Odd-degree spline interpolation at a biinfinite knot sequence

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Abstract It is shown that for an arbitrary strictly increasing knot sequence $\mathbf{t} = (t_i)_{-\infty}^{\infty}$ and for every i, there exists exactly one fundamental spline L_i (i.e., $L_i(t_j) = \delta_{ij}$, all j), of order $2r$ whose rth derivative is square integrable. Further, $L_i^{(r)}(x)$ is shown to decay exponentially as x moves away from t_i, at a rate which can be bounded in terms of r alone. This allows one to bound odd-degree spline interpolation at knots on bounded functions in terms of the global mesh ratio $M_{\mathbf{t}} := \sup_{i,j} \Delta t_i / \Delta t_j$.

A very nice result of Demko's concerning the exponential decay away from the diagonal of the inverse of a band matrix is slightly refined and generalized to (bi)infinite matrices.

1. Introduction. Let $\mathbf{t} := (t_i)_{-\infty}^{+\infty}$ be a biinfinite, strictly increasing sequence, set

$$t_{\pm\infty} := \lim_{i\to\pm\infty} t_i,$$

let $k = 2r$ be a positive, even integer, and denote by $\$_{k,\mathbf{t}}$ the collection of spline functions of order k (or, of degree $< k$) with knot sequence $\mathbf{t}$. Explicitly, $\$_{k,\mathbf{t}}$ consists of exactly those $k-2$ times continuously differentiable functions on

$$I := (t_{-\infty}, t_\infty)$$

that, on each interval (t_i, t_{i+1}), coincide with some polynomial of degree $< k$, i.e.,

$$\$_{k,\mathbf{t}} := \mathbb{P}_{k,\mathbf{t}} \cap C^{k-2} \;\text{ on } I = (t_{-\infty}, t_\infty).$$

We are particularly interested in *bounded* splines

$$m\$_{k,\mathbf{t}} := \$_{k,\mathbf{t}} \cap m(I),$$

i.e., in splines s for which

$$\|s\|_\infty := \sup_{t\in I} |s(t)|$$

is finite. It is obvious that the **restriction map**

$$R_{\mathbf{t}} : \$_{k,\mathbf{t}} \to \mathbb{R}^{\mathbb{Z}} : s \mapsto s|_{\mathbf{t}} := (s(t_i))_{-\infty}^{\infty}$$

carries $m\$_{k,\mathbf{t}}$ into the space $m(\mathbb{Z})$ of bounded, biinfinite sequences. We are interested in inverting this map, i.e., in interpolation. We consider the

Bounded interpolation problem. *To construct, for given $\alpha \in m(\mathbb{Z})$, some $s \in m\$_{k,\mathbf{t}}$ for which $s|_{\mathbf{t}} = \alpha$.*

We will say that the B.I.P. is **correct** (for the given knot sequence $\mathbf{t}$ if it has exactly one solution for every $\alpha \in m(\mathbb{Z})$.

We consider under what conditions on $\mathbf{t}$ the B.I.P. is correct. We also discuss the continuity properties of the map $\alpha \mapsto s_\alpha$ in case the B.I.P. is correct. We establish the following theorem.

Theorem 1. *If the global mesh ratio*

$$M_{\mathbf{t}} := \sup_{i,j} \Delta t_i / \Delta t_j$$

is finite, then $I = (-\infty, \infty)$, and $R_{\mathbf{t}}$ maps $m\$_{k,\mathbf{t}}$ faithfully onto $m(\mathbb{Z})$, i.e., for every bounded, biinfinite sequence α, there exists one and only one bounded spline $s_\alpha \in \$_{k,\mathbf{t}}$ for which $s_\alpha(t_i) = \alpha_i$, all i. Moreover,

$$\|s_\alpha\|_\infty \le \text{const}\|\alpha\|_\infty, \quad \text{all } \alpha \in m(\mathbb{Z}), \tag{1.1}$$

with const *depending only on k and $M_{\mathbf{t}}$.*

We note in passing the following immediate corollary.

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Corollary. *Denote by* $\overset{\circ}{\mathrm{C}}[a,b]$ *the space of continuous* $(b-a)$*-periodic functions on* R*. Given* $\boldsymbol{\tau} := (\tau_i)_0^n$ *with* $a = \tau_0 < \cdots < \tau_n = b$*, let* $\mathbf{t} = (t_i)_{-\infty}^{\infty}$ *be its "*$(b-a)$*-periodic extension", i.e.,*

$$t_{i+nj} := \tau_i + j(b-a) \ \text{ for } i = 1, \ldots, n \ \text{ and all } j \in \mathbb{Z}\,.$$

Denote by $\overset{\circ}{\$}_{k,\boldsymbol{\tau}}$ *the* $(b-a)$*-periodic functions in* $\$_{k,\mathbf{t}}$*. Then (as is well known), for every* $f \in \overset{\circ}{\mathrm{C}}[a,b]$*, there exists exactly one* $s_f \in \overset{\circ}{\$}_{k,\boldsymbol{\tau}}$ *that agrees with* f *at* $\tau_0, \tau_1, \ldots, \tau_n$*. Further, for some* const *depending only on the global mesh ratio* $M_{\boldsymbol{\tau}} = \max_{i,j} \Delta\tau_i/\Delta\tau_j$*,*

$$\|s_f\|_\infty \le \mathrm{const}\|f\|_\infty, \quad \text{all } f \in \overset{\circ}{\mathrm{C}}[a,b].$$

Indeed, if $s_f \in \$_{k,\mathbf{t}}$ agrees with $f \in \overset{\circ}{\mathrm{C}}[a,b]$ at $\mathbf{t}$, then so does its translate $s_f(\cdot - (b-a))$ which is also in $\$_{k,\mathbf{t}}$, and therefore must equal s_f, by the uniqueness of the interpolating spline. This shows that s_f is the interpolating spline in $\overset{\circ}{\$}_{k,\boldsymbol{\tau}}$ for f, and so $\|f_s\| \le \mathrm{const}\|f\|$ from (1.1).

For the case of *uniform* $\mathbf{t}$, $\mathbf{t} = \mathbb{Z}$ say, the problem of bounded interpolation has been solved some time ago by Ju. Subbotin [17]. In this case, the interpolation conditions $s_\alpha|_{\mathbf{t}} = \alpha$ establish a one-to-one and continuous correspondence between bounded splines and bounded sequences. Subbotin came upon the interpolating spline as a solution of the extremum problem of finding a function s with $s|_{\mathbf{t}} = \alpha$ and smallest possible $(k-1)$st derivative, measured in the supremum norm. Later, I. J. Schoenberg investigated the B.I.P. once more, this time as a special case of cardinal spline interpolation to sequences α that do not grow too fast at infinity [15], [16].

