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On Polynomial Ideals of Finite Codimension with Applications to Box Spline Theory

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ABSTRACT

We investigate the relation between an ideal I of finite codimension in the space Π of multivariate polynomials and ideals which are generated by lower order perturbations of some generators for I. Of particular intereest are the codimension of these ideals and the local approximation order of their kernels.

The discussion, stimulated by recent results in approximation theory, allows us to provide a simple analysis of the polynomial and exponential spaces associated with box splines. We describe their structure, dimension, local approximation order and an algorithm for their construction. The resulting theory is extended to subspaces of the above exponential/polynomial spaces.

AMS (MOS) Subject Classifications: primary 13A15, 41A15, 41A63; secondary 13F25, 13H15, 35G05.

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1. Introduction

In this note, we examine some specific properties of multivariate polynomial ideals of finite codimension (i.e., of transcendental dimension 0). We are interested in this topic because such ideals play a fundamental role in the theory of box splines and exponential box splines. In addition, the results of this investigation have applications to multivariate polynomial interpolation which are presented in [4], and, although the discussion of polynomial ideals here was stimulated by problems arise in approximation theory, some of the results may be found to be of independent interest.

In our discussion, the interplay between a polynomial ideal I and its homogeneous counterpart $I_\uparrow$ will be important. For its definition, we denote by $p_\uparrow$ the **leading term** (or **leading form**) of the polynomial p, i.e., for $p \neq 0$, the (unique) *homogeneous* polynomial for which

$$\deg(p - p_\uparrow) < \deg p.$$

For completeness, we take the zero polynomial to be its own leading term. With this, we define

$$I_\uparrow := \operatorname{span}\{p_\uparrow : p \in I\}$$

and verify that $I_\uparrow$ is again an ideal.

An ideal I is of **finite codimension** if its codimension in the space Π of all multivariate polynomials i.e., the dimension (equivalently the length) of the quotient space Π/I, is finite. The questions we are interested in concern the relationship between an ideal I and the associated $I_\uparrow$, or between a homogeneous ideal I and a lower order perturbation of it, i.e., an ideal generated by a set of polynomials F whose leading terms $F_\uparrow$ generate I. Special emphasis is given to the *kernels* associated with these ideals. By definition, the **kernel** of an ideal I is the set

$$\{f \in \mathcal{D}'(\mathbb{R}^s) : \; p(D)f = 0, \; \forall p \in I\}, \tag{1.1}$$

where $\mathcal{D}'(\mathrm{I\!R}^s)$ is the space of s-dimensional complex-valued distributions and $p(D)$ is the linear differential operator with constant coefficients induced by p. Kernels associated with ideals of finite codimension are finite-dimensional exponential spaces, i.e., are spanned by products of exponentials with polynomials.

The paper is laid out as follows. In section 2 we introduce the map $H \mapsto H_\downarrow$ which assigns to every finite-dimensional space of analytic functions its "limit at the origin" $H_\downarrow$, a scale-invariant polynomial space of the same dimension as H. Several aspects of this map and its range, that are needed later in the derivation of the properties of ideals' kernels, are reviewed. The (essentially known) background material on ideals of finite codimension is collected in section 3, where the varieties, multiplicity spaces, primary decompositions and the kernels of such ideals are examined. Results concerning the connection between an ideal I and $I_\uparrow$ or a lower order perturbation of I are presented in section 4. These results are used in section 5 in the examination of some aspects of box spline theory, while the same approach is exploited in the last section in the derivation of extensions and generalizations of various box spline results.

2. The "limit at the origin" of analytic function spaces

In this section, we consider finite-dimensional subspaces of the space

$$A_0$$

of all (germs of) functions analytic at the origin. (The power series space would do here as well). We single out the **least term** (or, the **initial form**, cf. [12;Chap. VII]) $f_\downarrow$ (read 'f least') of $f \in A_0$ and mean by this

$$f_\downarrow := T_j f,$$

with j the smallest integer for which $T_j f \neq 0$, with $T_j f$ the Taylor polynomial of degree $< j$ for f at the origin, i.e.,

$$T_j f := \sum_{|\alpha|<j} [\![\,]\!]^\alpha D^\alpha f(0), \tag{2.1}$$

and with $[\![]\!]^\alpha : x \mapsto x^\alpha/\alpha!$ the normalized power function. Consequently, with $j := \deg f_\downarrow$,

$$f_\downarrow = \lim_{t\to 0} f(t\cdot)/t^j, \tag{2.2}$$

(say, in the pointwise sense), as follows readily from L'Hôpital's rule.

We associate with a finite-dimensional subspace H of A_0 the polynomial space $H_\downarrow$ defined by

$$H_\downarrow = \mathrm{span}\{f_\downarrow : \ f \in H\}. \tag{2.3}$$

Note that $H_\downarrow$ is **scale-invariant** since it is spanned by homogeneous polynomials. The space $H_\downarrow$ has been considered in [4], and in what follows we recall from there various properties of this space of use in our subsequent discussion. For details we refer to [4], where an algorithm for the construction of $H_\downarrow$ from H is presented, the continuity of the map $H \mapsto H_\downarrow$ is examined, and some optimality properties of $H_\downarrow$ are established. The space H to which we intend to apply the results here is the kernel of a polynomial ideal, i.e., a D-invariant (=closed under differentiation) space; yet D-invariance plays no role in the results of this section, and therefore is not assumed.

Let H be a finite-dimensional subspace of A_0. We observe that, for $f \in H$, $\deg f_\downarrow = j$ if and only if $f \in (\ker_H T_j)\backslash(\ker_H T_{j+1})$, i.e., if and only if $f_\downarrow \in T_{j+1}(\ker_H T_j)\backslash 0$, with

$$\ker_H T_j := \ker(T_{j|H}). \tag{2.4}$$

Since

$$\dim T_{j+1}(\ker_H T_j) = \dim \ker_H T_j - \dim \ker_H T_{j+1},$$

we conclude the following.

(2.5) Proposition. *$H_\downarrow$ is a scale-invariant space of polynomials of the same dimension as H and admits the decomposition*

$$H_\downarrow = \sum_{j=0}^{\infty} T_{j+1}(\ker_H T_j) = \bigoplus_{j=0}^{\infty} T_{j+1}(\ker_H T_j).$$

Also, $(H_\downarrow)_\downarrow = H_\downarrow$.

Next, we consider the effect of multiplying all the elements of H by some $f \in A_0$. Since $(fg)_\downarrow = f_\downarrow g_\downarrow$, we deduce the following.

