallest eigenvalue of the truncated Gramian $G^0_{\Phi,s}$ does not determine the approximation order of the space provided, of course, that $\widehat{\Phi}(0) \neq 0$.
- The above example (i.e., of a case when the smallest eigenvalue of $G^0_{\Phi,s}$ determines the approximation order while there exists no good superfunction) is very much a multivariate phenomenon. It is not hard to prove that such a case is impossible in one variable, and we leave it as an exercise to the interested reader.

### 4.9 An application: approximation orders of smooth refinable functions

We provide in this section one of the most interesting applications of superfunction theory: lower bounds on approximation orders of smooth refinable vectors. We note that approximation orders of refinable vectors are treated in more detail in the next section. However, the current topic fits better into the realm of this section.

At base, our result will show that once $\Phi$ is refinable, and once $S_\Phi$ contains a single nonzero function $\psi$ from a certain class, the stationary ladder generated by $\Phi$ must provide an approximation order that corresponds to the class of $\psi$. Our definition of the "class" in question requires the Fourier transform of $\psi$ to decay (in a weak sense) at a certain rate.

This problem has rich history in the context of PSI ladders (see the introduction to [31]). A substantial treatment of the FSI case is given in [31]. However, that treatment is carried out under the assumption that the Gramian $G_\Phi$ is invertible at the origin. In contrast, we focus in this paper on the situation where there are multiple solutions to a single refinement equation, and in such a case the Gramian of any particular solution is *not* invertible at the origin. This understanding was our motivation to look for an alternative approach to that of [31]. It is useful to stress that, in general, refinable vectors that contain a smooth (even analytic!) function need not provide any positive approximation order at all. (An example of this type can be found in [31].) Thus, one must impose certain side conditions either on the vector $\Phi$ or on the function $\psi$.

Let $P$ be an $r \times r$ matrix whose entries are $2\pi$-periodic and measurable. Let $\Phi$ be a vector-valued function with $r$ components whose entries are in $W_2^s$ for some $s \in \mathbb{R}$. We say that $\Phi$ is **refinable** if the functional equation

$$\widehat{\Phi}(2\cdot) = P\widehat{\Phi} \tag{32}$$

is satisfied.

Our goal is to prove the following result. In the result, as well as elsewhere in this subsection, we use the following notation:

$$A := \{\omega \in \mathbb{R}^d : \ \frac{1}{2}\rho < |\omega| \le \rho\}. \tag{33}$$

Here, $\rho \in (0, \pi)$ is arbitrary, but fixed.

**Theorem 35** *Let $s \le 0$, and let $\Phi \subset W_2^s$ be a solution to (32). With $A$ as in (33), assume that there exists $f \in S_\Phi(W_2^s)$ with the following properties:*

1. *$|\widehat{f}|$ is bounded above as well as away from zero on $A$.*
2. *The numbers*
$$\lambda_m := \| \sum_{\alpha \in 2^m(2\pi\mathbb{Z}^d\backslash 0)} |\widehat{f}(\cdot + \alpha)|^2 \|_{L_\infty(A)}, \quad m \in \mathbb{Z}_+$$
*satisfy $\lambda_m = O(2^{-2mk})$, for some positive $k$.*

*Then $S_\Phi(W_2^s)$ provides approximation order $k$.*

We approach this result via the notion of the **dual equation**

$$v^*(2\cdot)P = v^*. \tag{34}$$

Here, $v$ is a vector-valued function with $r$ components. We require the dual equation to be valid in some (small) ball $U$ centered at the origin, and define the entries of the **dual vector** $v$ to be equal to 0 outside $[-\pi, \pi]^d \backslash U$. We then extend $v$ to a $2\pi$-periodic vector. Thus, $v$ is supported on $U + 2\pi\mathbb{Z}^d$, and satisfies (34) there.

We collect in the next lemma a few simple facts about dual vectors.

**Lemma 36** *Let $A$ be as in (33). Given any $v_0$ defined on $A$, equation (34) can be solved on the punctured disk*

$$U := \{\omega : 0 < |\omega| \le \rho\} \tag{35}$$

*so that the solution $v$ satisfies $v|_A = v_0$. Moreover, we have then, a.e. on $U$,*

$$v^*(\omega/2^m)\widehat{\Phi}(\omega/2^m + \alpha) = v^*(\omega)\widehat{\Phi}(\omega + 2^m\alpha), \quad \text{all } m \in \mathbb{Z}_+, \ \alpha \in 2\pi\mathbb{Z}^d.$$

*Proof.* We define $v$ by $v^*(\omega) := v^*(2\omega)P(\omega)$, for all $\omega \in 2^l A$, $l = -1, -2, \ldots$. Then $v$ clearly satisfies (34) (on $U$, and hence on $U + 2\pi\mathbb{Z}^d$).

The second part of the lemma is obtained by iterating $m$ times with

$$v^*\widehat{\Phi}(\cdot + \alpha) = v^*(2\cdot)P\widehat{\Phi}(\cdot + \alpha) = v^*(2\cdot)P(\cdot + \alpha)\widehat{\Phi}(\cdot + \alpha) =$$
$$v^*(2\cdot)\widehat{\Phi}(2(\cdot + \alpha)).$$

□

*Proof of Theorem 35.* By Corollary 3, $\widehat{f} = \tau^* \widehat{\Phi}$ for some $2\pi$-periodic $\tau$. Denoting by $v_0$ the restriction of $\tau$ to $A$, we extend $v_0$ to a dual vector $v$ by Lemma 36. Defining $\psi$ by

$$\widehat{\psi} := v^* \widehat{\Phi},$$

we have, by the same lemma, that, for a.e. on $A$, and for every nonnegative integer $m$,

$$|\widehat{\psi}(\omega/2^m)| = |\widehat{f}(\omega)|.$$

Thus, in view of our assumptions on $f$, we conclude that $|\widehat{\psi}|$ is bounded between two positive constants around the origin.

Next, we prove that $S_\psi(L_2)$ provides approximation order $k$. The argument will show, as a by-product, that $\psi \in L_2$.

Since $\widehat{\psi}$ is bounded away from 0 around the origin, it remains to prove, in view of Result 6, that $[\widehat{\psi}, \widehat{\psi}]^0 | \cdot |^{-2k}$ is bounded around the origin (with $[\widehat{\psi}, \widehat{\psi}]^0 := [\widehat{\psi}, \widehat{\psi}] - |\widehat{\psi}|^2$). Let $\omega \in A$, and $m$ a positive integer. Then, by the definition of $\psi$ and Lemma 36

$$[\widehat{\psi}, \widehat{\psi}]^0(\omega/2^m) = \sum_{\alpha \in 2\pi\mathbb{Z}^d \backslash 0} |\widehat{\psi}|^2(\omega/2^m + \alpha) = \sum_{\alpha \in 2\pi\mathbb{Z}^d \backslash 0} |\widehat{f}|^2(\omega + 2^m\alpha) \leq \lambda_m \leq C 2^{-2mk} \leq C|\omega/2^m|^{2k}.$$

Thus, $[\widehat{\psi}, \widehat{\psi}]^0 = O(| \cdot |^{2k})$, on the punctured ball $U$ of radius $\rho$ centered at the origin. Since $\widehat{\psi}$ is supported on $U + 2\pi\mathbb{Z}^d$, it follows that $\widehat{\psi}$, hence $\psi$, lies in $L_2$. Result 6 then applies to show that $\psi$ provides approximation order $k$ in $L_2$.

On the other hand, $\widehat{\psi} = v^* \widehat{\Phi}$, with $v$ measurable and $2\pi$-periodic. Since $\psi \in L_2$, and $s \leq 0$, we have that $\psi \in W_2^s$. Corollary 3 shows then that $\psi \in S_\Phi(W_2^s)$. Now, Proposition 11 implies that $\psi$ provides approximation order $k$ in $W_2^s$, hence $\Phi$ provides also approximation order $k$ in that space.
□

**Discussion 37** *As the proof of the theorem shows, there is in fact more freedom in the choice of $A$. It suffices to assume that $A$ is compact, that the intersection $A \cap 2A$ has measure zero, and that the union $\cup_{m=-\infty}^{0} A/2^m$ contains a neighborhood of the origin. The proof remains essentially the same.*

## 5 Vector refinement equations

In our studies so far, we considered SI spaces one at a time. There are situations, however, where several different SI spaces may stem from one common source. In cases of this type, it is important to study the resulting SI spaces in a cohesive, combined, way.