Little is known for more general knot sequences. The simplest case, $k = 2$, of piecewise linear interpolation is, of course, trivial. The next simplest case, $k = 4$, of cubic spline interpolation has been investigated in [6] where the above theorem can be found for this case.

The basic tool of the investigation in [6] is the exponential decay or growth of *nullsplines*. Nullsplines are therefore the topic of Section 2 of this paper, if only to admit defeat in the attempt to generalize the approach of [6]. We are more successful, in Section 3, in identifying, for each knot sequence $\mathbf{t}$ and each i, a particular **fundamental spline** L_i, i.e., a spline with $L_i(t_j) = \delta_{ij}$, that must figure in the solution of the B.I.P., if there is one at all (see Lemma 1 and 2). The argument is based on an idea of Douglas, Dupont and Wahlbin [12] as used in [7] and further clarified, simplified and extended by S. Demko [10]. It is also shown (in Lemma 3 and its corollary) that the rth derivative of a nontrivial nullspline must increase exponentially in at least one direction. The exponential decay of the fundamental spline L_i is used in Section 4 to prove Theorem 1. That section also contains a proof of the fact (Theorem 4) that the B.I.P. is solvable in terms of exponentially decaying fundamental splines, if it is correct at all. This fact is closely connected with S. Demko's results [10].

2. Nullsplines and fundamental splines. It is clear that the problem of finding, for an arbitrary given biinfinite sequence α, *some* spline $s \in \$_{k,\mathbf{t}}$ for which $s|_{\mathbf{t}} = \alpha$, always has solutions. In other words, it is clear that $R_{\mathbf{t}}$ maps $\$_{k,\mathbf{t}}$ *onto* $\mathbb{R}^{\mathbb{Z}}$. To see this, start with a polynomial p_0 of order k that satisfies $p_0(t_0) = \alpha_0$, $p_0(t_1) = \alpha_1$, and set $s = p_0$ on $[t_0, t_1]$. Now suppose that we have s already determined on some interval $[t_i, t_j]$ and let p_{j-1} be the polynomial that coincides with s on $[t_{j-1}, t_j]$. Then

$$p_j(t) := p_{j-1}(t) + (\alpha_{j+1} - p_{j-1}(t_{j+1}))\Big(\frac{t - t_j}{t_{j+1} - t_j}\Big)^{k-1}$$

is the *unique* polynomial of order k that takes on the value α_{j+1} at t_{j+1} and agrees with p_{j-1} $(k-1)$-fold at t_j. The definition

$$s = p_j \text{ on } [t_j, t_{j+1}]$$

therefore provides an extension of s to $[t_i, t_{j+1}]$, and, in fact, the only one possible. The extension to $[t_{i-1}, t_{j+1}]$ is found analogously. In this way, we find a solution inductively.

The argument shows that we can freely choose the interpolating spline on the interval $[t_0, t_1]$ from the k-2 dimensional linear manifold

$$\{p \in \mathbb{P}_k : p(t_0) = \alpha_0, \quad p(t_1) = \alpha_1\}$$

and that, with this choice, the interpolating spline is otherwise uniquely determined. In particular, the set of solutions for $\alpha = 0$, i.e., the kernel or nullspace of the restriction map $R_{\mathbf{t}}$, is a k-2 dimensional linear space, whose elements we call **nullsplines**. In other words, nullsplines are splines that vanish at all their knots.

The difficulty with the B.I.P. is therefore not the construction of some interpolating spline. Rather, the problem is interesting because we require an interpolating spline with certain additional characteristics or "side conditions", viz. that it be bounded. Nullsplines can be made to play a major role in the analysis of this problem.

For instance, the question of how many bounded solutions there are is equivalent to the question of how many bounded nullsplines there are. More interestingly, a well known approach to the construction of interpolants consists in trying to solve first the special problem of finding, for each i, a **fundamental spline**, i.e., a spline $L_i \in \$_{k,\mathbf{t}}$ for which

$$L_i(t_j) = \delta_{i-j}, \quad \text{all } j.$$

Such a spline consists (more or less) of two nullsplines joined together smoothly at t_i. Therefore, if one could prove that both nullsplines decay exponentially away from t_i, i.e.,

$$\|L_i\|_{(t_j, t_{j+1})} \le \text{const}_k \lambda^{|i-j|}, \quad \text{all } j,$$

at a rate $\lambda \in [0, 1)$ which is independent of i, then it would follow that the series

$$s_\alpha := \sum_{i=-\infty}^{\infty} \alpha_i L_i \tag{2.1}$$

converges uniformly on compact subsets of I and gives a solution s_α to the B.I.P. In fact, s_α then depends continuously on α, i.e.,

$$\|s_\alpha\|_\infty \le \text{const}_{k,\lambda} \|\alpha\|_\infty, \quad \text{all } \alpha \in m(\mathbb{Z})$$

for some $\text{const}_{k,\lambda}$ which does not depend on α.

The hope for such exponentially decaying fundamental functions is really not that farfetched. Such functions form the basis for Schoenberg's analysis in the case of equidistant knots, and they occur implicitly already in Subbotin's work. Further, a very nice result of S. Demko [10] to be elaborated upon in the next section (see also C. Chui's talk at this conference) shows that the bounded spline interpolant s_α to bounded data α is necessarily of the form (2.1) with exponentially decaying L_i in case s_α depends continuously on α.

In a rather similar way, nullsplines also occur in the discussion of interpolation error. If f is sufficiently smooth, and s_f is its spline interpolant, i.e., $s_f|_{\mathbf{t}} = f|_{\mathbf{t}}$, then one gets, formally at first, that

$$f(t) - s_f(t) = \int_{t_{-\infty}}^{t_\infty} K(t, s) f^{(k)}(s)\, \mathrm{d}s. \tag{2.2}$$

Here, the Peano kernel $K(t, \cdot)$ is a spline function of order k with knots $\mathbf{t}$ and an additional knot at the point t, and vanishes at all the knots $\mathbf{t}$. Hence, $K(t, \cdot)$ is again a function put together from two nullsplines. The exponential decay of these two nullsplines away from t is desirable here, since only with such a decay can (2.2) actually be verified for interesting functions f. But, I won't say anything more about this here.