(2.6) Proposition. *For any $f \in A_0$ satisfying $f(0) \neq 0$,*

$$\{fg: \ g \in H\}_\downarrow = H_\downarrow.$$

The interaction of differentiation with the map $H \mapsto H_\downarrow$ is determined by the fact that, for any $p \in \Pi$ and any $f \in A_0$,

$$p(D)f = p_\uparrow(D)f_\downarrow + \text{ terms of higher degree.} \tag{2.7}$$

This implies that $(p(D)f)_\downarrow = p_\uparrow(D)f_\downarrow$ in case $p_\uparrow(D)f_\downarrow \neq 0$ and so proves the following.

(2.8) Proposition. *For every $p \in \Pi$,*

$$p_\uparrow(D)H_\downarrow \subset (p(D)H)_\downarrow. \tag{2.9}$$

Of particular importance for subsequent applications is the following

(2.10)Corollary. *If $p(D)$ annihilates H, then $p_\uparrow(D)$ annihilates $H_\downarrow$.*

(2.8)Proposition also leads to

(2.11) Corollary. *If H is D-invariant, so is $H_\downarrow$.*

Proof: For every homogeneous polynomial p, we have $p(D)(H_\downarrow) \subset (p(D)H)_\downarrow$ by (2.8)Proposition, while $(p(D)H)_\downarrow \subset H_\downarrow$ since $p(D)H \subset H$ by assumption. ♠

Next, we show that the dual of H can be represented by $H_\downarrow$. For this purpose, recall that a space Λ of linear functionals is said to be **total** for H if the condition $\lambda f = 0 \ \forall \lambda \in \Lambda$ implies $f \notin H\backslash 0$. This implies that every $\mu \in H'$ can be represented by some $\lambda \in \Lambda$. If Λ is *minimally* total over H, then this representation is unique, i.e., H' is represented by Λ (cf. [4] for more details).

We are interested in using linear functionals of the form

$$p^* : f \mapsto \big(p(D)f\big)(0) = \sum_\alpha D^\alpha p(0) D^\alpha f(0)/\alpha! \tag{2.12}$$

with $p \in \Pi$. These are continuous linear functionals on A_0 (when equipped with the topology of pointwise convergence of the coefficients, hence *a fortiori* with respect to Krull's topology) and even on $A :=$ the space of all power series (with D^α being formal differentiation). The map $p \mapsto p^*$ is linear and one-one, hence provides a linear embedding of Π in the dual of A_0. In fact, Π (in this identification) is the continuous dual of A (hence of its dense subspace A_0).

(2.13) Theorem. *For any finite-dimensional linear subspace H of A_0, the linear space $\overline{H}^*_\downarrow$ is minimally total for H.*

Proof: For any $f \in H\backslash 0$, $p := f_\downarrow \in H_\downarrow$ and $\overline{p}^* f = \overline{p}^* p > 0$. This implies that the only $f \in H$ with $p^* f = 0$ for all $p \in \overline{H}_\downarrow$ is $f = 0$, i.e., $\overline{H}^*_\downarrow$ is total for H. On the other hand, since $\dim \overline{H}^*_\downarrow = \dim H_\downarrow = \dim H$, no proper subspace of $\overline{H}_\downarrow$ could be total for H. ♠

The fact that $\overline{H}^*_\downarrow$ can be used to represent the dual of H is of use in the determination of the local approximation order of H (at 0). By definition, the **local approximation order** of H is the largest integer d for which, for every $f \in C^\infty(\mathbb{R}^s)$, there exists $h \in H$ so that

$$(f - h)(x) = O(\|x\|^d) \quad \text{as } x \to 0.$$

(2.14) Corollary. *The local approximation order at 0 of a finite-dimensional subspace H of A_0 equals the largest integer d for which $\Pi_{<d} \subset H_\downarrow$.*

Proof: Let d be the local approximation order from H.

Having $(f - h)(x) = O(\|x\|^d)$ as $x \to 0$ is the same as having $\deg(f - h)_\downarrow \ge d$. If, in particular, $f \in \Pi_{<d}$, then this can only happen if $h_\downarrow = f_\downarrow$. Since $\Pi_{<d} = (\Pi_{<d})_\downarrow$, this shows that $\Pi_{<d} \subset H_\downarrow$.

For the converse, let T_H be the projector on $C^\infty(\mathbb{R}^s)$ onto H with respect to $\overline{H}^*_\downarrow$ (i.e., such that $\lambda f = \lambda T_H f,\ \forall \lambda \in \overline{H}^*_\downarrow$). Since we have $\lambda(f - T_H f) = 0$ for every $f \in C^\infty(\mathbb{R}^s)$ and $\lambda \in \overline{H}^*_\downarrow$, the assumption $\Pi_{<k} \subset H_\downarrow$ implies $(f - T_H f)(x) = O(\|x\|^k)$ and hence $k \le d$.
♠

3. Ideals of finite codimension

This section is devoted to some background material on polynomial ideals of finite codimension. Most of the results here are known (cf. e.g., [8; Chap. IV,§ 2, esp. pp.176ff] or [9;Thm. 89]) and the proofs are provided primarily for the sake of completeness.

Let I be an ideal in the ring Π of polynomials in s variables over $\mathbb{C}$. The codimension of I is the dimension of the quotient space Π/I. Equivalently, it is the dimension of its **annihilator**, i.e., the dimension of $\{\lambda \in \Pi' : \ \lambda f = 0, \ \forall f \in I\}$. It is therefore also the dimension of the orthogonal complement of I in the space A of all formal power series with respect to the pairing

$$A \times \Pi \to \mathbb{C} : (f, p) \mapsto f^* p := p^* f \ \ (= p(D) f(0)), \tag{3.1}$$

using the fact that A can be identified with Π'.

An important subset of the annihilator of any ideal I is provided by the **variety** of I, i.e., the pointset

$$\mathcal{V}_I := \{\theta \in \mathbb{C}^s : \ p(\theta) = 0, \ \forall p \in I\}. \tag{3.2}$$

For, $I \subseteq \bigcap_{\theta \in \mathcal{V}_I} \ker[\theta]$, with $[\theta] : p \mapsto p(\theta)$ point-evaluation at θ, hence

$$\operatorname{codim} I \ge \# \mathcal{V}_I, \tag{3.3}$$

using the fact that point-evaluations at any finite set of points are linearly independent over Π.

We now use the primary decomposition of an ideal to show that, with the appropriate notion of 'multiplicity', these point-evaluations *span* the annihilator of I in case I has finite codimension. Precisely, we show that

$$I = \bigcap_{\theta \in \mathcal{V}_I, p \in P_\theta} \ker[\theta] p(D),$$

with $P_\theta := \{p \in \Pi : p(D) f(\theta) = 0, \forall f \in I\}$.

We begin with the following observation.