The best examples of this type are the multiple vector-valued solutions to refinement equations, and this is, indeed, the topic of the current section. Let us start with the requisite definitions.

Let $P$ be an $r \times r$ square matrix whose entries are, $2\pi$-periodic (measurable) functions (defined on $\mathbb{R}^d$). The functional equation

$$\widehat{\Phi}(2\cdot) = P\widehat{\Phi}, \tag{36}$$

is a **vector refinement equation**, $P$ is a **refinement (matrix) mask**, and a solution $\Phi$ is a **refinable vector**. Here, the entries of the vector $\Phi$ are (measurable) functions, or, more generally, tempered distributions, defined on $\mathbb{R}^d$. The rows and columns of the matrix $P$ are, thus, indexed by either the integers $1, \ldots, r$, or, more directly, by the entries of $\Phi$. In this generality, the equation (36) has, as a rule, infinitely many linearly independent solutions. Indeed, if $P$ is nonsingular around zero, then a solution $\widehat{\Phi}$ can be chosen arbitrarily on a set $A$ of the 'dyadic annulus' type introduced in Discussion 37 and then continued to the rest of the Fourier domain using the recipe

$$\begin{aligned} \widehat{\Phi}(2\omega) &:= P(\omega)\widehat{\Phi}(\omega), \quad \omega \in 2^j A, \quad j = 0, 1, 2, \ldots \\ \widehat{\Phi}(\omega) &:= P^{-1}(\omega)\widehat{\Phi}(2\omega), \quad \omega \in 2^j A, \quad j = -1, -2, \ldots. \end{aligned}$$

Most of the solutions of the above type will decay very slowly (will not be even in $L_1(\mathbb{R}^d)$). In contrast, if we assume the entries of $P$ to consist of trigonometric polynomials, and if we correspondingly insist on compactly supported solutions, then the solution space is necessarily finite-dimensional (as explained in detail in the next section). The special instance when the compactly supported solution space is one-dimensional is quite well understood (see, e.g., [5], [9], [17], and [23]). We are therefore primarily interested in the case when there are multiple (in a nontrivial sense) compactly supported solutions to the equation (36). We denote by

$$R(P)$$

the linear space of all the solutions of (36) whose entries are compactly supported distributions.

The core of study here is the connection between properties of the refinement mask $P$ and its corresponding solution(s) $\Phi$. Shift-invariant spaces enter the discussion in a very natural way. For example, if the solution vector $\Phi$ lies in $L_2$, then one has the inclusions

$$S_\Phi(L_2) \subset \mathcal{D}(S_\Phi(L_2)),$$

with $\mathcal{D}$ the dilation operator $f \mapsto f(2\cdot)$. Due to the above inclusion, we refer to the SI spaces $S_\Phi$ generated by a refinable $\Phi$ as a *refinable SI space*.

We start our study in this section with the problem of *existence* of compactly supported solutions to (36). Our second, and main, topic is the characterization of the approximation orders of the FSI space generated by the solutions $\Phi$ to the refinement equation. This study is based on the premise that, in the case where multiple solutions to the same equation exist, the objective should be the interplay among those solutions, and not only the individual properties of each one of them. In this course of study, we introduce the notions of the *combined Gramian* and the *coherent approximation order* and connect them with the (i) the approximation orders of the SI spaces generated by the solutions to the equations, (ii) the polynomial reproduction property of the mask $P$, and (iii) the sum rules satisfied by $P$. Finally, we already provided (in §4.9) lower bounds on the approximation order of a refinable SI space in terms of the smoothness of the smoothest function in that space.

## 5.1 Compactly supported solutions to the refinement equation

The structure of the compactly supported solutions of (36) was first completely described in [20]. We now restate the main result of that paper and provide a different proof for it. We use the partial order $\le$ on $\mathbb{Z}^d$, defined by

$$a \le b \Longleftrightarrow a - b \in \mathbb{Z}^d_+.$$

Also, given a nonnegative integer $N$, we set

$$\mathcal{Z}_N := \{\alpha \in \mathbb{Z}^d_+ : \ |\alpha| := \alpha_1 + \ldots + \alpha_d \le N\}.$$

Finally, we recall that the definition of the monomial differential operator $D^\gamma$ includes the normalization factor $1/\gamma!$.

**Theorem 38** *Given an $r \times r$-matrix $P$ whose entries are trigonometric polynomials, set*

$$N := \max\{n : 2^n \in \mathrm{spec}(P(0))\}.$$

*Then the map*

$$\Phi \mapsto ((D^\alpha \widehat{\Phi})(0))_{\alpha \in \mathcal{Z}_N},$$

*is a bijection between the collection $R(P)$ of all compactly supported solutions of (36) and the kernel* $\ker L$ *of the map*

$$L : \mathbb{C}^r \times \mathcal{Z}_N \to \mathbb{C}^r \times \mathcal{Z}_N : (w_\alpha)_{\alpha \in \mathcal{Z}_N} \mapsto (2^{|\alpha|} w_\alpha - \sum_{0 \le \beta \le \alpha} (D^{\alpha-\beta} P)(0)\, w_\beta : \ \alpha \in \mathcal{Z}_N).$$

*Proof.* Let $\Phi$ be a compactly supported distributional solution to (36), and denote $w_\alpha := (D^\alpha \widehat{\Phi})(0)$, $\alpha \in \mathbb{Z}_+^d$. Since the vector-valued function $\widehat{\Phi}$ is entire, the vectors $w_\alpha$ are all well-defined. Moreover, one easily concludes from the relation (36) (by applying $D^\alpha$ to both sides of that identity, expanding the right hand side with the aid of Leibniz' formula, and evaluating the result at 0) that the sequence $(w_\alpha)_{\alpha \in \mathbb{Z}_+^d}$ solves the infinite triangular system

$$2^{|\alpha|} w_\alpha = \sum_{0 \le \beta \le \alpha} (D^{\alpha-\beta} P)(0) w_\beta, \qquad \alpha \in \mathbb{Z}_+^d. \tag{37}$$

In particular, $(w_\alpha)_{\alpha \in \mathcal{Z}_N} \in \ker L$.

Conversely, let $w := (w_\alpha)_{\alpha \in \mathcal{Z}_N} \in \ker L$. Then $w$ extends uniquely to a solution to (37) (in order to solve uniquely for $w_\alpha$ in (37) one needs the matrix $2^{|\alpha|} I - P(0)$ to be invertible, which is indeed the case for every $|\alpha| > N$, by our assumption of $\mathrm{spec}\, P(0)$).

Let $\|\cdot\|$ be any vector norm on $\mathbb{C}^r$. The operator norm on $\mathbb{C}^{r \times r}$ subordinate to $\|\cdot\|$ will be denoted in the same way. We claim that for some constant $A > 0$,

$$\|w_\alpha\| \le A^{|\alpha|}/\alpha!, \qquad \text{all} \quad \alpha \in \mathbb{Z}_+^d. \tag{38}$$

Let us see first that (38) yields the existence of a suitable solution to (36).

With (38) in hand, we define (with $()^\alpha$ the normalized monomial) $g := \sum_{\alpha \in \mathbb{Z}_+^d} \alpha! ()^\alpha\, w_\alpha$, and observe that (each of the entries of) $g$ is entire of exponential type, i.e., it satisfies

$$\|g(\omega)\| \le \mathrm{e}^{\widetilde{A}|\omega|}, \qquad \text{all} \quad \omega \in \mathbb{C}^d, \tag{39}$$

where $|\cdot|$ denotes an arbitrary norm on $\mathbb{C}^d$. We need further to show that each of the entries of $g$ is the Fourier transform of a compactly supported distribution, which, by the Paley-Wiener-Schwartz Theorem (Theorem 19.3 from [34, p. 375] and Exercise 7.4 from [30, p. 216]) amounts to showing that (in addition to (38)) the restriction of $g$ to $\mathbb{R}^d$ has slow growth at $\infty$. In order to prove the requisite slow growth, we follow an argument from [23]: Denoting

$$C_1 := \sup_{\xi \in \mathbb{R}^d} \|P(\xi)\| \quad \text{and} \quad C_2 := \sup_{1 \le |\xi| \le 2} \|g(\xi)\|,$$

we pick $\omega \in \mathbb{R}^d$, such that $1 \le |\omega| < 2$. By the construction of $g$, $g(2\cdot) = Pg$, and hence, for every positive $n$, $g(2^n \omega) = P(2^{n-1}\omega) \cdots P(\omega) g(\omega)$. Consequently,

$$\|g(2^n \omega)\| \le C_2 C_1^n \le C_2 (2^n |\omega|)^{\log_2(C_1)},$$

a bound that evidently establishes the sought-for slow growth.