Based on my experience with [6], I had at one time considerable hope that the exponential decay of nullsplines could be proved with the help of the following considerations. A nullspline $s \in \$_{k,\mathbf{t}}$ is determined on the interval $[t_i, t_{i+1}]$ as soon as one knows the vector

$$\widehat{s}_i := (s'(t_i), \ldots, s^{(k-2)}(t_i)/(k-2)!)$$

since one knows that $s(t_i) = s(t_{i+1}) = 0$. One can therefore compute $\widehat{s}_{i+1}$ from $\widehat{s}_i$ in a linear manner. Specifically,

$$\widehat{s}_{i+1} = -A(\Delta t_i)\widehat{s}_i,$$

with $A(h)$ the matrix of the form

$$A(h) := \operatorname{diag}(1, h^{-1}, \ldots, h^{-k+3}) A \operatorname{diag}(1, h, \ldots, h^{k-3})$$

and $A = A(1)$ the matrix

$$A := (\binom{k-1}{i} - \binom{j}{i})_{i,j=1}^{k-2}.$$

This means that $A(h)$ has many nice properties. For instance, $A^{-1}(h) = A(-h)$, and $A(h)$ is an oscillation matrix in the sense of Gantmacher and Krein.

In the special cubic case, $k = 4$, $A(h)$ has the simple form

$$A(h) = \begin{pmatrix} 2 & h \\ 3/h & 2 \end{pmatrix}$$

and allows therefore the conclusion that $\widehat{s}_i$ grows exponentially either for increasing or else for decreasing index i, at a rate of at least 2. This observation goes back to a paper by Birkhoff and the author [1].

The transformation $A(h)$ has been studied in much detail in the case of equidistant knots in a paper by Schoenberg and the author [8], and also, in more generality, by C. Micchelli [14]. But, such exponential decay or growth for nullsplines on an arbitrary knot sequence has so far not been proved. S. Friedland and C. Micchelli [13] have obtained from such considerations results concerning the maximal allowable **local** mesh ratio

$$m_{\mathbf{t}} := \sup_{|i-j|=1} \Delta t_i / \Delta t_j.$$

3. Exponential decay of the rth derivative of fundamental splines and nullsplines of order $k = 2r$. We base the arguments in this section on the best approximation property of spline interpolation. To recall, the rth divided difference of a sufficiently differentiable function f at the points $t_i, \ldots, t_{i+r}$ can be represented by

$$[t_i, \ldots, t_{i+r}]f = \int M_i(t) f^{(r)}(t)\, \mathrm{d}t / r!$$

with $M_i = M_{i,r,\mathbf{t}}$ a B-spline of order r,

$$M_i(t) := r[t_i, \ldots, t_{i+r}](\cdot - t)_+^{r-1},$$

normalized to have unit integral. Further, $\{s^{(r)} : s \in \$_{2r,\mathbf{t}}\} = \$_{r,\mathbf{t}}$ while, by a theorem of Curry and Schoenberg [9],

$$\$_{r,\mathbf{t}} = \{\sum_i \beta_i M_i : \beta \in \mathbb{Z}^{\mathbb{R}}\} \text{ on } I,$$

where we take the biinfinite sum **pointwise**, i.e.,

$$(\sum_i \beta_i M_i)(t) := \sum_i \beta_i M_i(t), \quad \text{all } t \in \mathbb{R}.$$

This, makes good sense since

$$M_i(t) \geq 0, \text{ with strict inequality iff } t_i < t < t_{i+r}.$$

Lemma 1. *Let $\mathcal{L}_i := \{L \in \$_{2r,\mathbf{t}} : L(t_j) = \delta_{i-j}, \quad \text{all } j\}$. Then $\mathcal{L}_i$ has exactly one element in common with $\mathbb{L}_2^{(r)}(I)$. We denote this element by*

$$L_i$$

and call it **the** *ith fundamental spline for the knot sequence* $\mathbf{t}$*. Further, with the abbreviations*

$$\overline{h} := \sup_j \Delta t_j, \quad \underline{h} := \inf_j \Delta t_j, \tag{3.1}$$

we have

$$\|L_i^{(r)}\|_2 \le \mathrm{const}_r \overline{h}^{1/2}/\underline{h}^r \tag{3.2}$$

for some constant const_r *depending only on* r*.*

Proof: We first prove that $\mathcal{L}_i$ contains *at most* one element in $\mathbb{L}_2^{(r)}(I) = \{f \in C^{r-1}(I) : f^{(r-1)}$ abs.cont., $f^{(r)} \in \mathbb{L}_2(I)\}$. Since $\mathcal{L}_i - \mathcal{L}_i = \ker R_{\mathbf{t}}$, it is sufficient to prove that the only nullspline in $\mathbb{L}_2^{(r)}$ is the trivial nullspline. For this, let $s \in \ker R_{\mathbf{t}} \cap \mathbb{L}_2^{(r)}(I)$. Then, by the introductory remarks for this section,

$$s^{(r)} = \sum_j \beta_j M_j \text{ for some } \beta \in \mathbb{R}^{\mathbb{Z}}, \quad s^{(r)} \in \mathbb{L}_2, \quad \text{and} \quad \int M_j s^{(r)} = 0 \text{ for all } j.$$

But, by a theorem in [3], there exists a positive constant D_r which depends only on r so that, for $1 \le p \le \infty$, and for all $\gamma \in \mathbb{R}^{\mathbb{Z}}$,

$$D_r^{-1}\|\gamma\|_p \le \| \sum_j \gamma_j((t_{j+r} - t_j)/r)^{1-1/p} M_j\|_p \le \|\gamma\|_p. \tag{3.3}$$

Here, $\|\gamma\|_p := (\sum_j |\gamma|^p)^{1/p}$, while, for f on I, $\|f\|_p := (\int_I |f|^p)^{1/p}$. This shows the sequence $(\widehat{M_j})$ given by

$$\widehat{M_j} := ((t_{j+r} - t_j)/r)^{1/2} M_j, \quad \text{all } j, \tag{3.4}$$

is a Schauder basis for $\$_{r,\mathbf{t}} \cap \mathbb{L}_2$. Therefore, $\sum_j \gamma_j \widehat{M_j}$ converges $\mathbb{L}_2$ to the spline function in $\mathbb{L}_2$ it represents. But this means that our particular spline $s^{(r)}$ is in the $\mathbb{L}_2$-span of (M_i), yet orthogonal to every one of the M_i, which means that $s^{(r)}$ vanishes identically. But then, since s vanishes more than r times, s itself must vanish identically.