(3.4) Proposition. *An ideal with finite variety is primary if and only if its variety consists of a single point. Furthermore, if $\mathcal{V}_I = \{\theta\}$, then the shifted ideal $E^\theta I := \{p(\cdot + \theta) : \ p \in I\}$ contains all monomials of sufficiently high degree.*

Proof: If $\#\mathcal{V}_I > 1$, then one can find two polynomials p and q which do not vanish on $\mathcal{V}_I$ while their product does (i.e., this variety is reducible). By Hilbert's Nullstellensatz a power of pq lies in I while on the other hand no power of p or q lies in I, hence I is not primary. The converse is obtained by a similar argument since a one-point variety is trivially irreducible.

To prove the second part of the proposition, we may assume without loss that $\theta = 0$. Thus the Nullstellensatz implies that I contains powers of each of the coordinate polynomials, and these powers generate all monomials of sufficiently high degree. ♠

With each $\theta \in \mathcal{V}_I$, we associate the polynomial space

$$P_\theta := P_{I,\theta} := \{p \in \Pi : \ p^* E^\theta f = p(D)f(\theta) = 0, \ \forall f \in I\}, \tag{3.5}$$

where E^θ is the shift operator

$$E^\theta f = f(\cdot + \theta). \tag{3.6}$$

The space P_θ, as well as its dimension, are usually referred to as the **multiplicity** of θ. In words, the multiplicity (space) of θ is the orthogonal complement in Π of $E^\theta I$ with respect to the pairing (3.1). Therefore

$$I \subset \bigcap_{\theta \in \mathcal{V}_I, p \in P_\theta} \ker p^* E^\theta. \tag{3.7}$$

Since I is an ideal, P_θ is D-invariant. Indeed, using the identity

$$p(D)(fg) = \sum_\beta ([\![D]\!]^\beta p)(D) f \ [\![D]\!]^\beta g \tag{3.8}$$

(which follows directly from Leibniz' Formula $[\![D]\!]^\alpha (fg) = \sum_{\beta+\gamma=\alpha} [\![D]\!]^\beta f \ [\![D]\!]^\gamma g$), one finds that, for $f \in I$ and $g = (\cdot - \theta)^\alpha$, $fg \in I$ and $[\![D]\!]^\beta g(\theta) = \delta_{\beta,\alpha}$, hence, for any $p \in P_\theta$,

$$(D^\alpha p) f(\theta) = p(D)(fg)(\theta) = 0.$$

The variety of I together with the multiplicity spaces characterizes the ideal. We first prove this claim with respect to a primary ideal:

(3.9) Lemma. *If P is a finite-dimensional D-invariant polynomial space, then*

$$I_P := I_{P,\theta} := \bigcap_{q \in P} \ker q^* E^\theta \tag{3.10}$$

is the unique ideal with variety $\{\theta\}$ and multiplicity P. Further,

$$\mathrm{codim} I_P = \dim P. \tag{3.11}$$

Proof: Since I_P is defined as the orthogonal complement of the finite dimensional space $\{q^* E^\theta : q \in P\}$ of linear functionals, (3.11) holds. For the rest, assume without loss that $\theta = 0$ (which can always be achieved by a shift). First, it follows from (3.8) and the D-invariance of P that I_P is an ideal. Also, since $\Pi_0 \subset P$ (by D-invariance and nontriviality of P), all polynomials in I_P vanish at $\theta = 0$, while, from the fact that P is finite-dimensional, it follows that $P \subset \Pi_k$ for some k, hence I_P contains all monomials $()^\alpha$ with $|\alpha| > k$, and therefore 0 is the only common zero of I_P. Now, we know that $I_P \cap \Pi_k$ is the orthogonal complement of P in Π_k (with respect to the pairing (3.1)) and vice versa, which means that the multiplicity space of I_P at 0 lies in P, while (3.10) ensures the converse inclusion. Hence P is the multiplicity of I_P's sole zero.

If J is an ideal with variety $\{0\}$, then by (3.4)Proposition J contains all monomials of sufficiently high degree k. This means that $f \in J$ if and only if $T_{k+1} f \in J$. Thus J as well as its multiplicity space (at 0) are uniquely determined by $J \cap \Pi_k$. Reversing this last argument, we see that the multiplicity space of J identifies $J \cap \Pi_k$ and hence J in a unique way. ♠

This last lemma is used now in the description of the primary decomposition of an ideal with finite variety:

(3.12) Proposition. *If I is an ideal with finite variety, then the primary decomposition of I takes the form*

$$I = \bigcap_{\theta \in \mathcal{V}_I} I_\theta, \tag{3.13}$$

where I_θ is the unique primary ideal with variety $\{\theta\}$ and the (finite-dimensional) multiplicity space $P_{I,\theta}$.

Proof: From (3.4)Proposition we know that the (unique irredundant) primary decomposition of I has the form

$$I = \bigcap_{\theta \in \mathcal{V}_I} J_\theta \tag{3.14}$$

with $\mathcal{V}_{J_\theta} = \{\theta\}$, all θ. Hence, to finish the proof of (3.13), it is, by (3.9)Lemma, sufficient to show that the multiplicity

$$Q_\theta := P_{J_\theta,\theta} = \{q \in \Pi : \ q(D)f(\theta) = 0, \ \forall f \in J_\theta\}$$

for the sole zero θ of J_θ is finite-dimensional and equals $P_{I,\theta}$. The fact that Q_θ is finite-dimensional follows from (3.4)Proposition, since J_θ has a single-point variety. To prove that also $Q_\theta = P_{I,\theta} =: P_\theta$, we note that Q_θ is contained in P_θ (since I is contained in J_θ). If now

$$Q_\theta \neq P_\theta,$$

then there would be a smallest j so that $Q_\theta \cap \Pi_j \neq P_\theta \cap \Pi_j$, thus providing us with a $p \in P_\theta \backslash Q_\theta$ for which $D^\alpha p \in Q_\theta$ for all $\alpha \neq 0$. There would, therefore, exist $f \in J_\theta$ for which

$$p(D)f(\theta) \neq 0. \tag{3.15}$$

Yet, since the other primary ideals in (3.14) do not have θ in their variety, we could now find g so that $fg \in \prod_{\mathcal{V}_I} J_\tau \subset \cap_{\mathcal{V}_I} J_\tau = I$ and $g(\theta) \neq 0$. But now, using (3.8) one more time,

$$0 = p(D)(fg)(\theta) = p(D)f(\theta)g(\theta),$$

hence $p(D)f(\theta) = 0$, contradicting (3.15). ♠

The last proposition is equivalent to the statement

$$\Pi/I = \bigoplus_{\theta \in \mathcal{V}_I} \Pi/I_\theta,$$

which is no more than the fact that each Artinian ring is the sum of primary rings (cf. [12;Thm. 3, p.205]).