It remains to prove (38). To that end, we pick $N_0 \in \mathbb{N}$ so that

$$2^{-N_0}\|P(0)\| < 1, \qquad \left(\frac{3}{4}\right)^{N_0} \frac{1}{1-2^{-N_0}\|P(0)\|} \le 1.$$

Since $P$ is a matrix of trigonometric polynomials, there exists $A > 0$ such that $\|(D^\alpha P)(0)\| \le (A/2)^{|\alpha|}/\alpha!$, all $\alpha$. Moreover, by modifying $A$ if need be, we may assume that $A$ satisfies the estimate (38) for every $\alpha \in \mathcal{Z}_{N_0}$.

In order to prove (38) for $|\alpha| > N_0$, we may assume, by induction, that (38) holds for all $\beta$ with $|\beta| < |\alpha|$. Then, by (37),

$$\|(2^{|\alpha|}I - P(0))w_\alpha\| \le \sum_{0\le\beta<\alpha} \frac{A^{|\alpha-\beta|}}{2^{|\alpha-\beta|}(\alpha-\beta)!}\frac{A^{|\beta|}}{\beta!} = A^{|\alpha|}\sum_{|\beta|<|\alpha|}\frac{1}{2^{|\alpha-\beta|}(\alpha-\beta)!\beta!} \le \frac{A^{|\alpha|}}{\alpha!}\left(\frac{3}{2}\right)^{|\alpha|},$$

hence

$$\|w_\alpha\| \le \left(\frac{3}{4}\right)^{|\alpha|}\frac{A^{|\alpha|}}{\alpha!(1-2^{-|\alpha|}\|P(0)\|)} \le \frac{A^{|\alpha|}}{\alpha!}.$$

This proves (38), and the proof is thus complete. □

The theorem can be extended to refinement equations more general than (36). For example, we can replace the dilation by 2 by a dilation by any matrix $M$ which is *expansive*, i.e., its spectrum lies outside the closed unit disc.

**Theorem 39** *Given an $r \times r$-matrix $P$ whose entries are trigonometric polynomials and an expansive $d \times d$ matrix $M$, set*

$$N := \max\{n : 0 \in \operatorname{spec}\{M^n - P(0)\}\}.$$

*Then, the map*

$$\Phi \mapsto ((D^\alpha\widehat{\Phi})(0))_{\alpha\in\mathcal{Z}_N},$$

*is a bijection between the collection of all (compactly supported) solutions of the refinement equation $\widehat{\Phi}(M\cdot) = P\widehat{\Phi}$ on the one hand, and the kernel $\ker L$ of the map*

$$L : \mathbb{C}^r \times \mathcal{Z}_N \to \mathbb{C}^r \times \mathcal{Z}_N : (w_\alpha)_{\alpha\in\mathcal{Z}_N} \mapsto (M^{|\alpha|}w_\alpha - \sum_{0\le\beta\le\alpha} D^{\alpha-\beta}P(0)w_\beta : \ \alpha \in \mathcal{Z}_N),$$

*on the other hand.*

*Proof.* Analogous to that of Theorem 38. □

If the refinement equation is inhomogeneous, viz., a given function is added to its right-hand side, then any solution to it is a sum of its specific solution and a solution to the corresponding homogeneous refinement equation. This allows for generalizations of Theorems 38 and 39 to inhomogeneous equations as well. For the exact statement, see [20].

If $P(0)$ is "regular" in the sense that 1 is the largest dyadic eigenvalue of it, the characterization of Theorem 38 is much simpler: every right 1-eigenvector of $P(0)$ gives rise to a solution $\Phi \in R(P)$. However, it is easy to generate examples when the largest dyadic eigenvalue of $P(0)$ is $> 1$: for example one can replace $P$ by $2P$. In that case, the solutions in $R(2P)$ are obtained by differentiating suitably the solutions in $R(P)$. We close this section with two results related to the current discussion. In the first, we describe a general set up in which the solution space $R(P)$ is decomposed into the sum of derivatives of solutions to "regular" refinement equations. In the second result, we provide an example when such decomposition does not hold. Since the discussion here is somewhat tangential to our main study of approximation orders, we skip the proof of the following theorem.

**Theorem 40** *Given an $r\times r$ refinement mask $P$, let $R(P)$ denote the space of compactly supported solutions to (36), and let $N$ be the maximal integer $n$ for which $2^n \in \operatorname{spec}(P(0))$. Suppose, that that we can find two $r \times r$ function-valued matrices $T$ and $\tilde{P}$ such that: (i) $T$ is analytic and invertible around the origin, (ii) the entries of $\tilde{P}$ are trigonometric polynomials, (iii) the matrix $T(2\cdot)P - \tilde{P}T$ has a zero of order $N+1$ at the origin, and (iv) the Taylor expansion of degree $N$ of $\tilde{P}$ around the origin is block-diagonal and the spectrum of each block evaluated at zero intersects the set $\{2^j : j = 0, \ldots, N\}$ at no more than one point. Let $\Phi$ be in $R(P)$, and assume that each of the entries of $\widehat{\Phi}$ has a zero of order $l$ at the origin. Then $\Phi$ admits a representation*

$$\Phi = \sum_{j=l}^{N} p_j(D)\Phi_j, \tag{40}$$

*with $\Phi_j \in R(P/2^j)$, $\widehat{\Phi}_j(0) \neq 0$, and $p_j$ is a homogeneous polynomial of degree $j$, $j = l, \ldots, N$.*

As mentioned before, the solution space $R(P)$ does not always have such structure, as the following counterexample demonstrates.

**Example 41** *For some masks $P$, the decomposition (40) from Theorem 40 is not valid.*

*Proof.* Let $d = 2$ and let the mask $P$ satisfy the following conditions:

$$P(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix}, \quad (D^{(0,1)}P)(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (D^{(1,0)}P)(0) = 0.$$

Let us show that the following inclusion fails:

$$\operatorname{span}[(D^\alpha \widehat{\Phi}(0))_{|\alpha|\leq N} : \Phi \in R(P), \widehat{\Phi}(0) = 0] \subseteq \sum_{j=1}^{d} \operatorname{span}[(D^\alpha \left( ()^{e_j} \widehat{\Phi} \right)(0))_{|\alpha|\leq N} : \Phi \in R\left(\frac{1}{2}P\right)]. \tag{41}$$

Here $e_j$ is the vector in $\mathbb{Z}_+^d$ with 1 in position $j$ and zeros elsewhere. By Theorem 38, this is equivalent to the fact that (40) fails.

The sequences $w$ in $\mathbb{C}^{r\times\binom{N+d}{N-1}}$ (in our case $r = 3$, $d = N = 2$) indexed by $\alpha$, $|\alpha| < N$, we envision as 'long' vectors with the components $w_\alpha$, each of length $r$, all stacked together in some fixed order, e.g., in the graded lexicographic order of the $\alpha$'s.

The relation (41) is, again by Theorem 38, equivalent to the following:

$$\ker L_0 \subseteq \sum_{j=1}^{d} \ker L_j, \tag{42}$$

where

$$L_0 : \mathbb{C}^{r\times\binom{N+d}{N-1}} \to \mathbb{C}^{r\times\binom{N+d}{N-1}} : (w_\alpha) \mapsto (2^{|\alpha|}I - P(0))w_\alpha - \sum_{0<\beta<\alpha} D^{\alpha-\beta}P(0)w_\beta, \qquad 1 \le |\alpha| \le N,$$

$L_j : \mathbb{C}^{r\times\binom{N+d}{N-1}} \to \mathbb{C}^{r\times\binom{N+d}{N-1}}$:

$$(w_\alpha) \mapsto \begin{cases} (2^{|\alpha|}I - P(0))w_\alpha - \sum_{e_j\le\beta<\alpha} D^{\alpha-\beta}P(0)w_\beta, & \text{if } \alpha \ge e_j \\ w_\alpha & \text{otherwise,} \end{cases} \qquad 1 \le |\alpha| \le N.$$

Now, (42) fails iff its dual statement

$$\operatorname{ran} L_0^* \supseteq \cap_{j=1}^d \operatorname{ran} L_j^* \tag{43}$$

fails. Here (43) is obtained from (42) by taking orthogonal complements on both sides and using the property $\ker A = (\operatorname{ran} A^*)^\perp$, which holds, in particular, for any linear map acting on a finite-dimensional Hilbert space.