Next, we prove that $\mathcal{L}_i$ contains *at least* one element on $\mathbb{L}_2^{(r)}(I)$. For this, we recall from [5] that there exists, for any given $\alpha \in \mathbb{R}^{\mathbb{Z}}$, a function g that is locally in $\mathbb{L}_2^{(r)}$ and satisfies $g|_{\mathbf{t}} = \alpha$, and whose rth derivative satisfies

$$\|g^{(r)}\|_2 \le D_r \big(\sum_j (t_{j+r} - t_j)([t_j, \ldots, t_{j+r}]\alpha)^2\big)^{1/2}, \tag{3.5}$$

with D_r the same constant mentioned in (3.3). Here, a number $[t_j, \ldots, t_{j+r}]\alpha$ stands for the rth divided difference at the points $t_j, \ldots, t_{j+r}$ of any function f for which $f|_{\mathbf{t}} = \alpha$. In this way, we obtain for the specific sequence $\alpha = (\delta_{i-j})_{j=-\infty}^{\infty}$ a function $g \in \mathbb{L}_2^{(r)}$ for which

$$g(t_i) = \delta_{i-j}, \quad \text{all } j,$$

while $\|g^{(r)}\|_2$ is bounded by the right side of (3.5). Note that, for the specific sequence $\alpha = (\delta_{i-j})$, this bound becomes

$$\begin{aligned}\|g^{(r)}\|_2 &\le D_r \big(\sum_{j=i-r}^{i} (t_{j+r} - t_j)[1/\prod_{\substack{n=j\\ n\ne i}}^{j+r} (t_i - t_n)]^2\big)^{1/2} \\ &\le \mathrm{const}_r (\overline{h})^{1/2}/\underline{h}^r.\end{aligned}$$

Now let $\widehat{g}$ be any element in $\mathbb{L}_2^{(r)}$ so that $\widehat{g}^{(r)}$ is the $\mathbb{L}_2$-approximation to $g^{(r)}$ from $\$_{r,\mathbf{t}} \cap \mathbb{L}_2$. This makes sense since (3.3) insures that $\$_{r,\mathbf{t}} \cap \mathbb{L}_2$ is a closed subspace of $\mathbb{L}_2(I)$. Then

$$[t_j, \ldots, t_{j+r}]\widehat{g} = \int M_j \widehat{g}^{(r)}/r! = \int M_j g^{(r)}/r! = [t_j, \ldots, t_{j+r}]g,$$

all j, while $\|\widehat{g}^{(r)}\|_2 \le \|g^{(r)}\|_2$. But this means that, for an appropriate polynomial p of order r,

$$(\widehat{g} + p)(t_j) = g(t_j) = \delta_{i-j}, \quad \text{all } j,$$

while still $\|(\widehat{g} + p)^{(r)}\|_2 \le \|g^{(r)}\|_2 \le \mathrm{const}_r \overline{h}^{1/2}/\underline{h}^r$. This shows that $L := \widehat{g} + p$ is a function of the desired kind. □

We continue to use the inequality (3.3) and the abbreviation $\widehat{M}_j = ((t_{j+r} - t_j)/r)^{1/2} M_j$, and come now to what I consider to be the main point of this paper.

Lemma 2. *If β is the sequence of coefficients for $L_i^{(r)}$ with respect to the basis $(\widehat{M}_j)$ for $\$_{r,\mathbf{t}}$, i.e., if $L_i^{(r)} = \sum_j \beta_j \widehat{M}_j$, and*

$$\beta_j^{(n)} := \begin{cases} 0 & , \quad |j-i| < n, \\ \beta_j & , \quad |j-i| \geq n, \end{cases} \qquad n = 0, 1, 2, \ldots,$$

then there exist const_r *and $\lambda_r \in [0,1)$ depending only on r so that*

$$\|\beta^{(n)}\|_2 \leq \mathrm{const}_r \|\beta\|_2 \lambda_r^n, \quad n = 0, 1, 2, \ldots \,. \tag{3.6}$$

The inequalities (3.3) allow us to conclude from Lemma 2 the exponential decay of $L_i^{(r)}$ in the following form.

Corollary. *For some* const_r, *and some $\lambda_r \in [0,1)$ depending only on r, and for all i and n,*

$$\|L_i^{(r)}\|_{2,(t_{-\infty}, t_{i-n})} + \|L_i^{(r)}\|_{2,(t_{i+n}, t_{\infty})} \leq \mathrm{const}_r \|L_i^{(r)}\|_2 \lambda_r^n.$$

Proof of Lemma 2. Let

$$A := \left(\int \widehat{M}_i \widehat{M}_j\right)$$

be the Gram matrix for our appropriately normalized B-spline basis of $\$_{r,\mathbf{t}}$. A proof of the lemma can be obtained directly from the fact that the elements of the inverse matrix for A decay exponentially away from the diagonal at a rate that can be bounded in terms of r and independently of $\mathbf{t}$. This is proved in [7] with the aid of a nice inequality due to Douglas, Dupont and Wahlbin [12]. But, between the time I proved Lemma 2 this way and the delivery of this talk, S. Demko wrote a paper [10] in which he demonstrated that such arguments use actually very little specific information about splines. Using the inequality of Douglas, Dupont and Wahlbin, he proved the following nice

Theorem (S. Demko)**.** *Let $A := (a_{ij})$ be an invertible band matrix (of finite order). Explicitly, assume that, for some m, $a_{ij} = 0$ whenever $|i-j| > m$, and that, for some positive $\underline{K}$ and $\overline{K}$, and some $p \in [1, \infty]$,*

$$\underline{K}\|x\|_p \leq \|Ax\|_p \leq \overline{K}\|x\|_p, \quad \text{all } x.$$

Then the entries of the inverse $A^{-1} =: (b_{ij})$ satisfy

$$|b_{ij}| \leq \mathrm{const}\lambda^{|i-j|}, \quad \text{all } i, j,$$

for some const *and some $\lambda \in [0,1)$ which depend only on m, p, $\underline{K}$ and $\overline{K}$. In particular, these constants do not depend on the order of the matrix A.*

The interested reader will have no difficulty in proving this theorem after a study of the following proof of Lemma 2, a proof which makes essential use of Demko's ideas, even though the inequality of Douglas, Dupont and Wahlbin fails to make an explicit appearance. In the bargain, the reader will thereby obtain explicit estimates for const and λ (which Demko did not bother to compute).

We note that the specific matrix $A = (\int \widehat{M}_i \widehat{M}_j)$ is a band matrix, of band width $m = r - 1$ in the sense that $\int \widehat{M}_i \widehat{M}_j = 0$ for $|i-j| > r-1$. Also, we conclude from (3.3) that the sequence-to-sequence transformation

$$\alpha \mapsto A\alpha$$

induces a linear map on $\ell_2(\mathbb{Z})$ to $\ell_2(\mathbb{Z})$ which we also call A and which is bounded and boundedly invertible. Specifically, one obtains from (3.3) that

$$\kappa := \|A\|_2 \|A^{-1}\|_2 \leq D_r^2. \tag{3.7}$$

Here, $\|B\|_2 := \sup\{\|Ba\|_2/\|\alpha\|_2 : \alpha \in \ell_2(\mathbb{Z})\}$, as usual.