Note that (3.9)Lemma implies that, for each I_θ in (3.13),

$$I_\theta = \bigcap_{q \in P_{I,\theta}} \ker q^* E^\theta,$$

and hence (3.12)Proposition yields that

$$I = \bigcap_{\theta \in \mathcal{V}_I, q \in P_\theta} \ker q^* E^\theta, \tag{3.16}$$

which shows in particular that I is of finite codimension. We have thus proved

(3.17) Corollary. *A polynomial ideal I has a finite variety if and only if it has finite codimension. Also*

$$\text{codim} I \leq \sum_{\theta \in \mathcal{V}_I} \dim P_{I,\theta} = \sum_{\theta \in \mathcal{V}_I} \text{codim} I_\theta.$$

Equation (3.16) is a special case of the following result [10]: if I is a polynomial ideal and $\mathbf{\Theta} \subset \mathcal{V}_I$ intersects each of the varieties of the primary ideals in a primary decomposition of I, then $I = \bigcap_{\theta \in \mathbf{\Theta}, q \in P_\theta} \ker q^* E^\theta$.

Each ideal I induces a set

$$I(D) := \{p(D) : \ p \in \Pi\}$$

of differential operators with constant coefficients. The **kernel** of I is the set

$$\{f \in \mathcal{D}'(\mathbb{R}^s) : \ p(D)f = 0, \ \forall p \in I\}$$

of all distributions that are being annihilated by $I(D)$. Note that the kernel is D-invariant.

As we show later, the finite codimension of I implies that its kernel lies in A_0. We therefore find it more convenient to focus now on the space

$$I\perp := \{f \in A_0 : \ p(D)f = 0, \ \forall p \in I\}, \tag{3.18}$$

i.e, on the intersection of the kernel with A_0.

Since, for any $f \in A_0$, $p(D)f = 0$ if and only if $(()^\alpha p(D))f(0) = 0$ for all α, and since $I = \{()^\alpha p : \ \alpha \in \mathbb{Z}^s, \ p \in I\}$, we conclude that (with f^* as in (3.1))

$$I\perp = \{f \in A_0 : f^* p = 0, \forall p \in I\}.$$

This, together with the identity

$$(q(D)p)(\theta) = q^* E^\theta p = p^*(e_\theta q) = p(D)(e_\theta q)(0) \tag{3.19}$$

valid for any polynomials q and p, implies the following

(3.20) Proposition. *$f \in A_0$ lies in $I\perp$ if and only if $f^* \in \Pi'$ annihilates I. In particular, the exponential $e_\theta q$ lies in $I\perp$ if and only if the linear functional $q^* E^\theta$ annihilates I.*

(3.21) Corollary. *An ideal I has a finite variety if and only if $I\perp$ is finite-dimensional. In this case (with I_θ as in (3.13))*

$$\dim I\perp = \mathrm{codim} I = \sum_{\theta \in \mathcal{V}_I} \mathrm{codim} I_\theta, \tag{3.22}$$

$$I\perp = \bigoplus_{\theta \in \mathcal{V}_I} e_\theta P_{I,\theta}, \tag{3.23}$$

and

$$I = (I\perp)\perp := \{p \in \Pi : \ p(D)f = 0 \ \forall f \in I\perp\}. \tag{3.24}$$

Proof: Since A_0 is embedded in Π', (3.20)Proposition implies that there is an isomorphism between $I\perp$ and the orthogonal complement of I, a space which was identified in (3.16). The dimension of this latter space matches the codimension of I, hence indeed $\dim I\perp = \mathrm{codim} I$. Since exponentials of different frequencies are linearly independent, we conclude that (3.23) holds, hence $\dim I\perp = \sum_{\theta\in\mathcal{V}_I} \dim P_\theta$, and the rest of (3.22) follows from (3.9)Lemma.

For the last equality, let $p \in (I\perp)\perp$. Then, by the above arguments (when applied to the ideal $p\Pi$ rather than I) and (3.20)Proposition, the ideal $p\Pi$ is being annihilated by all the functionals in the orthogonal complement of I, thus $p\Pi \subset I$. ♠

The question whether the exponentials in the kernel of I are dense in it is a fundamental one in the theory of linear differential operators with constant coefficients (cf. [11]). In the special case of interest here, viz. when I has a finite variety, the kernel contains only exponentials, hence coincides with $I\perp$:

(3.25) Corollary. *The kernel of an ideal of finite variety is a finite-dimensional exponential space admitting the direct sum decomposition (3.23).*

Proof: let I be the ideal in question. By (3.23) we only need to prove that the kernel K of I lies in A_0 (and hence coincides with $I\perp$). Furthermore, since every finite-dimensional D-invariant space of distributions is an exponential space (cf. [3; Th. 1.3]), it is sufficient to prove that K is finite-dimensional. This will be established the moment we show that $\dim(K \cap C^\infty(\mathrm{I\!R}^s)) < \infty$, since by [3; Lem. 3.2], it ensures that $K \subset C^\infty(\mathrm{I\!R}^s)$.

Since $\mathcal{V}_I$ is finite, we can choose, for each j, a polynomial which is constant in all variables but the jth and which vanishes on $\mathcal{V}_I$. By the Nullstellensatz, some power p_j of this polynomial must lie in I, and hence K lies in the kernel of $p_j(D)$, a constant coefficient differential operator involving only differentiation in the jth variable. It follows that every element of $K \cap C^\infty(\mathrm{I\!R}^s)$ is in the span of the exponentials $()^\alpha e_\theta$, where θ_j is a root of p_j of multiplicity $> \alpha_j$, all j. In particular, $\dim(K \cap C^\infty(\mathrm{I\!R}^s)) < \infty$. ♠

Because of the above result, we do not distinguish in the sequel between the kernel of an ideal of finite codimension I and $I\perp$ and use the same notation and terminology for both.

We conclude this section with another corollary of the results here of use subsequently:

(3.26) Corollary. *Let G be a generating set for the ideal J, let θ be some zero of J, and let I be the ideal generated by $\{gh_g : g \in G\}$, with each h_g nonzero at θ. Then $P_{J,\theta} = P_{I,\theta}$. In particular, $J = I_\theta$ (cf. (3.13)) in case θ is the only zero of J.*

Proof: Since $I \subset J$, we only need to show that $P_{I,\theta} \subset P_{J,\theta}$. For this, let $p \in P_{I,\theta}$. By D-invariance, $D^\beta p \in P_{I,\theta}$ for all β, while, for each g, $1/h_g$ is analytic near θ. Therefore, by (3.8), (with $\theta = 0$ for notational convenience)

$$p^* g = p^*((gh_g)/h_g) = \sum_\beta (D^\beta p)^*(gh_g)\ ([\![D]\!]^\beta)^*(1/h_g) = 0 \qquad \forall g \in G.$$

♠

4. Ideals with finite codimension: perturbation

We call the ideal J in Π a **perturbation** of the ideal I in case $F_\uparrow = G_\uparrow$ for some generating sets F and G of I and J, respectively.

Here we are mainly interested in two types of perturbations:

(a) I is an arbitrary ideal and the perturbed ideal is the corresponding homogeneous ideal $I_\uparrow = \mathrm{span}\{p_\uparrow : \ p \in I\}$.