Now let $w_{(0,1)} := \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \qquad w_{(1,1)} := \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$. Then

$$w := \begin{bmatrix} w_{(0,1)} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = L_{(0,1)}^* \begin{bmatrix} 0 \\ 0 \\ 0 \\ w_{(1,1)} \\ 0 \end{bmatrix} = L_{(1,0)}^* \begin{bmatrix} w_{(1,0)} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

but, by direct calculation, the vector $w$ is not in the range of $L_0^*$. □

## 5.2 Coherent approximation orders

The general theory of approximation orders of FSI spaces (§4) focuses on the individual space $S_\Phi$ and its properties. In contrast, when studying the solutions of the refinement equation (36), we believe that the focus should be on the interplay among the various solutions, in other words on their "common ground". An attempt to establish a theory that treats simultaneously all the solutions of (36) should be done with care: it is easy to show that different solutions of the same refinement equation may have completely different properties, as the following discussion makes clear.

**Discussion 42** *For $j = 1, \ldots, r$, let $\phi_j$ be a (scalar-valued) refinable function with (scalar) mask $p_j$. That is, each $p_j$ is $2\pi$-periodic and $\widehat{\phi}_j(2\cdot) = p_j\widehat{\phi}_j$. Define $P := \operatorname{diag}(p_1, \ldots, p_r)$, $\Phi := (\phi_j)_{j=1}^r$. Then $\Phi$ is a refinable vector with mask $P$. For each fixed $j$, the vector $\Phi_j$ whose $j$th entry is $\phi_j$ and all other entries are $0$ is refinable with respect to $P$. Since we may select the original refinable elements $(\phi_j)$ in a completely arbitrary manner, it is clear that the different solutions $(\Phi_j)_j$ to the same refinement equation may be very different one from the other.*

This discussion reveals another difficulty that arises when dealing with different solutions to the same refinement equation: with $G_j$ the Gramian of $\Phi_j$, that Gramian is *singular* at the origin. It is well-known that this is not an accident:

**Result 43 ([23])** *Let $\Phi \subset L_2$ be a compactly supported refinable vector with Gramian $G_\Phi$. If $G_\Phi(0)$ is invertible, then the spectral radius $\varrho(P(0))$ of $P(0)$ is equal to 1, 1 is the only eigenvalue on the unit circle, and 1 is a simple eigenvalue.*

That is, the Gramian of a refinable function is invertible at zero only if the spectrum of $P(0)$ is of a special nature, which, in particular, implies that the refinement equation has a *unique* solution. We note that the analysis of the approximation order of this case (viz., a refinable vector whose Gramian is invertible at the origin) is carried out in [5] and [17] and is *not* among our objectives here (although we will recall those results momentarily).

In order to deal with all the solutions of a fixed refinement equation in a combined fashion, we introduce first the notions of the *combined Gramian* and the *coherent approximation order* of the solutions. Let $P$ be a refinement mask, and let $(\Phi_1, \ldots, \Phi_n)$ be a basis for the solution space $R(P)$ of the underlying refinement equation (36). Assuming that, for some $s \in \mathrm{I\!R}$ and for every $j = 1, \ldots, n$, $\Phi_j \subset W_2^s(\mathrm{I\!R}^d)$, we define the **combined Gramian** $G_{R(P),s}$ of the refinement equation (36) to be the *sum* of the individual Gramians:

$$G_{R(P),s} := \sum_{j=1}^{n} G_{\Phi_j,s}.$$

Although the above definition depends on the particular basis that we choose for the solution space, our subsequent analysis of $G_{R(P),s}$ is independent of the basis' choice for the following reason. Let $B = (\Phi_1, \ldots, \Phi_n)$ be a basis for $R(P)$. We consider $B$ as an $r \times n$ matrix. Thanks to the identity

$$\sum_{l=1}^{n} \widehat{\Phi}_l \widehat{\Phi}_l^{\,*} = \widehat{B}\widehat{B}^*, \tag{44}$$

we conclude that

$$G_{R(P),s} = \sum_{\alpha \in 2\pi \mathbb{Z}^d} (\widehat{B}\widehat{B}^* |\cdot|^{2s})(\cdot + \alpha).$$

A new basis for $R(P)$ can be written as $BM$, with $M$ an $n \times n$ constant matrix. Thus the combined Gramian for the new basis has the form

$$\tilde{G}_{R(P),s} := \sum_{\alpha \in 2\pi \mathbb{Z}^d} (\widehat{B} M M^* \widehat{B}^* |\cdot|^{2s})(\cdot + \alpha).$$

Therefore, for some constants $c$, $C > 0$,

$$c v^* \tilde{G}_{R(P),s} v \le v^* G_{R(P),s} v \le C v^* \tilde{G}_{R(P),s} v,$$

for any vector $v$. Using the above inequalities, one can easily check that all our subsequent results are independent of the choice of $B$. We also use the notion of the truncated combined Gramian:

$$G^0_{R(P),s} := \sum_{j=1}^{n} G^0_{\Phi_j,s}.$$

**Definition.** Let $P$ be a refinement mask whose solution space $R(P)$ lies in $W_2^s(\mathrm{I\!R}^d)$. Let $G_{R(P),s}$ be the corresponding combined Gramian. We say that $R(P)$ (or, in short, $P$) provides

**coherent approximation order** $k$ if the following condition holds: there exists a neighborhood $\Omega$ of 0 such that

$$\text{the function } \mathcal{M}_{P,s,k} : \omega \mapsto \frac{1}{|\omega|^{2k-2s}} \inf_v \frac{v^* G^0_{R(P),s}(\omega) v}{v^* G_{R(P),s}(\omega) v} \quad \text{belongs to} \quad L_\infty(\Omega).$$

Here, $G^0_{R(P),s}$ is the truncated version of the combined Gramian, i.e., the expression obtained after subtracting from $G_{R(P),s}$ the terms indexed by $\alpha = 0$ (cf. the analogous definition of the truncated Gramian from Theorem 21).

While Theorem 21 provides ample motivation for the above definition (specifically, it shows that the coherent approximation order coincides with the usual approximation order notion in case the solution space of (36) is one-dimensional), we note that the coherent notion of approximation order does not translate immediately into any clear statement on the approximation order of the individual solutions.

**Discussion 44** *Let us continue with the example in Discussion 42. We observe that in the case discussed there, $G_{R(P),s} = \operatorname{diag}(G_{\phi_1,s}, \ldots, G_{\phi_n,s})$, with $G_{\phi_j,s}$ the (scalar) Gramian of $\phi_j$, i.e., $[\widehat{\phi}_j, \widehat{\phi}_j]$ in the $L_2$-case. It follows easily then that the coherent approximation order matches or exceeds the approximation order provided by $\phi_j$ (for any value of $j$).*

In order to advance our discussion, we consider vectors $v$ that realize the coherent approximation order $k$. That is, with $\Omega \subset \mathbb{R}^d$ some neighborhood of the origin,

$$\Omega \ni \omega \mapsto v(\omega) \in \mathbb{C}^r$$

is measurable, and, a.e. on $\Omega$,

$$\frac{v^*(\omega) G^0_{R(P),s}(\omega) v(\omega)}{v^*(\omega) G_{R(P),s}(\omega) v(\omega)} = O(|\omega|^{2k-2s}). \tag{45}$$

We call such $v$ a **universal supervector** (of order $k$). A vector $v$ is a **regular universal supervector** if (45) can be replaced by the conditions that, near the origin,

$$\frac{v^* G_{R(P),s} v}{v^* v} \sim |\cdot|^{2s} \quad \text{and} \quad \frac{v^* G^0_{R(P),s} v}{v^* v} = O(|\cdot|^{2k}). \tag{46}$$

A regular universal supervector is clearly a universal supervector.