We now claim that,

$$\text{(3.8)} \qquad \text{for all } n \geq 2r, \quad \|\beta^{(n)}\|_2^2 \leq (\kappa^2/(1+\kappa^2))\|\beta^{(n-2m)}\|_2^2$$

which, with the **t**-independent estimate (3.7) for κ, establishes the lemma (with $\lambda_r \leq (\kappa/(1+\kappa^2)^{1/2})^{1/2m}$).

For the proof of (3.8), we consider without loss of generality only the specific function L_0. We note that

$$\begin{aligned}(A\beta)_i = \int \widehat{M}_i L_0^{(r)} &= r!((t_{i+r} - t_i)/r)^{1/2}[t_i, \ldots, t_{i+r}]L_0 \\ &= 0 \ \text{ unless } t_i \leq t_0 \leq t_{i+r}.\end{aligned}$$

Therefore,

$$\text{(3.9)} \qquad \operatorname{supp} A\beta \subseteq [-r, 0],$$

where, for any biinfinite sequence α, we use the abbreviation

$$\operatorname{supp} \alpha := \{i \in \mathbb{Z} : \alpha_i \neq 0\}.$$

We claim that, for $n \geq m$,

$$\text{(3.10)} \qquad \operatorname{supp} A\beta^{(n)} \subseteq (-n-m, n+m) \backslash (-n+m, n-m).$$

Indeed, $\operatorname{supp}(\beta^{(n)} - \beta \subseteq (-n, n)$, hence $\operatorname{supp} A(\beta^{(n)} - \beta) \subseteq (-n-m, n+m)$ which also contains $\operatorname{supp} A\beta = [-r, 0]$, therefore

$$\operatorname{supp} A\beta^{(n)} \subseteq (-n-m, n+m).$$

On the other hand, $\operatorname{supp} \beta^{(n)} \subseteq \mathbb{Z} \backslash (-n, n)$, therefore also

$$\operatorname{supp} A\beta^{(n)} \subseteq \mathbb{Z} \backslash (-n+m, n-m).$$

It follows from (3.10) that, for $n \geq 2r$,

$$\text{(3.11)} \qquad \operatorname{supp} A\beta^{(n)} \cap \operatorname{supp} A\beta^{(n-2m)} = \emptyset,$$

therefore

$$\|A\beta^{(n)}\|_2^2 \leq \|A\beta^{(n)}\|_2^2 + \|A\beta^{(n-2m)}\|_2^2 = \|A(\beta^{(n)} - \beta^{(n-2m)})\|_2^2.$$

But then

$$\begin{aligned}\|A^{-1}\|_2^{-1}\|\beta^{(n)}\|_2 \leq \|A\beta^{(n)}\|_2 &\leq \|A\big(\beta^{(n)} - \beta^{(n-2m)}\big)\|_2 \\ &\leq \|A\|_2\|\beta^{(n)} - \beta^{(n-2m)}\|_2,\end{aligned}$$

i.e.,

$$\begin{aligned}\|\beta^{(n)}\|_2^2 &\leq \kappa^2\|\beta^{(n)} - \beta^{(n-2m)}\|_2^2 \\ &= \kappa^2\big(\|\beta^{(n-2m)}\|_2^2 - \|\beta^{(n)}\|_2^2\big)\end{aligned}$$

which proves our earlier claim (3.8). □

It is clear that the argument provides the exponential decay of the form (3.6) and with $\lambda \le (\kappa/(1+\kappa^2)^{1/2})^{1/2m}$ for any sequence β in $\ell_2(\mathbb{Z})$ for which $A\beta$ has finite support. In particular, one obtains such exponential decay for the sequence $\gamma^{(i)}$ for which $A\gamma^{(i)} = (\delta_{i-j})$, i.e., for the ith row of the matrix inverse of A. Further, it is clear that (3.11) implies $\|A\beta^{(n)}\|_p^p \le \|A\beta^{(n)}\|_p^p + \|A\beta^{(n-2m)}\|_p^p$ for any $1 \le p < \infty$, hence, the argument carries at once from $\ell_2(\mathbb{Z})$ over to any $\ell_p(\mathbb{Z})$ with $1 \le p < \infty$. Demko obtains such exponential decay also for $p = \infty$ by considering the transposed matrix A^T for which then automatically

$$\|A^T\|_1 \|(A^T)^{-1}\|_1 = \|A\|_\infty \|A^{-1}\|_\infty$$

due to the finite order of the matrix he considers. This switch requires a word or two in the infinite case, as follows. As one easily checks, if a (bi)infinite matrix (a_{ij}) gives rise to a bounded linear map A on ℓ_∞, then its transpose gives a bounded linear map B on ℓ_1, and the adjoint of B is then necessarily A itself. This implies that, if a matrix (a_{ij}) gives rise to a bounded linear map on ℓ_∞ that is boundedly invertible, then its inverse can also be represented by a matrix, viz. the transpose of the matrix that represents the inverse of the linear map on ℓ_1 given by the transpose of (a_{ij}). Of course, exponential decay away from the diagonal is unchanged when going over to the transpose.

These comments establish the following

Theorem 2. *Let M be a finite, infinite or biinfinite "interval" in $\mathbb{Z}$, let $1 \le p \le \infty$, and let $q := \min\{p, p/(p-1)\}$. Let $(a_{ij})_{i,j \in M}$ be a matrix with band width $m := \sup\{|i-j| : a_{ij} \ne 0\}$, and assume that (a_{ij}) induces a bounded linear map A on $\ell_p(M)$. If A is boundedly invertible, then A^{-1} is also given by a matrix, (b_{ij}) say, and*

$$|b_{ij}| \le \mathrm{const}\,\lambda^{|i-j|}, \quad \text{all } i, j,$$

with

$$\lambda := (\kappa/(1+\kappa^q)^{1/q})^{1/2m}, \quad \mathrm{const} \le \|A^{-1}\|_p/\lambda^{2m}, \quad \kappa := \|A\|_p \|A^{-1}\|_p.$$

We add one more remark. With the appropriate interpretation of "bandedness", the above argument carries through even for matrices that are not banded in the straightforward sense. As a typical example, consider the Gram matrix for a local support basis of some space of functions of *several* variables. Then there is no ordering of that basis for which the corresponding Gram matrix is appropriately banded. But, if we follow the geometry of the underlying problem and think of the Gram "matrix" as acting on functions on some *multidimensional* index set M having an appropriate metric $|\cdot|$ (instead of on $\mathbb{Z}$), then the statement and the proof of Theorem 2 go through otherwise unchanged. We do not pursue this point here further, but alert the reader to Desloux's fine paper [11] in which such considerations can be uncovered once one knows what to look for.