(b) I is a *homogeneous* ideal, and the perturbed ideal J is generated by a perturbation of a (finite) set of *homogeneous* generators for I.

Here and elsewhere, a **homogeneous** ideal is an ideal admitting a homogeneous set of generators. Note that I is homogeneous if and only if it **stratifies** (or **is graded**) i.e., is the sum of its homogeneous components. Also, an ideal is homogeneous if and only if it is scale-invariant if and only if its kernel is scale-invariant.

The first result here characterizes homogeneous ideals of finite codimension:

(4.1) Proposition. *An ideal I of finite codimension is homogeneous if and only if its kernel $I\perp$ is a finite-dimensional scale-invariant polynomial space.*

Proof: Since we assume that I is of finite codimension, (3.21)Corollary yields that its kernel is a finite-dimensional exponential space. Since the only exponentials that can lie in a finite-dimensional scale-invariant space are polynomials, our claim follows. ♠

Employing (3.21)Corollary, we conclude

(4.2) Corollary. *An ideal I of finite codimension is homogeneous if and only if 0 is its only zero and the corresponding multiplicity space is scale-invariant.*

In particular, every homogeneous ideal of finite codimension is primary.

We turn now to the main part of the discussion here.

(4.3) Theorem. *Let I be an ideal of finite codimension. Then*

(a) $\mathrm{codim} I = \mathrm{codim} I_\uparrow$;

(b) $I_\uparrow\perp = (I\perp)_\downarrow$.

Proof: By (2.5)Proposition and (3.22)

$$\dim (I\perp)_\downarrow = \dim I\perp = \mathrm{codim} I, \tag{4.4}$$

while

$$\dim I_\uparrow\perp = \mathrm{codim} I_\uparrow, \tag{4.5}$$

hence (b) implies (a). To prove (b), we first show that $(I\perp)_\downarrow \subset (I_\uparrow)\perp$. For this, it is sufficient to show that $p(D)(I\perp)_\downarrow = \{0\}$ for every *homogeneous* $p \in I_\uparrow$. But, for such p, there exists $q \in I$ with $q_\uparrow = p$, while by definition of $I\perp$, $q(D)$ annihilates $I\perp$ and hence by (2.10)Corollary (when applied to $H = I\perp$) $p(D) = q_\uparrow(D)$ annihilates $(I\perp)_\downarrow$.

To complete the proof of (b), it suffices to show that $\dim(I\perp)_\downarrow \geq \dim I_\uparrow\perp$, which, in view of (4.4), is equivalent to

$$\mathrm{codim} I \geq \dim I_\uparrow\perp. \tag{4.6}$$

To prove (4.6), let $p \in I\backslash 0$. Then $p_\uparrow \in I_\uparrow$ and therefore $p_\uparrow$ annihilates $I_\uparrow\perp$. Since $p_\uparrow$ does not annihilate $\overline{p}_\uparrow$, we conclude that $\overline{p}_\uparrow \notin I_\uparrow\perp$, and hence neither is $\overline{p}$. We have shown that $\overline{I} \cap I_\uparrow\perp = \{0\}$, and (4.6) thus follows. ♠

We cannot expect to maintain the equality $\mathrm{codim} I = \mathrm{codim} J$ for an arbitrary lower order perturbation J of I (see the Example below). Still, we have:

(4.7) Theorem. *Let I be a homogeneous ideal of finite codimension. Let J be an ideal obtained by perturbing a set of homogeneous generators of I. Then*

$$\mathrm{codim} I \geq \mathrm{codim} J; \tag{4.8}$$

$$J\perp_\downarrow \subset I\perp. \tag{4.9}$$

Proof: Let G be the finite set of (homogeneous) generators of I, whose perturbation F generates J. Then $G = F_\uparrow \subset J_\uparrow$, hence

$$I \subset J_\uparrow, \tag{4.10}$$

which yields that

$$\mathrm{codim} I \geq \mathrm{codim} J_\uparrow, \tag{4.11}$$

and

$$J_\uparrow\perp \subset I\perp. \tag{4.12}$$

Application of (4.3)Theorem (a) (resp. (b)) to (4.11) (resp. (4.12)) thus yields (4.8) (resp. (4.9)). ♠

It is known that equality holds in (4.8) and (4.9) (for small perturbations) in case I is a homogeneous ideal generated by s generators (cf. [1; Chap. 5]). For an arbitrary homogeneous ideal, though, the inequality (4.8) (or, equivalently, the inclusion (4.9)) is strict. Here is an illustration.

(4.13) Example. Let I be the bivariate ideal generated by the monomials $()^{2,0}, ()^{0,2}, ()^{1,1}$. Then $\mathrm{codim} I = 3$. Yet if we perturb the given generators by adding a non-zero constant to each of them, the resulting perturbed ideal will generically have codimension 0; in any case, it will always have codimension < 3.

The map $I \mapsto I_\uparrow$ sets up an equivalence relation between ideals. (4.3)Theorem shows this to be the same equivalence relation as the one set up by the map $I \mapsto I\perp_\downarrow$. Each resulting equivalence class contains exactly one homogeneous ideal. If this ideal is trivial (i.e., $I = \{0\}$ or $I = \Pi$), then there are no other elements in the equivalence class. At this moment we do not know whether there exist other isolated homogeneous ideals (in the sense that they comprise their entire equivalence class). A closely related question is whether the equivalence class of a given homogeneous ideal contains an ideal with only simple zeros (whose kernel therefore is spanned by pure exponentials). In this connection we note the following

(4.14) Proposition. *Let I_1, I_2 be two ideals with only simple zeros. If the zeros of one can be obtained from the zeros of the other by translations and dilations, then the ideals are equivalent.*

Proof: First, let us assume that $\mathcal{V}_{I_1}$ is obtained from $\mathcal{V}_{I_2}$ by a translation by a. Set $H_j := I_j\perp$, $j = 1, 2$. From (3.23) we infer that each H_j is spanned by pure exponentials and that $H_1 = e_a H_2$. Applying (2.6)Proposition, we conclude that

$$H_{1\downarrow} = H_{2\downarrow},$$

and (4.3)Theorem thus yields the desired result. The proof for the dilation case is the same, with (2.2) replacing (2.6)Proposition. ♠

The same results hold also for the non-simple case provided that the multiplicity spaces together with the variety of one of the ideals are obtained from the other by a translation or dilation. Note also that, due to their D-invariance, a translation will not change the multiplicity spaces of an ideal.

5. Applications to box splines

As mentioned before, our discussion of the correspondence between ideals and homogeneous ideals of finite codimension was primarily stimulated by the theory of box splines and aimed at getting a better insight into that theory. Indeed, the results of the previous sections do provide painless proofs for some of the highlights of box spline theory, as well as invite natural extensions of them.