**Discussion 45** *The existence of a universal supervector is implied (almost automatically) by the definition of coherent approximation order. The proof of this fact parallels the proof of the superfunction existence (Theorem 22) and is therefore omitted. The regularity of a universal supervector $v$ may be implied by either of the following two stronger assumptions:*

*(1) The combined Gramian $G_{R(P),s}$ is invertible a.e. around the origin, and the norm of its inverse there satisfies*

$$\|G^{-1}_{P,s}\| = O(|\cdot|^{-2s}).$$

*Indeed, in that case $v^* G_{P.s} v \geq c v^* v |\cdot|^{2s}$ for some positive constant $c$, since $\|G^{-1}_{P,s}\|$ is proportional to the reciprocal of the smallest eigenvalue $\rho_{\min}(G_{R(P),s})$ of $G_{R(P),s}$ and $v^* G_{R(P),s} v \geq \rho_{\min}(G_{R(P),s}) v^* v$. On the other hand, if $v$ is a universal supervector, then*

$$v^* G_{p,s} v \leq \text{const}\, v^* \sum_{\Phi \in B} \widehat{\Phi} \widehat{\Phi}^* |\cdot|^{2s} v \leq \text{const}\, |\cdot|^{2s},$$

*where the last inequality follows from the fact that all $\Phi \in B$ are compactly supported, so their Fourier transforms are bounded around the origin. Therefore, the first, hence all the conditions in (46) are satisfied.*

*(2) For one of the solutions $\Phi \in R(P)$, $|v^*\widehat{\Phi}|/|v| \geq c > 0$, a.e. in some neighborhood of the origin. Indeed, then*

$$v^* \sum_{\Phi \in B} \widehat{\Phi}\widehat{\Phi}^* |\cdot|^{2s} v \sim |\cdot|^{2s} v^* v$$

*and the conditions (46) follow from the fact that $v$ is a universal supervector.*

We now connect among the notions of coherent orders, approximation orders, and regular universal supervectors. It is worthwhile to note that the following result does not invoke the fact that $R(P)$ comprises the solutions to (36). We do not even need the fact that the individual vectors in $R(P)$ are refinable.

**Theorem 46** *Assume that the refinement mask $P$ provides coherent approximation order $k$ in $W_2^s(\mathbb{R}^d)$. Then*

*(a) Let $S_P \subset W_2^s(\mathbb{R}^d)$ be the shift-invariant space generated by $R(P)$ (i.e., it is the smallest closed shift-invariant subspace of $W_2^s(\mathbb{R}^d)$ that contains each entry of each vector in $R(P)$). Then $S_P$ is an FSI space and provides approximation order $k$.*

*(b) Let $v$ be a regular universal supervector of order $k$ that is bounded in a neighborhood of the origin. Let $\Phi$ be a solution of the refinement equation. Then*

*(i) $v^* G^0_{\Phi,s} v = O(|\cdot|^{2k})$ around the origin. In particular, the function $\psi$ defined by $\widehat{\psi} := v^*\widehat{\Phi}$ satisfies the Strang-Fix conditions of order $k$.*

*(ii) If, for some positive $c$, $|v^*\widehat{\Phi}| \geq c$ a.e. in some neighborhood of the origin, then $S_\Phi$ provides approximation order $k$. Moreover, with $\psi \in S_\Phi$ defined by $\widehat{\psi} := v^*\widehat{\Phi}$, the PSI space $S_\psi$ already provides that approximation order.*

*Proof.* (a) Let $B$ be a basis for $R(P)$. Then $S_P = S_F$, with $F$ any vector that contains all the entries from all the vectors $b \in B$. Hence $S_P$ is FSI.

Now, let $v : [-\pi, \pi]^d \to \mathbb{C}^r$ be a vector that realizes the coherent approximation order $k$, i.e., a.e. on $[-\pi, \pi]^d$,

$$\frac{v^* G^0_{R(P),s} v}{v^* G_{R(P),s} v} \leq c|\cdot|^{2k-2s}. \tag{47}$$

If follows that for a.e. $\omega \in [-\pi, \pi]^d$, there exists $\Phi \in B$ such that

$$\frac{v^*(\omega) G^0_{\Phi,s}(\omega) v(\omega)}{v^*(\omega) G_{\Phi,s}(\omega) v(\omega)} \leq c|\omega|^{2k-2s}. \tag{48}$$

This allows us to represent $[-\pi, \pi]^d$ as the disjoint union of sets $\Omega_\Phi$, $\Phi \in B$ such that (48) holds for every $\Phi \in B$ and a.e. $\omega \in \Omega_\Phi$. We need, furthermore, to ensure that these sets are *measurable.* We argue the measurability as follows. First, since $v$ is measurable, so are the functions from the left-hand side of (48). Therefore, the function

$$f_{\min} \quad : \quad \omega \mapsto \min_{\Phi \in B} \frac{v^*(\omega) G^0_{\Phi,s}(\omega) v(\omega)}{v^*(\omega) G_{\Phi,s}(\omega) v(\omega)}$$

is also measurable. Thus, once we define $(\Omega_\Phi)$ by

$$\Omega_\Phi := \{\omega \in [-\pi,\pi]^d : f_{\min}(\omega) = \frac{v^*(\omega) G^0_{\Phi,s}(\omega) v(\omega)}{v^*(\omega) G_{\Phi,s}(\omega) v(\omega)}\}, \qquad \Phi \in B,$$

we obtain the requisite measurability.

Now, let $\tau_\Phi$, $\Phi \in B$, be the $2\pi$-periodic extensions of the characteristic functions of $\Omega_\Phi$, $\Phi \in B$. Defining $\Phi_0$ via its Fourier transform as follows:

$$\widehat{\Phi}_0 := \sum_{\Phi \in B} \tau_\Phi \widehat{\Phi},$$

we conclude from Corollary 3 that each of the entries of $\Phi_0$ lie in $S_{P,s}$. Consequently, $S_{\Phi_0,s} \subset S_{P,s}$. On the other hand, the definition of $\Phi_0$ implies that, a.e. on $[-\pi,\pi]^d$,

$$\frac{v^* G^0_{\Phi_0,s} v}{v^* G_{\Phi_0,s} v} \le c|\cdot|^{2k-2s}. \tag{49}$$

This, in view of Theorem 21, shows that $S_{\Phi_0}$ provides approximation order $k$ (in $W_2^s(\mathrm{IR}^d)$), *a fortiori* its superspace $S_P$ provides that order.

(b): The regularity of the supervector $v$ implies that, around the origin,

$$\frac{v^* G^0_{R(P),s} v}{v^* v} = O(|\cdot|^{2k})$$

Since $v$ is assumed bounded, we conclude that

$$v^* G^0_{R(P),s} v = O(|\cdot|^{2k}),$$

and therefore

$$v^* G^0_{\Phi,s} v = O(|\cdot|^{2k})$$

for *every* $\Phi \in R(P)$. This proves the first part of (b)(i), while the second part follows from the fact that each of the summands $v^*|\cdot{+}\alpha|^{2s}(\widehat{\Phi}\widehat{\Phi}^*)(\cdot{+}\alpha)\, v$ (that together make up $v^* G^0_{\Phi,s}\, v$) is nonnegative, hence has to vanish to order $2k$ as well.

As to (b)(ii), the analysis above shows that the function $\psi$ defined by $\widehat{\psi} := v^* \widehat{\Phi}$ satisfies

$$[\widehat{\psi}, \widehat{\psi}]^0_s = [\widehat{\psi}, \widehat{\psi}]_s - |\widehat{\psi}|^2 |\cdot|^{2s} = O(|\cdot|^{2k})$$

(near the origin). Since we further assume here that $|\widehat{\psi}| \ge c > 0$ around the origin, we also conclude that $[\widehat{\psi}, \widehat{\psi}]_s \ge c^2 |\cdot|^{2s}$ there. Thus, (5) of Theorem 7 holds, and that theorem implies that $S_\psi(W_2^s(\mathrm{IR}^d))$ provides approximation order $k$. □

The first part of Theorem 46 leads to the following conclusion:

**Corollary 47** *Let $P$ be a refinement mask and let $S_P \subset W_2^s(\mathrm{IR}^d)$ be the corresponding SI space. If $P$ provides a coherent approximation order $k$, then there exists $\psi \in S_P$ for which the PSI space $S_\psi \subset W_2^s(\mathrm{IR}^d)$ provides approximation order $k$.*

**Remark.** Note that the combined Gramian $G_{R,s}$ can be defined for any *finite-dimensional* space $R$ of distributional solutions to the refinement equation (36). Likewise, the notion of (regular) universal supervectors makes sense with respect to any such space $R$ . The requirement that $R$ be the space of all compactly supported solutions actually plays no role in the results of this section. The only condition used is that, for each $\Phi \in R$, its Fourier transform $\widehat{\Phi}$ be bounded around the origin. Therefore, all results of this section are applicable to this more general setup.