We finish this section with the observation that the rth derivative of a nontrivial nullspline must increase exponentially in at least one direction. The argument is rather similar to the proof of Lemma 2. We continue to denote by A the specific matrix $(\int \widehat{M}_i \widehat{M}_j)$ and recall

$$\kappa := \|A\|_2 \|A^{-1}\|_2 \le D_r^2. \tag{3.7}$$

Lemma 3. *If $\sum_i \beta_i \widehat{M}_i$ is the rth derivative of a nullspline in $\$_{2r,\mathbf{t}}$ and $i \le j$ are arbitrary indices, then*

$$(1+\kappa^2) \sum_{\nu=i}^{j} |\beta_\nu|^2 \le \kappa^2 \sum_{\nu=i-2m}^{j+2m} |\beta_\nu|^2.$$

Proof: Define β', β'' by

$$\beta'_\nu := \begin{cases} \beta_\nu & , \quad i \le \nu \le j \\ 0 & , \quad \text{otherwise} \end{cases}, \quad \beta''_\nu := \begin{cases} \beta_\nu & , \quad i-2m \le \nu \le j+2m \\ 0 & , \quad \text{otherwise} \end{cases},$$

so that the inequality to be proved reads

$$(1+\kappa^2)\|\beta'\|_2^2 \le \kappa^2 \|\beta''\|_2^2. \tag{3.12}$$

We have

$$\operatorname{supp} A\beta' \subseteq [i-m, j+m]$$

while

$$\operatorname{supp}(\beta - \beta'') \subseteq \mathbb{Z} \backslash [i-2m, j+2m],$$

therefore, with $A\beta = 0$,

$$\operatorname{supp} A\beta'' = \operatorname{supp} A(\beta - \beta'') \subseteq \mathbb{Z} \backslash [i-m, j+m] \subseteq \mathbb{Z} \backslash \operatorname{supp} A\beta'.$$

Consequently,

$$\|A^{-1}\|_2^{-1} \|\beta'\|_2 \le \|A\beta'\|_2 \le \|A(\beta' - \beta'')\|_2 \le \|A\|_2 \|\beta' - \beta''\|_2,$$

or, with $\kappa = \|A\|_2 \|A^{-1}\|_2$,

$$\|\beta'\|_2^2 \le \kappa^2 \|\beta' - \beta''\|_2^2 = \kappa^2 (\|\beta''\|_2^2 - \|\beta'\|_2^2)$$

which implies (3.12). □

Corollary. *Let $\sum_i \beta_i \widehat{M}_i$ be the rth derivative of a nullspline s in $\$_{2r,\mathbf{t}}$ and set*

$$a_j := \sum_{2mj < i \le 2m(j+1)} |\beta_i|^2, \quad \text{all } j \in \mathbb{Z},$$

with $m := r - 1$, as before. Then

$$\sum_{i<\nu<j} a_\nu \le \kappa^2 (a_i + a_j), \quad \text{for all } i < j. \tag{3.13}$$

Therefore, for all μ, and either for all $i > \mu$ or for all $i < \mu$,

$$a_i \ge \text{const}_\mu \wedge^{|i-\mu|}$$

with

$$\text{const}_\mu := \frac{1}{2} a_\mu / (\kappa^2 \wedge)$$

and

$$\wedge := (1 + \kappa^2)/\kappa^2 > 1.$$

Proof: Assertion (3.13) follows at once from the lemma. The second assertion of the corollary is less obvious. For its proof, assume without loss that $\mu = 0$. From (3.13),

$$\wedge^{i-1} a_0 \le \sum_{-i<\nu<i} a_\nu, \quad i = 1, 2, 3, \ldots\ .$$

Therefore,

$$\wedge^i c \le \sum_{-i<\nu<i} a_\nu, \quad i = 1, 2, 3, \ldots \tag{3.14}$$

with

$$c := a_0 \ / \wedge .$$

Let now $\text{const}_0 = \frac{1}{2} c/\kappa^2$, as defined above, and assume that the inequality

$$a_i \ge \text{const}_0 \wedge^{|i|}$$

is violated for some $i > 0$ while also

$$a_{-j} < \mathrm{const}_0 \wedge^j \tag{3.15}$$

for some positive j which we assume without loss of generality to be no less than i. Then, we can also assume that j is the smallest index $\geq i$ for which (3.15) holds. We obtain from (3.13) that

$$\sum_{-j<\nu<i} a_\nu \leq \kappa^2(a_{-j} + a_i) < \kappa^2 \mathrm{const}_0(\wedge^j + \wedge^i) = \frac{1}{2}c(\wedge^j + \wedge^i). \tag{3.16}$$

On the other hand, by (3.13) and by the choice of j,

$$\begin{aligned}\sum_{-j<\nu<i} a_\nu &= \sum_{-j<\nu\leq -i} a_\nu + \sum_{-i<\nu<i} a_\nu \\ &\geq \mathrm{const}_0\big(\wedge^j - 1 - (\wedge^i - 1)\big)/(\wedge - 1) + c\wedge^i \\ &= \frac{1}{2}c(\wedge^j - \wedge^i) + c\wedge^i = \frac{1}{2}c(\wedge^j + \wedge^i)\end{aligned}$$

which contradicts (3.16), and so finishes the proof. In the second last equality, we used the fact that $\wedge - 1 = (\kappa^2 + 1)/\kappa^2 - 1 = 1/\kappa^2$. □

Remark. It is easy to see that, in the corollary, $a_{\mu-1} + a_\mu \neq 0$ for any μ in case the nullspline s is not trivial. For if, e.g., $a_{-1} = a_0 = 0$, then $s^{(r)}$ would vanish on $[t_{-2m+1+(r-1)}, t_{2m-(r-1)}] = [t_{2-r}, t_{r-1}]$, hence s would be a polynomial of degree $< r$ on that interval and vanish $2(r-1)$ times there, therefore would have to vanish identically there. But then, we would have $s = 0$ by the considerations in Section 2. We can therefore conclude from the corollary that, *for a nontrivial nullspline s,*

$$a_i \geq \wedge^{|i|}$$

either for all $i > 1$ or else for all $i < -1$, with a_i and $\wedge$ as in the corollary and $\mathrm{const} := \frac{1}{2}\max\{a_{-1}, a_0\}/(\kappa\wedge)^2 > 0$.