Box splines will not be defined here or elsewhere in this paper. The object of our investigation here is an ideal and its corresponding kernel which are associated with a box spline. To define these ideals, let Γ be a finite multiset of linear polynomials. We use the notation

$$\gamma(x) =: x_\gamma \cdot x - \lambda_\gamma \tag{5.1}$$

to indicate the linear and constant terms of $\gamma \in \Gamma$ and assume that, for all $\gamma \in \Gamma$,

$$x_\gamma \in \mathbb{R}^s \backslash 0, \quad \lambda_\gamma \in \mathbb{C}, \tag{5.2}$$

and that

$$\operatorname{span}\{x_\gamma\}_{\gamma\in\Gamma} = \mathbb{R}^s;$$

in particular, $\#\Gamma \geq s$.

With

$$x_K := \{x_\gamma : \ \gamma \in K\}$$

for $K \subset \Gamma$, we define the set of "bases" in Γ by

$$\mathbb{B}(\Gamma) := \{B \subset \Gamma : \ x_B \text{ is a basis for } \mathbb{R}^s\}. \tag{5.3}$$

Each $B \in \mathbb{B}(\Gamma)$ is associated with the unique common zero $\theta_B \in \mathbb{C}^s$ of the polynomials in B. This gives rise to the map

$$b : \mathbb{B}(\Gamma) \to \mathbb{C} : B \mapsto \theta_B, \tag{5.4}$$

whose image

$$\boldsymbol{\Theta}(\Gamma) := \operatorname{ran} b$$

strongly depends on the sequence

$$\lambda_\Gamma := (\lambda_\gamma)_{\gamma \in \Gamma}$$

of constant terms, while the cardinality of this map's domain does not. Every choice of λ_Γ provides a decomposition

$$\mathbb{B}(\Gamma) = \dot{\bigcup}_{\theta \in \boldsymbol{\Theta}(\Gamma)} \mathbb{B}(\Gamma_\theta) \tag{5.5}$$

of that domain, with

$$\Gamma_\theta := \{\gamma \in \Gamma : \ \gamma(\theta) = 0\}. \tag{5.6}$$

Note that for each $B \in \mathbb{B}(\Gamma_\theta)$ we have $\theta_B = \theta$ and hence

$$\boldsymbol{\Theta}(\Gamma_\theta) = \{\theta\}, \tag{5.7}$$

which shows that

$$\mathbb{B}(\Gamma_\theta) = b^{-1}(\theta).$$

In general we have

$$\#\boldsymbol{\Theta}(\Gamma) \le \#\mathbb{B}(\Gamma);$$

yet (for fixed x_Γ and variable λ_Γ) the map b is *generically* 1-1, hence generically

$$\#\boldsymbol{\Theta}(\Gamma) = \#\mathbb{B}(\Gamma). \tag{5.8}$$

For reasons to be discussed soon, we refer to this generic situation as “the simple case”.

We now construct an ideal I_Γ with variety $\boldsymbol{\Theta}(\Gamma)$ and multiplicity at each θ equal to $\#b^{-1}(\theta)$. For this let

$$\mathbb{K}(\Gamma) := \{K \subset \Gamma : \ K \cap B \neq \emptyset, \ \forall B \in \mathbb{B}(\Gamma)\}. \tag{5.9}$$

Setting

$$p_K := \prod_{\gamma \in K} \gamma \tag{5.10}$$

for $K \subset \Gamma$, we define I_Γ to be the ideal generated by

$$\{p_K : \ K \in \mathbb{K}(\Gamma)\}. \tag{5.11}$$

Our first aim is to analyze the variety of I_Γ: given $\theta \in \mathbf{\Theta}(\Gamma)$, the set Γ_θ contains at least one "basis" and thus has a non-empty intersection with each $K \in \mathbb{K}(\Gamma)$; hence $\theta \in \mathcal{V}_{I_\Gamma}$. Conversely, if $\theta \notin \mathbf{\Theta}(\Gamma)$, then Γ_θ contains no basis, hence $\Gamma \backslash \Gamma_\theta$ intersects every basis and thus lies in $\mathbb{K}(\Gamma)$, while on the other hand $p_{\Gamma\backslash\Gamma_\theta}(\theta) \neq 0$; hence $\theta \notin \mathcal{V}_{I_\Gamma}$. We conclude that

$$\mathcal{V}_{I_\Gamma} = \mathbf{\Theta}(\Gamma). \tag{5.12}$$

More information about I_Γ is recorded in the following theorem.

(5.13) Theorem. *The ideal I_Γ is of finite codimension. Its primary decomposition takes the form*

$$I_\Gamma = \bigcap_{\theta \in \mathbf{\Theta}(\Gamma)} I_{\Gamma_\theta}. \tag{5.14}$$

Furthermore, if for some $\theta \in \mathcal{V}_I$, $\#\Gamma_\theta = s$, then θ is a simple zero of the variety.

Proof: Combining (5.12) with the fact that $\mathbf{\Theta}(\Gamma)$ is a finite set, we see that $\mathcal{V}_{I_\Gamma}$ is finite. Application of (3.17)Corollary thus yields that I_Γ is indeed of finite codimension.

¿From the fact that $\mathcal{V}_{I_{\Gamma_\theta}} = \mathbf{\Theta}(\Gamma_\theta) = \{\theta\}$, we obtain that the right hand of (5.14) is indeed a primary decomposition. Now, for $K \in \mathbb{K}(\Gamma)$ we have $p_{K\backslash(K\cap\Gamma_\theta)}(\theta) \neq 0$, while $\{p_{K\cap\Gamma_\theta} : \ K \in \mathbb{K}(\Gamma)\} = \{p_K : \ K \in \mathbb{K}(\Gamma_\theta)\}$. Hence (3.26) (with I replaced by I_{Γ_θ}, $G = \{p_{K\cap\Gamma_\theta} : \ K \in \mathbb{K}(\Gamma)\}$ and $h_{p_{K\cap\Gamma_\theta}} = p_{K\backslash K\cap\Gamma_\theta}$) together with (3.12)Proposition shows that (5.14) is indeed the primary decomposition of I_Γ.