### 5.3 Universal supervectors and sum rules

#### 5.3.1 Known results: singleton solutions in $L_2(\mathbb{R}^d)$

The characterizations to-date of the approximation power of refinable vectors are confined to the $L_2$ setup, and assume, at a minimum, that the Gramian of the (necessarily unique) compactly supported solution is invertible at zero (as well as at several additional points). These characterizations allow one to deduce the approximation order provided by the refinable vector directly from the mask. Two relevant notions in this context are *Condition* $Z_k$ and the *sum rules*. We begin with the definition of the former.

**Definition.** Given $k > 0$, we say that the refinement mask $P$ **satisfies Condition** $Z_k$ if there exists a vector $v$ of trigonometric polynomials such that, for each $l \in E$, the vector $v^*(2\cdot)P - \delta_{l,0}v^*$ has a zero or order $k$ at $\pi l$, while $v(0) \neq 0$. Here,

$$E := \{0,1\}^d, \tag{50}$$

is the set of vertices of the $d$-unit cube.

**Result 48 ([5], [17])** *Let $P$ be a refinement mask, and assume that $\dim R(P) = 1$. Let $\Phi$ be the unique solution of (36), and assume that $\Phi \subset L_2(\mathbb{R}^d)$ is compactly supported, and that $G_\Phi$ is invertible at the origin. Then, for $k \in \mathbb{N}$:*

*I. If $P$ satisfies Condition $Z_k$ then $S_\Phi$ provides approximation order $k$.*

*II. If $S_\Phi$ provides approximation order $k$ and if $G_\Phi$ is invertible at each point of $\pi E$, then $P$ satisfies Condition $Z_k$.*

We note that the compact support assumption on $\Phi$ in the above-quoted result can be weakened: the essential needed information is about the behavior of the Gramian around $E$. We refer to [5] and [15] for more details.

Condition $Z_k$ is written on the Fourier domain. It can be equivalently expressed on the "space" domain. The equivalent space-based formulations of Condition $Z_k$ are colloquially known as the *sum rules*. We provide, for completeness, the two frequently used versions of these sum rules. The second is taken from [5] (see also [14]), while the first is borrowed from [17].

**Result 49** *Let $\Phi \subset L_2$ be compactly supported with trigonometric refinement mask $P$. Let $v$ be a vector of trigonometric polynomials. Then the following conditions are equivalent:*

*(a) $P$ satisfies Condition $Z_k$ with respect to the current $v$.*

*(b) The pair $v, P$ satisfies the 1st version of sum rules: with $(v_\gamma)$ and $(P_\gamma)$ the Fourier coefficients of $v$ and $P$, respectively,*

$$\sum_{\sigma\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d} v^*_{\sigma-\gamma}P_{l+2\sigma}q(l+2\gamma) = 2^{-d}\sum_{\gamma\in\mathbb{Z}^d} v^*_{-\gamma}q(\gamma), \qquad l \in E, \quad q \in \Pi_{k-1}.$$

*(Note that $v_\gamma$ is a vector, $P_\gamma$ is a matrix, and $q(\gamma)$ is a scalar.)*

(c) *The pair $v, P$ satisfies the 2nd version of sum rules: With $v^\alpha = D^\alpha v(0)$, $\alpha \in \mathbb{Z}_+^d$, the Taylor coefficients of $v$ at the origin, we have:*

$$\sum_{\beta \leq \alpha} 2^{|\alpha-\beta|}(v^{\alpha-\beta})^*(D^\beta P)(\pi l) = \delta_{l,0}(v^\alpha)^*, \quad l \in E, \ |\alpha| < k.$$

*Proof.* The second version of the sum rules is equivalent to Condition $Z_k$, as seen by applying $D^\alpha$ to $v^*(2\cdot)P - \delta_{l,0}v^*$, expanding the first term by Leibniz' rule, and evaluating the result at $\pi l$, $l \in E$.

The equivalence of the first version to Condition $Z_k$ can be argued as follows: first, recall that for a finitely supported $s \in \mathbb{C}^{\mathbb{Z}^d}$, its Fourier series $\widehat{s}$ has a zero of order $k$ at the origin iff $\Pi_{k-1}$ lies in the kernel of the functional

$$\lambda_s : q \mapsto (s * q)(0),$$

where $s * q$ is either the semi-discrete convolution or the discrete convolution (the statement is true for each of the two choices) of the sequence $s$ and the polynomial $q$. Let $m \in E$, and let $s_m$ be the (vector-valued) Fourier coefficients of the function $v^*(2\cdot)P(\cdot + \pi m) - \delta_{m,0}v^*$. Thus, Condition $Z_k$ tells that, for every $q \in \Pi_{k-1}$ and for every $m \in E$,

$$0 = \lambda_{s_m}(q) = \sum_{\gamma,\eta \in \mathbb{Z}^d} v_{-\gamma}^* P_\eta q(2\gamma + \eta)e^{\pi i \eta \cdot m} - \delta_{m,0} \sum_{\gamma \in \mathbb{Z}^d} v_{-\gamma}^* q(\gamma).$$

Fixing some $l \in E$, we can write $\eta = l' + l + 2\sigma$, for suitable $\sigma \in \mathbb{Z}^d$, and $l' \in E$. Thus,

$$0 = \sum_{l' \in E} \sum_{\gamma,\sigma} v_{-\gamma}^* P_{l'+l+2\sigma} q(2\gamma + 2\sigma + l' + l)e^{\pi i(l'+l)\cdot m} - \delta_{m,0} \sum_{\gamma} v_{-\gamma}^* q(\gamma).$$

Multiplying the two sides of the last display by $e^{-\pi i l \cdot m}$ and summing over $m$, we obtain

$$0 = \sum_{l' \in E} \sum_{\gamma,\sigma} v_{-\gamma}^* P_{l'+l+2\sigma} q(2\gamma + 2\sigma + l' + l) \sum_{m \in E} e^{\pi i l' \cdot m} - \sum_{\gamma} v_{-\gamma}^* q(\gamma).$$

Thus,

$$0 = 2^d \sum_{\gamma,\sigma} v_{-\gamma}^* P_{l+2\sigma} q(2\gamma + 2\sigma + l) - \sum_{\gamma} v_{-\gamma}^* q(\gamma).$$

Replacing $\gamma$ by $\gamma - \sigma$ finishes the proof. □

### 5.3.2 New results: multiple solutions in $W_2^s$

Our analysis of the multiple solution case is based on drawing a connection between universal regular supervectors on the one hand, and Condition $Z_k$ (together with its associated sum rules) on the other hand. This approach requires some limited regularity formulated in terms of the combined Gramian $G_{R,s}$ of the space $R$ of refinable distributions (see Theorem 53 for the precise assumption).

So, let $P$ be an $r \times r$ refinement mask and let $R$ be a finite-dimensional space of solutions to (36) lying in some Sobolev space: $R \subset W_2^s(\mathbb{R}^d)$ (in the sense that every entry of every vector in $R$ lies in $W_2^s(\mathbb{R}^d)$). Let $k$ be a positive number. We consider vector-valued functions $v$, that together with $P$ and $R$ satisfy the following assumptions:

**Assumptions 50** *1. There exists a neighborhood $\Omega$ of the origin, such that, a.e. on $\Omega$,*

$$\|G_{R,s}(\cdot+\pi l)\| = \begin{cases} O(1), & l\in E\backslash 0,\\ O(|\cdot|^{2s}), & l=0. \end{cases} \tag{51}$$

2. *The entries of $v$ are $2\pi$-periodic and measurable. Moreover, $v^*v$ is bounded, and bounded away from zero, around the origin.*
3. *For some $k>0$, and for every $\Phi\in R$, the function $\phi$ defined by $\widehat{\phi} := v^*\widehat{\Phi}$ satisfies condition (11).*

Note that Assumptions 50 are valid (regardless of the value of $k$) whenever $R$ contains only compactly supported solutions, and $v$ is a vector-valued trigonometric polynomial, provided that $v(0)\neq 0$.