The argument for this corollary would have been simpler had I been able to prove that every β with $A\beta = 0$ can be written as a sum $\beta = \beta' + \beta''$ with $\sum_{i\geq 0} |\beta_i'|^2 < \infty$ and $\sum_{i\leq 0} |\beta_i''|^2 < \infty$, and $A\beta' = A\beta'' = 0$.

A minor variation of the arguments for Lemma 3 and its corollary allow the following conclusion of independent interest in the study of linear difference equations.

Theorem 3. *Let $A = (a_{ij})$ be a biinfinite matrix which represents a linear map, also denoted by A, on $\ell_p(\mathbb{Z})$ for some $p \in [1, \infty)$ which is bounded and bounded below, i.e., there exist positive $\underline{K}$ and $\overline{K}$ so that*

$$\underline{K}\|\alpha\|_p \leq \|A\alpha\|_p \leq \overline{K}\|\alpha\|_p \quad \text{for all } \alpha \in \ell_p(\mathbb{Z}).$$

If A is a band matrix, i.e., if

$$m := \sup\big\{|1 - j| : a_{ij} \neq 0\big\} < \infty,$$

then any nontrivial sequence β for which $A\beta = 0$ must increase exponentially either for increasing or for decreasing i. Explicitly, there exist an index μ and a positive $\mathrm{const}_{\beta,\mu}$, so that, either for all $i > \mu$ or else for all $i < \mu$,

$$\sum_{2mi<j\leq 2m(i+1)} \beta_j^p \geq \mathrm{const}_{\beta,\mu} \wedge^{|i-\mu|}$$

with

$$\wedge := (1 + \kappa^p)/\kappa^p \quad \text{and } \kappa := \overline{K}/\underline{K}.$$

Thanks are due to Allan Pinkus for questioning the necessity of an additional assumption in an earlier version of this theorem.

4. Exponential decay of the fundamental spline. Assume that the knot sequence is such that the B.I.P. is correct, i.e., has exactly one solution $s_\alpha \in m\$_{k,\mathbf{t}}$ for every $\alpha \in m(\mathbb{Z})$. This means that the restriction map $R_{\mathbf{t}}$, when restricted to $m\$_{k,\mathbf{t}}$, is one-one, onto, and clearly bounded with respect to the sup-norm. One verifies directly (else see (4.2) below) that $m\$_{k,\mathbf{t}}$ is a closed subspace of $m(I)$, hence complete. The Open Mapping Theorem therefore provides the conclusion that $R_{\mathbf{t}}$ is **boundedly** invertible. This means the existence of some const so that

$$\|s_\alpha\|_\infty \le \text{const}\|\alpha\|_\infty, \quad \text{all } \alpha \in m(\mathbb{Z}). \tag{4.1}$$

Let $N_i = N_{i,k,\mathbf{t}}$ be the ith B-spline of order k for the knot sequence $\mathbf{t}$, normalized so that

$$N_i(t) := \big([t_{i+1}, \ldots, t_{i+k}] - [t_i, \ldots, t_{i+k-1}]\big)(\cdot - t)_+^{k-1}$$

and so, comparing with the B-splines introduced at the beginning of Section 3,

$$N_{i,k,\mathbf{t}} = \big((t_{i+k} - t_i)/k\big) M_{i,k,\mathbf{t}}.$$

From (3.3), or already from [2],

$$D_k^{-1}\|\beta\|_\infty \le \|\sum_i \beta_i N_i\|_\infty \le \|\beta\|_\infty, \quad \text{all } \beta \in m(\mathbb{Z}), \tag{4.2}$$

for some positive constant D_k depending only on k and not on $\mathbf{t}$.

Since $(N_i)_{-\infty}^{\infty}$ is a basis for $\$_{k,\mathbf{t}}$ (in the sense described in the preceding section), it follows that $s \in \$_{k,\mathbf{t}}$ satisfies $s|_{\mathbf{t}} = \alpha$ if and only if its B-spline coefficient sequence β satisfies

$$\sum_j N_j(t_i)\beta_j = \alpha_i, \quad \text{all } i, \tag{4.3}$$

while $s \in \$_{k,\mathbf{t}}$ is bounded if and only if its corresponding B-spline sequence β is bounded, by (4.2). We conclude that the B.I.P. has exactly one solution for every $\alpha \in m(\mathbb{Z})$ iff the matrix

$$A := (N_j(t_i))$$

maps ℓ_∞ faithfully onto ℓ_∞. We collect these facts in the following

Theorem 4. *The bounded interpolation problem is correct if and only if the matrix*

$$A = (N_j(t_i))$$

provides a faithful linear map from $\ell_\infty(\mathbb{Z})$ onto $\ell_\infty(\mathbb{Z})$. If one or the other of these conditions holds, then A, being trivially bounded, is boundedly invertible. Since A is also a band matrix, of band width $m := r - 1$, it then follows from Theorem 2 that the inverse of A is also given by a matrix, (b_{ij}) say, and that

$$|b_{ij}| \le \text{const}\lambda^{|i-j|}, \quad \text{all } i, j,$$

with

$$\lambda := (\kappa/(1+\kappa))^{1/2m}, \quad \text{const} \le \kappa/\lambda^{2m}, \quad \kappa := \|A^{-1}\|_\infty$$

since $\|A\|_\infty = 1$. In particular, for all i, the function

$$L_i := \sum_j b_{ij} N_j$$

is then a fundamental spline that decays exponentially at the rate λ, and the solution s_α of the B.I.P. for given $\alpha \in m(\mathbb{Z})$ is given by

$$s_\alpha = \sum_j \alpha_j L_j,$$

a series that converges uniformly on compact subsets of I.

