In ODEs, it is important to understand the solutions to the first-order ODE

$$Dx(t) = Ax(t), \quad x(0) = c,$$

in which A is a linear map on some finite-dimensional vector space, V, and, correspondingly, $x : \mathbb{R} \to V$ is a V-valued function to be determined.

The formal solution is

$$x(t) = \exp(tA)c,$$

with

$$\exp(B) := \sum_{j=0}^{\infty} B^j/j!$$

well-defined for every linear map B on V, but such a formal expression doesn't give much insight.

Let $\oplus_j V_j$ be a finest A-invariant direct sum decomposition for V, and let A_j be the restriction of A to V_j. Assuming the underlying field to be algebraically closed, there is some λ_j in the spectrum of A_j, hence $B := A - \lambda_j$ has a nontrivial kernel, while the sequence $(\ker B^r : r = 0, 1, \ldots)$ is increasing, hence must become stationary. If q is the smallest natural number for which $\ker B^q = \ker B^{q+1}$, then $\operatorname{ran} B^q \cap \ker B^q$ is trivial, hence V_j is the direct sum of $\ker B^q$ and $\operatorname{ran} B^q$ and, the direct sum decomposition being finest, it follows that $\operatorname{ran} B^q$ must be trivial, i.e., B is nilpotent. Thus, on V_j, $A = \lambda_j + B$, the sum of a constant (hence diagonable trivially and also commuting with any linear map on V_j) and a nilpotent. Since the direct sum decomposition is A-invariant, it follows that $A = D + N$, with D diagonable, and N nilpotent, and $DN = ND$.

It follows that, on V_j and with $N_j := A_j - \lambda_j$ nilpotent of order q_j,

$$\exp(tA) = \exp(t\lambda_j + tN_j) = \exp(t\lambda_j)\exp(tN_j) = \exp(t\lambda_j) \sum_{i<q_j} (tN_j)^i/i!.$$

In particular, if $q_j = 1$, then $\exp(tA)$ reduces on V_j to multiplication by the number $\exp(t\lambda_j)$.

To this, Mike Crandall has the following to say.

Let p be any monic polynomial that annihilates A and factor it, i.e.,

$$p =: \prod_{j=1}^{m} (\cdot - \lambda_j)^{m_j} =: p_1 \cdots p_m.$$

(If p is of minimal degree, then the spectrum of A is necessarily the set $\{\lambda_j : j = 1{:}m\}$, but that matters only when we are looking for m and the m_j here to be as small as possible). Define

$$V_j := \ker p_j(A), \qquad j = 1{:}m,$$

and

$$\ell_i := \prod_{j \neq i} p_j, \qquad i = 1{:}m.$$

Since the ℓ_i have no zeros in common, any nontrivial polynomial of minimal degree in

$$\mathcal{I}(\ell_i : i = 1{:}m) := \sum_i \ell_i \Pi$$

must be of degree 0 (since, otherwise, by the Euclidean algorithm, there would be a polynomial of positive degree dividing each of the ℓ_i, hence its zeros (sure to exist since we are over $\mathbb{C}$) would be common to all the ℓ_i). In particular,

$$1 = \sum_i \ell_i q_i$$

for some $q_i \in \Pi$.

It follows that

$$1 = P_1 + \cdots + P_m,$$

with

$$P_i := \ell_i(A) q_i(A), \qquad i = 1{:}m,$$

linear maps that commute with $r(A)$ for any $r \in \Pi$. Further, P_i vanishes on each $V_j = \ker p_j(A)$ for $j \neq i$ (since ℓ_i contains the factor p_j for each such j), hence $P_i = 1$ on V_i. On the other hand, $\operatorname{ran} P_i \subset V_i$ since

$$p_i(A) P_i = p(A) q_i(A) = 0.$$

Consequently, $\operatorname{ran} P_i = V_i$ and $P_i = 1$ on its range, hence P_i is a linear projector, onto V_i, all i, and so,

$$P_i P_j = \delta_{ij}.$$

It follows that

$$V = \oplus_i V_i.$$

Further,

$$N := A - \sum_i \lambda_i P_i = \sum_i (A - \lambda_i) P_i$$

is nilpotent since $P_i P_j = 0$ for $i \neq j$, hence

$$N^q = \sum_i (A - \lambda_i)^q P_i = 0$$

for $q \geq \max_i m_i$.