**Theorem 51** *Let $P$ be an $r\times r$ refinement mask, let $R\subset W_2^s(\mathbb{R}^d)$ be a finite-dimensional space of solutions to (36), and let $v$ be a vector-valued function, so that $P$, $R$ and $v$ satisfy Assumptions 50 with respect to some $k>0$. If $P$ and $v$ satisfy Condition $Z_k$, then $v^*G^0_{R,s}v$ has a zero of order $2k$ at the origin, and, in addition, for every $\Phi\in R$, $v^*\widehat{\Phi}-(v^*\widehat{\Phi})(0)=O(|\cdot|^k)$, provided that $v^*\widehat{\Phi}$ is smooth at the origin. In particular:*

*(a) If $v^*G_{R,s}v\sim|\cdot|^{2s}$ a.e. around the origin, then $v$ is a universal supervector (with respect to $R$) of order $k$, and hence $R$ provides coherent approximation order $k$. This, for example, is the case if $G_{R,s}$ is invertible (a.e.) in a neighborhood of the origin, and $\|G^{-1}_{R,s}\|=O(|\cdot|^{-2s})$ a.e. there.*

*(b) If, for some $\Phi\in R$, $|v^*\widehat{\Phi}|\geq c>0$ a.e. around the origin, then (a) applies, and we further conclude that $S_\Phi$ provides approximation order $k$.*

*Proof.* Let $\alpha\in 2\pi\mathbb{Z}^d\backslash 0$, and let $m:=m(\alpha)\geq 1$ be the smallest integer for which $\alpha/2^m\notin 2\pi\mathbb{Z}^d$. Let $\Phi\in R$. We prove, by induction on $m(\alpha)$, that $v^*\widehat{\Phi}(\cdot+\alpha)=O(|\cdot|^k)$. For $m=1$, we choose $l\in E\backslash 0$ such that $2\pi l-\alpha\in 4\pi\mathbb{Z}^d$. Since

$$v^*\widehat{\Phi}(\cdot+\alpha)=v^*P\left(\frac{\cdot}{2}+\pi l\right)\widehat{\Phi}\left(\frac{\cdot+\alpha}{2}\right),$$

the claim follows from the fact that $v^*P(\frac{\cdot}{2}+\pi l)=O(|\cdot|^k)$. For $m>1$, we write

$$v^*\widehat{\Phi}(\cdot+\alpha)=v^*\left(\frac{\cdot}{2}\right)\widehat{\Phi}\left(\frac{\cdot+\alpha}{2}\right)+(v^*P\left(\frac{\cdot}{2}\right)-v^*\left(\frac{\cdot}{2}\right))\widehat{\Phi}\left(\frac{\cdot+\alpha}{2}\right).$$

By Condition $Z_k$, $v^*P(\cdot/2)-v^*(\cdot/2)=O(|\cdot|^k)$. In addition, since $m(\alpha/2)=m-1$, the induction hypothesis yields that $v^*(\cdot/2)\widehat{\Phi}((\cdot+\alpha)/2)=O(|\cdot|^k)$, too.

Now, fix $\Phi\in R$ and define $\widehat{\psi}:=v^*\widehat{\Phi}$. Since $v$ is bounded and $\psi\in W_2^s$, we can invoke Corollary 3 and conclude that $\psi\in S_\Phi(W_2^s)$. Since $\psi$ satisfies (11) (as stipulated in Assumptions 50), and since, by our argument above, $\widehat{\psi}(\cdot+\alpha)=O(|\cdot|^k)$, for every $\alpha\in 2\pi\mathbb{Z}^d\backslash 0$, we see that

$$\sum_{\alpha\in 2\pi\mathbb{Z}^d\backslash 0}|\widehat{\psi}(\cdot+\alpha)|^2|\cdot+\alpha|^{2s}=O(|\cdot|^{2k}). \tag{52}$$

However, the left-hand-side in the above equality is $v^* G^0_{\Phi,s} v$, and, hence, by summing (52) over a basis of $R$ we obtain that $v^* G^0_{R(P),s} v = O(|\cdot|^{2k})$.

Next, with $\psi$ as above, the case $l = 0$ in Condition $Z_k$ together with the boundedness of $\widehat{\Phi}$ around the origin (the latter is embedded in Assumptions 50) imply that

$$\widehat{\psi}(2\cdot) - \widehat{\psi} = (v^*(2\cdot)P - v^*)\widehat{\Phi} = O(|\cdot|^k).$$

Once we assume $\widehat{\psi}$ to be smooth around the origin, the above implies that $\widehat{\psi} - \widehat{\psi}(0) = O(|\cdot|^k)$, as claimed.

The proofs of (a) and (b) are straightforward, hence are omitted. $\square$

Since the case of compactly supported solutions and a trigonometric polynomial $v$ is of central importance here, we record separately the statement of Theorem 51 for this case.

**Corollary 52** *Let $P$ be a trigonometric polynomial refinement mask, let $R(P) \subset W_2^s$ be the space of all compactly supported solutions to the refinement equation (36), and let $v$ be a vector-valued trigonometric polynomial. Suppose that $P$ and $v$ satisfy Condition $Z_k$ for some $k > 0$. Then:*

1. *For each $\Phi \in R(P)$, the (compactly supported) $\psi$ defined by $\widehat{\psi} := v^*\widehat{\Phi}$ satisfies the Strang-Fix conditions of order $k$, and in addition, $\widehat{\psi} - \widehat{\psi}(0) = O(|\cdot|^k)$. Consequently, if $v^*(0)\widehat{\Phi}(0) \neq 0$, then $S_\Phi(W_2^s)$ provides approximation order $k$, and $\psi \in S_\Phi(W_2^s)$ is a corresponding superfunction.*

2. *If $v^*(0)\widehat{\Phi}(0) \neq 0$ for some $\Phi \in R(P)$, then $P$ provides coherent approximation order $k$, and $v$ is a corresponding universal regular supervector.*

At present, we do not know whether Condition $Z_k$ is necessary for the provision of coherent approximation order $k$ in case $\dim R(P) > 1$. The results of this type that we are able to prove make strong assumptions on the mask $P$. Below is one such result. The stringent assumption here is that $P(0) = I$. In what follows, we use the notation $G^0_{R(P),s}$ introduced in §5.2 for the truncated combined Gramian. We prove the result only for $s = 0$, although it extends to other values of $s$ (at a cost of a few technical details and more awkward notation).

**Theorem 53** *Let $P$ be an $r \times r$ trigonometric polynomial refinement mask and let $R(P)$ be the space of compactly supported solutions to (36). Suppose that $R(P) \subset L_2$ and that the combined Gramian $G_{R(P)}$ satisfies Assumption 50.1, is smooth around each $l \in E$ and is boundly invertible around each $l \in E$. If $P(0) = I$, the following conditions are equivalent:*

*(a) $P$ satisfies Condition $Z_k$ with some vector $v$ satisfying Assumptions 50.2 and 50.3.*

*(b) There exists a regular universal supervector $v$ of order $k$ for the space $R(P)$.*

*In addition, a regular universal supervector $v$ of order $k$ can be always chosen so that, for every $\Phi \in R(P)$,*

$$v^*\widehat{\Phi} - (v^*\widehat{\Phi})(0) = O(|\cdot|^k).$$

*Proof.* In view of Theorem 51, we only need to prove the implication (b) $\Longrightarrow$ (a).

We start the proof by noting the identities

$$G_{R(P)}(2\cdot) = \sum_{l\in E} (PG_{R(P)}P^*)(\cdot + \pi l), \tag{53}$$

and

$$G^0_{R(P)}(2\cdot) = PG^0_{R(P)}P^* + \sum_{l\in E\backslash 0} (PG_{R(P)}P^*)(\cdot + \pi l). \tag{54}$$

The first identity is straightforward (and is quite well-known; cf. [23]). The second one is obtained by the subtraction of the identity

$$\sum_{\Phi\in B} (\widehat{\Phi}\widehat{\Phi}^*)(2\cdot) = \sum_{\Phi\in B} P\widehat{\Phi}\widehat{\Phi}^* P^*$$

from the first one. Here, $B$ is the basis for $R(P)$ that was used to define $G_{R(P)}$.