We prove the last statement only for $\theta = 0$; the general case is obtained by shifting the ideal. Assuming that $\theta = 0 \in \mathbf{\Theta}(\Gamma)$, we suppose that $\#\Gamma_\theta = s$. This means that Γ_θ comprises the unique element of $\mathbb{B}(\Gamma_\theta)$ and hence each of the s homogeneous linearly independent linear polynomials in Γ_θ lies in I_{Γ_θ}. It follows that the ideal I_{Γ_θ} contains all homogeneous linear polynomials, hence the zero at the origin (of I_{Γ_θ}, and hence of I_Γ) is indeed simple. ♠

Note that in the simple case (see (5.8) above) every Γ_θ (with $\theta \in \mathbf{\Theta}(\Gamma)$) consists of s elements. Hence, for "simple" Γ, the variety of I_Γ consists of $\#\mathbf{\Theta}(\Gamma) = \#\mathbb{B}(\Gamma)$ simple zeros (therefore the epithet "simple" for such Γ). Combining this with (3.21)Corollary, we conclude

(5.15) Corollary. *Assume that* Γ *is simple. Then*

$$\operatorname{codim} I_\Gamma = \#\mathbf{\Theta}(\Gamma) = \#\mathbb{B}(\Gamma), \tag{5.16}$$

and

$$I_\Gamma\perp = \operatorname{span}\{e_\theta\}_{\theta\in\mathbf{\Theta}(\Gamma)}. \tag{5.17}$$

We now use (5.15)Corollary to derive the following result about the general ideal I_Γ. The simple proof this result admits is striking when compared to the original proofs (cf. [5;§2], [3;§2], [6;§3]).*

(5.18) Theorem.

$$\operatorname{codim} I_\Gamma \geq \#\mathbb{B}(\Gamma). \tag{5.19}$$

Proof: Combining the primary decomposition (5.14) with (5.5) and (3.22), it suffices to prove the theorem with respect to each Γ_θ. So we may assume without loss that I_Γ admits a single-point variety and, by shifting this zero to the origin, that Γ is homogeneous.

We now consider perturbations of I_Γ induced by lower order perturbations of the polynomials in Γ (which means adding constant terms to some of the γ's in Γ). Generically, each such perturbation results in a "simple" set $\widetilde{\Gamma}$, hence, in view of (5.15), application of (4.7)Theorem yields the desired result. ♠

As a matter of fact, it is known [6], [7] that equality holds in (5.19), yet that does not seem to follow easily from the type of arguments we employ here. For completeness, we provide here a proof for the converse inequality of (5.19), which is a specialization of the argument given in [6;Th. 3.1] to the present situation.

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* Actually, the theory of box splines focuses on $I_\Gamma\perp$ rather than on I_Γ itself. The proofs in the references are therefore of the equivalent inequality $\dim I_\Gamma\perp \geq \#\mathbb{B}(\Gamma)$.

(5.20) Result.

$$\mathrm{codim} I_\Gamma \le \#\mathrm{I\!B}(\Gamma). \tag{5.21}$$

Proof: The proof is done by induction on $\#\Gamma \ge s$. In case $\#\Gamma = s$, Γ is simple, hence (5.21) follows from from (5.15).

For notational convenience, we set $H(\Gamma)$ for the kernel of I_Γ, and also assume throughout this proof that $\mathrm{I\!K}(\Gamma)$ consists of the *minimal* subsets $K \subset \Gamma$ with the property $K \cap B \ne \emptyset$, $\forall B \in \mathrm{I\!B}(\Gamma)$.

Now, in view of (3.22), (5.21) is equivalent to

$$\dim H(\Gamma) \le \#\mathrm{I\!B}(\Gamma). \tag{5.22}$$

Assume $\#\Gamma > s$. Then there exists $\gamma \in \Gamma$ such that

$$\mathrm{span}\, x_{\Gamma_\gamma} = \mathrm{I\!R}^s,$$

where $\Gamma_\gamma := \Gamma\backslash\gamma$. With this define the map

$$T : H(\Gamma) \to \times_{K\in\mathrm{I\!K}(\Gamma_\gamma)} H(\Gamma\backslash K) : f \mapsto (p_K(D)f)_{K\in\mathrm{I\!K}(\Gamma_\gamma)},$$

where by convention $H(\Gamma\backslash K) = \{0\}$ in case $K \in \mathrm{I\!K}(\Gamma)$. One checks that $p_K(D)H(\Gamma) \subset H(\Gamma\backslash K)$ to verify that the map is well defined.

Now,

$$\ker T = \bigcap_{K\in\mathrm{I\!K}(\Gamma_\gamma)} \ker p_K(D)_{|H(\Gamma)} \subset H(\Gamma_\gamma),$$

and so, by the induction hypothesis, (when applied to each one of the sets $\Gamma\backslash K$, $K \in \mathrm{I\!K}(\Gamma_\gamma)$, and to Γ_γ),

$$\begin{aligned}
\dim H(\Gamma) &= \dim\ker T + \dim\mathrm{ran}\, T \\
&\le \dim H(\Gamma_\gamma) + \sum_{K\in\mathrm{I\!K}(\Gamma_\gamma)} \dim H(\Gamma\backslash K) \\
&\le \#\mathrm{I\!B}(\Gamma_\gamma) + \sum_{K\in\mathrm{I\!K}(\Gamma_\gamma)} \#\mathrm{I\!B}(\Gamma\backslash K) \\
&= \#\mathrm{I\!B}(\Gamma).
\end{aligned}$$

The last equality follows from the fact that each element B of $\mathbb{B}(\Gamma)$ either lies in $\mathbb{B}(\Gamma_\gamma)$ (i.e., in case $\gamma \notin B$), or belongs to exactly one of the sets $\mathbb{B}(\Gamma\backslash K)$ (take $K = \{\widetilde{\gamma} \in \Gamma_\gamma : x_{\widetilde{\gamma}} \notin \operatorname{span} x_{B\backslash\gamma}\}$). ♠

Combining (5.19) and (5.21) we conclude that indeed

$$\mathrm{codim} I_\Gamma = \#\mathbb{B}(\Gamma). \tag{5.23}$$

To elaborate more on the connection between an ideal I_Γ and its homogeneous counterpart, we define $\Gamma_\uparrow$ to be the multiset of linear homogeneous polynomials obtained when replacing λ_Γ by 0. Then I_Γ is a lower order perturbation of $I_{\Gamma_\uparrow}$. Since $\#\mathbb{B}(\Gamma) = \#\mathbb{B}(\Gamma_\uparrow)$, we see that the perturbation I_Γ of $I_{\Gamma_\uparrow}$ preserves codimension, and hence, in view of (4.3)Theorem and (4.10),

$$I_{(\Gamma_\uparrow)} = (I_\Gamma)_\uparrow. \tag{5.24}$$

The results of the previous sections provide us with further important information about the kernel $H(\Gamma) = I_\Gamma\perp$ of I_Γ. First, (3.23) together with (5.14) shows that

$$H(\Gamma) = \bigoplus_{\theta\in\mathbf{\Theta}(\Gamma)} H(\Gamma_\theta), \tag{5.25}$$

with each $H(\Gamma_\theta)$ being of the form $e_\theta P_\theta$, and P_θ the multiplicity space for both I_Γ and I_{Γ_θ}, and in particular a polynomial space. By (3.22) and (5.23), $\dim H(\Gamma) = \#\mathbb{B}(\Gamma)$.