We do not know conditions that are both necessary and sufficient for the correctness of the B.I.P. . Since correctness implies boundedness of the map $\alpha \mapsto s_\alpha$, we obtain* from [4; Lemma of Section 2] the necessary condition that the local mesh ratio

$$m_{\mathbf{t}} = \sup_{|i-j|=1} \Delta t_i / \Delta t_j$$

be finite. If the local mesh ratio is indeed finite, then a simple sufficient condition for uniqueness of the interpolating bounded spline is the condition that

$$I = (-\infty, \infty). \tag{4.4}$$

This is connected with the fact that, with $k = 2r$, the rth derivative of any nontrivial nullspline grows exponentially in at least one direction, as described in Lemma 3 and its corollary. Precisely, we have the following

Lemma 4. *If $m_{\mathbf{t}} := \sup_{|i-j|=1} \Delta t_i/\Delta t_j < \infty$, and there exists a bounded nontrivial nullspline s in $\$_{k,\mathbf{t}}$, then either $t_{-\infty} > -\infty$ or $t_\infty < \infty$.*

Proof: Let $s = \sum_i \gamma_i N_{i,k}$ be the nontrivial bounded nullspline in $\$_{k,\mathbf{t}}$. Its jth derivative is then $s^{(j)} = \sum_i \gamma_i^{(j)} N_{i,k-j}$, with

$$\gamma_i^{(j)} := \begin{cases} \gamma_i & , \quad j = 0 \\ (k-j)(\gamma_i^{(j-1)} - \gamma_{i-1}^{(j-1)})/(t_{i+k-j} - t_i) & , \quad j > 0. \end{cases}$$

This implies the estimate

$$|\gamma_i^{(j)}| \le \frac{k!}{(k-j-1)!} 2^j \max\left\{|\gamma_{i-j}|, \dots, |\gamma_i|\right\}/(t_{i+k-j} - t_i)^j \tag{4.5}$$

(see, e.g., [4], for similar considerations). Write now the rth derivative of s in terms of the somewhat differently normalized B-splines $\widehat{M}_i := (r/(t_{i+r} - t_i))^{1/2} N_{i,r,\mathbf{t}}$ introduced in Section 2,

$$s^{(r)} = \sum_i \beta_i \widehat{M}_i.$$

Then $\beta_i = \gamma_i^{(r)}((t_{i+r} - t_i)/r)^{1/2}$, so that, from (4.5),

$$|\beta_i| \le \text{const}_r \|\gamma\|_\infty / (t_{i+r} - t_i)^{r-1/2}. \tag{4.6}$$

By the corollary to Lemma 3 (in Section 3), we may assume, without loss of generality, the existence of a positive const so that, with $m = r - 1$,

$$\sum_{2mj < i \le 2m(j+1)} |\beta_i|^2 \ge \text{const}\wedge^j, \quad j = 2, 3 \dots$$

where $\wedge := (1 + \kappa^2)/\kappa^2 > 1$ and $\kappa \le D_r^2$, the latter a certain constant independent of $\mathbf{t}$. In conjunction with (4.6), this implies that

$$\begin{aligned} \text{const}\wedge^j &\le \text{const}_{r,\gamma} \max\left\{(t_{i+r} - t_i)^{1-2r} : 2mj < i \le 2m(j+1)\right\} \\ &\le \text{const}_{r,\gamma} (m_{\mathbf{t}})^{r^2} \min\left\{(t_{i+r} - t_i)^{1-2r} : 2mj < i \le 2m(j+1)\right\}, \end{aligned}$$

where we have used the fact that

$$m_{\mathbf{t}}^{-|i-j|} \le (t_{i+r} - t_i)/(t_{j+r} - t_j) \le m_{\mathbf{t}}^{|i-j|}.$$

It follows that

$$t_{i+r} - t_i \le \text{const}\wedge^{-i/(2r)}, \quad i = 2r, \quad 2r+1, \dots$$

and therefore

$$t_\infty = t_{2r} + \sum_{i=0}^{\infty} (t_{(i+1)r} - t_{ir}) < \infty.$$

□

* 1995: is this really true?

We note in passing that the argument also establishes uniqueness in case either $t_{-\infty}$ or t_∞ is finite as long as the local mesh ratio is $< \rho$ for some ρ that is greater than 1 and depends on $\wedge$.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Since the global mesh ratio $M_{\mathbf{t}} = \sup_{i,j} \Delta t_i / \Delta t_j$ is finite, then, in particular, $I = (-\infty, \infty)$ and Lemma 4 implies that $R_{\mathbf{t}}$ maps $m\$_{k,\mathbf{t}}$ one-one to $m(\mathbb{Z})$.

Next, we prove that, for each i, the fundamental spline function L_i introduced in Lemma 1 decays exponentially away from t_i, i.e., for all j and all $x \in [t_j, t_{j+1}]$,

$$|L_i(x)| \le \text{const} \lambda^{|i-j|} \tag{4.7}$$

for some const depending only on k and $M_{\mathbf{t}}$, and some $\lambda \in [0, 1)$ which depends only on k. It suffices to consider $j \ge i$. We have $L_i(t_n) = 0$ for $n \ne i$, therefore

$$\begin{aligned} L_i(x) &= (x - t_{j+1}) \cdots (x - t_{j+r}[x, t_{j+1}, \ldots, t_{j+r}] L_i \\ &= \prod_{n=1}^{r} (x - t_{j+n}) \int r[x, t_{j+1}, \ldots, t_{j+r}](\cdot - t)_+^{r-1} L_i^{(r)}(t) \,\mathrm{d}t / r!. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} &\int r[x, t_{j+1}, \ldots, t_{j+r}](\cdot - t)_+^{r-1} L_i^{(r)}(t) \,\mathrm{d}t \\ &\qquad\qquad \le (r/(t_{j+r} - x))^{1/2} \|L_i^{(r)}\|_{2,[x,t_{j+r}]}, \end{aligned}$$

making use of (3.3), so that, from the corollary to Lemma 2,

$$|L_i(x)| \le \text{const}_r (\overline{h}^r / \underline{h}^{1/2}) \text{const}_r \|L_i^{(r)}\|_2 \lambda^{j-i}$$

with $\lambda \in [0, 1)$ depending only on k, and

$$\overline{h} := \sup_n \Delta t_n, \quad \underline{h} := \inf_n \Delta t_n.$$

But now, from Lemma 1,

$$\|L_i^{(r)}\|_2 \le \text{const}_r \overline{h}^{1/2} / \underline{h}^r,$$

and (4.7) follows.

The exponential decay of all fundamental splines L_i at a rate that does not depend on i now allows us to construct an interpolant s_α in $\$_{k,\mathbf{t}}$ for arbitrary $\alpha \in m(\mathbb{Z})$, in the form

$$s_\alpha = \sum_i \alpha_i L_i,$$

that satisfies

$$\|s_\alpha\|_\infty \le \text{const} \|\alpha\|_\infty$$

and therefore is in $m\$_{k,\mathbf{t}}$. □

It is clear that the argument for Theorem 1 shows the existence of a number $\rho > 1$ (which depends on k and on the λ of Lemma 2) so that the conclusions of Theorem 1 hold even if we only know that the *local* mesh ratio is less than ρ. A quick analysis of the constants involved shows this provable ρ to converge to 1 very fast as k increases.

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