Let $\tau$ be any $2\pi$-periodic vector-valued function that satisfies the condition

$$\tau^* G^0_{R(P)}\tau = O(|\cdot|^{2k}), \tag{55}$$

near the origin. Then $(\tau^* G^0_{R(P)}\tau)(2\cdot) = O(|\cdot|^{2k})$ near the origin. Thus the evaluation at $\tau(2\cdot)$ of the quadratic form in the right-hand-side of (54) leads to a function which has a zero of order $2k$ at the origin. Since each summand there is nonnegative, it follows that, for every $l \in E\backslash 0$,

$$\tau^*(2\cdot)(PG_{R(P)}P^*)(\cdot + \pi l)\tau(2\cdot) = O(|\cdot|^{2k}).$$

However, $G_{R(P)}$ is assumed to be boundly invertible around $\pi l$, hence we must have that $\tau^*(2\cdot)P(\cdot + \pi l) = O(|\cdot|^k)$, near the origin, for every $l \in E\backslash 0$. In addition,

$$\tau^*(2\cdot)(PG^0_{R(P)}P^*)\tau(2\cdot) = O(|\cdot|^{2k}). \tag{56}$$

Now, let $v$ be a universal supervector. Then, (55) is satisfied for $\tau := v$, hence $v$ satisfies the requirements in Condition $Z_k$ with respect to each $l \neq 0$. It remains to modify $v$ (if need be) so that Condition $Z_k$ be satisfied at $l = 0$, too. Note that so far we have not used out special assumption on $P$. Still, we already know that (56) is satisfied for $\tau := v$.

In order to complete our argument, we assume that $P = I + O(|\cdot|^k)$ near the origin. We will revisit this condition after completing the main part of the proof. This additional condition, when applied to (56) leads (once we take into account the boundedness and self-adjointness of $G^0_{R(P)}$) to

$$\tau^*(2\cdot)G^0_{R(P)}\tau(2\cdot) = O(|\cdot|^{2k}). \tag{57}$$

Thus, we proved that (55) implies (57), and hence, since (55) is satisfied for $\tau := v$, we conclude that

$$v^*(2^n\cdot)G^0_{R(P)}v(2^n\cdot) = O(|\cdot|^{2k}) \qquad \text{for all } n \in \mathbb{N}. \tag{58}$$

Our previous analysis then implies that $v^*(2^n\cdot)P(\cdot + \pi l) = O(|\cdot|^k)$, for every integer $n \geq 1$, and every $l \in E\backslash 0$.

Now, by forming a suitable finite linear combination of $v(2^n\cdot)$, $n = 0, 1, \ldots$, we can construct a vector $u$ such that $u(0) = v(0)$, while $u - u(0) = O(|\cdot|^k)$ at the origin. Clearly, $u^*P(\cdot + \pi l) = O(|\cdot|^k)$, for every $l \in E\backslash 0$. Since both $u - u(0)$ and $P - I$ have a $k$-fold zero at the origin, we conclude that $u^*(2\cdot)P - u^* = O(|\cdot|^k)$. Thus $u$ satisfies Condition $Z_k$.

We finally contend that there is no loss of generality in the assumption that $P - I = O(|\cdot|^k)$ around the origin. Indeed, consider the transformation $P \mapsto T^{-1}(2\cdot)PT$, where $T$ is a trigonometric polynomial-valued matrix, such that $T(0)$ is invertible. For each $\omega$ in some small neighborhood of the origin, we have the linear isomorphism defined on $\{\widehat{\Phi}(\omega) : \Phi \in R(P)\}$ by

$$\widehat{\Phi}(\omega) \mapsto T(\omega)\widehat{\Phi}(\omega).$$

Since each $\widehat{\Phi} \in R(P)$ is entire, the isomorphisms induce a corresponding one between the spaces $R(P)$ and $R(T^{-1}(2\cdot)PT)$. Moreover, Condition $Z_k$ is invariant under this isomorphism, since the vector $v^*(2\cdot)P - \delta_{l,0}v^*$ vanishes to order $k$ at $\pi l$ for each $l \in E$ if and only if so does the vector $(v^*T)(2\cdot)(T^{-1}(2\cdot)PT) - \delta_{l,0}(v^*T)$.

So, if we can show that we can choose a matrix-valued polynomial $T$ (of degree smaller than $k$) so that $T(0)$ is invertible and

$$PT = T(2\cdot) + O(|\cdot|^k), \tag{59}$$

then our claim will follow. To this end, let $T(0) = I$ and let the derivatives $(D^\alpha T)(0)$ be defined inductively, according to the partial order of multi-integers $\alpha$, as solutions to the equation

$$\sum_{0\le\beta\le\alpha} (D^{\alpha-\beta}P)(0)(D^\beta T)(0) = 2^{|\alpha|}(D^\alpha T)(0), \qquad 0 < |\alpha| < k. \tag{60}$$

This system is obtained by differentiating (59) at the origin and is equivalent to (59). For a fixed $\alpha$, the values $(D^\beta T)(0)$, $\beta < \alpha$ are already chosen, and the coefficient of the term $D^\alpha T(0)$ is $2^{|\alpha|} - 1 \neq 0$ (since $P(0) = I$). Thus, (60) has a solution $(D^\alpha T)(0)$. Thus, $P$ can be assumed to be within $O(|\cdot|^k)$ of the origin. This completes the proof. □

#### 5.3.3 Coherent polynomial reproduction

We restrict our attention again to the space $R(P)$ of compactly supported solutions to the refinement equation (36). We show that universal supervectors for $R(P)$ are also ultimately connected with polynomial reproduction using the shifts of any compactly supported solution from $R(P)$. In short, we show that universal supervectors provide universal polynomial reproduction schemes:

**Theorem 54** *Let $P$ be a refinement mask whose space of compactly supported solutions $R(P)$ lies in $W_2^s(\mathbb{R}^d)$. Let $v$ be a vector-valued trigonometric polynomial such that, for some $k > 0$,* any one *of the following conditions holds:*

1. *$v$ satisfies Condition $Z_k$.*

2. *$v$ and $P$ satisfy the 1st version of the sum rules*

3. *$v$ and $P$ satisfy the 2nd version of the sum rules*

4. *$v$ is a regular universal supervector of order $k$.*

*Let $a$ be the vector-valued sequence of the Fourier coefficients of $v^*$, and let $\Phi \in R(P)$. Then, with $a_1, \ldots, a_r$ the entries of $a$ and $\phi_1, \ldots, \phi_r$ the entries of $\Phi$, the map*

$$T_\Phi : q \mapsto \sum_{i=1}^{r} \phi_i *' (a_i *' q) =: \Phi *' (a *' q)$$

*maps $\Pi_{<k}$ into itself. The map is surjective (hence degree preserving) if and only if $v^*(0)\widehat{\Phi}(0) \neq 0$.*

*Proof.* By Result 49, conditions 1 through 3 are equivalent one to the other, and, by Theorem 51, each of them implies 4. Condition 4, in turns, implies that, for any $\Phi \in R(P)$, the compactly supported function $\psi$ defined by $\widehat{\psi} := v^*\widehat{\Phi}$ satisfies the Strang-Fix conditions of order $k$. The discussion preceding Theorem 17 now implies that the semi-discrete convolution operator $\psi *'$ reproduces polynomials of total degree at most $k-1$ and, moreover, preserves the degree if $\psi$ is nondegenerate, i.e., $\widehat{\psi}(0) \neq 0$. But $\psi$ itself is nothing but the semidiscrete convolution $\Phi *' \widehat{v}^*$. Since the convolution $*'$ is associative, we see that the function $\Phi *' (\widehat{v}^* *' q)$ is a polynomial in $\Pi_{k-1}$ whenever $q \in \Pi_{k-1}$; moreover, it has exactly the same degree as $q$ whenever $v^*(0)\widehat{\Phi}(0) = \widehat{\psi}(0) \neq 0$. $\square$

## Acknowledgments

We thank Carl de Boor for his critical reading of an earlier draft of this paper and Gautam Bharali for a very helpful discussion of our earlier approach to section 4.9.

# Index

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