A result of special significance follows from (5.24) when combined with (4.3)Theorem(b):

$$H(\Gamma)_\downarrow = H(\Gamma_\uparrow). \tag{5.26}$$

This result describes the (usually very complicated) kernel of a homogeneous I_Γ in terms of the "limit at the origin" of the exponentials in the kernel of a "simple" perturbation of I_Γ, thus allowing us to introduce an algorithm for the construction of the kernel of a *homogeneous* ideal.

(5.27) Algorithm. *Assume Γ is homogeneous. The following two-step algorithm would compute $H(\Gamma) = I_\Gamma\perp$:*

Step 1. Pick a "simple" lower order perturbation $\widetilde{\Gamma}$ and compute $\mathbf{\Theta}(\widetilde{\Gamma})$.

Step 2. Construct $(\exp_{\mathbf{\Theta}(\widetilde{\Gamma})})_\downarrow$ *with*

$$\exp_{\mathbf{\Theta}(\widetilde{\Gamma})} := \mathrm{span}\{e_\theta\}_{\theta\in\mathbf{\Theta}(\widetilde{\Gamma})} = H(\widetilde{\Gamma}).$$

A simple Gram-Schmidt-like algorithm for the construction of a basis for $H_\downarrow$ (which is orthogonal with respect to the pairing (3.1), and with H being an arbitrary finite-dimensional subspace of A_0) is described in [4; §5].

The construction above easily extends to a general (i.e., non-homogeneous) Γ. One only has to make use of the direct sum decomposition of $H(\Gamma)$ in (5.25) and the fact that, for each Γ_θ, $H(\Gamma_\theta) = e_\theta H(\Gamma_{\theta\uparrow})$. The algorithm above can then be applied to construct each $H(\Gamma_{\theta\uparrow})$, i.e., each of the multiplicity spaces $\{P_\theta\}_{\theta\in\mathbf{\Theta}(\Gamma)}$.

The local approximation order of the space $H(\Gamma)$ can also be deduced from (5.26). Indeed, an application of (2.14) to (5.26) yields

(5.28) Corollary. *The local approximation of $H(\Gamma)$ equals the largest d for which $\Pi_{<d} \subset H(\Gamma_\uparrow)$. In particular, the local approximation order of $H(\Gamma)$ matches that of $H(\Gamma_\uparrow)$.*

We remark that, although the *structure* of $H(\Gamma_\uparrow)$ may be quite involved, the number d in (5.28)Corollary can be easily determined in terms of the geometry induced by x_Γ (cf. [2] and the discussion of this issue in the next section).

6. Extensions

Here we introduce a generalization of the box spline ideals and discuss to what an extent the results about box splines ideals remain valid in this more general setting. The notations and terminology used in the previous section are retained here as well. We mention that the type of generalization here is in some sense opposite to the one discussed in [6]: here (5.19) is the inequality that holds in the more general setting, where there (5.21) is the one which is valid in general.

Let $\mathbb{B}_1$ be a subset of $\mathbb{B}(\Gamma)$, and define correspondingly

$$\mathbb{K}_1 := \{K \subset \Gamma :\ K \cap B \neq \emptyset,\ \forall B \in \mathbb{B}_1\}. \tag{6.1}$$

Let $I_\Gamma(\mathbb{B}_1)$ be the ideal generated by

$$\{p_K : \ K \in \mathbb{K}_1\}.$$

Note that $\mathbb{K}(\Gamma) \subset \mathbb{K}_1$, and hence

$$I_\Gamma(\mathbb{B}_1) \supset I_\Gamma, \tag{6.2}$$

and

$$I_\Gamma(\mathbb{B}_1)\perp \subset I_\Gamma\perp.$$

Is it still true that

$$\mathrm{codim} I_\Gamma(\mathbb{B}_1) = \#\mathbb{B}_1 \ ?$$

For a "simple" Γ, the answer is affirmative:

(6.3) Proposition. *Assume Γ is simple. Let $\mathbb{B}_1$ be an arbitray subset of $\mathbb{B}(\Gamma)$. Then, with $I_\Gamma(\mathbb{B}_1)$ as above,*

$$\mathrm{codim} I_\Gamma(\mathbb{B}_1) = \#\mathbb{B}_1.$$

Proof: Since Γ is simple, then, by (5.13)Theorem and (5.15)Corollary, $\mathcal{V}_{I_\Gamma}$ consists of finitely many simple zeros, and hence, by (6.2), the same is true for $\mathcal{V}_{I_\Gamma(\mathbb{B}_1)}$. Thus, by (3.21)Corollary, the codimension of $I_\Gamma(\mathbb{B}_1)$ coincides with the cardinality of its variety. Now, one checks that this variety consists exactly of the $\#\mathbb{B}_1$ points

$$b(\mathbb{B}_1) = \{\theta_B : \ B \in \mathbb{B}_1\}. \tag{6.4}$$

♠

Yet, for non-simple Γ, the answer in general is negative, as shown by the following example:

(6.5) Example. Assume that Γ consists of four bivariate linear homogeneous polynomials p_1, p_2, p_3, p_4, such that the pairs $\{p_j, p_{j+1}\}$, $j = 1, 3$, are linearly independent. Define

$$\mathbb{B}_1 := \{\{p_1, p_2\}, \ \{p_3, p_4\}\}.$$

Here, $\#\mathbb{B}_1 = 2$, yet each element of $\mathbb{K}_1$ (i.e., each subset of Γ that intersects both of the "bases" in $\mathbb{B}_1$) has cardinality ≥ 2, and thus each of the generators of $I_\Gamma(\mathbb{B}_1)$ is a homogeneous polynomial of degree ≥ 2. We conclude that Π_1 is perpendicular to $I_\Gamma(\mathbb{B}_1)$, hence

$$\mathrm{codim} I_\Gamma(\mathbb{B}_1) \geq \dim \Pi_1 = 3.$$

Therefore $\mathrm{codim} I_\Gamma(\mathbb{B}_1) > \#\mathbb{B}_1$.

However, a result like (5.19) does hold in this general setting, with the idea of lower order perturbations still providing a quite simple proof:

(6.6) Theorem. *Let $\mathbb{B}_1$ be an arbitrary subset of $\mathbb{B}(\Gamma)$, and let $\mathbb{K}_1$ and $I_\Gamma(\mathbb{B}_1)$ be as above. Then*

$$\mathrm{codim} I_\Gamma(\mathbb{B}_1) = \dim I_\Gamma(\mathbb{B}_1)\perp \geq \#\mathbb{B}_1.$$

Proof: The equality in the statement of the theorem is merely (3.22), so we need only to prove the inequality claim. As in (5.14), one recognizes $\cap_{\theta\in\boldsymbol{\Theta}(\Gamma)} I_{\Gamma_\theta}(\mathbb{B}_1 \cap \mathbb{B}(\Gamma_\theta))$ to be the primary decomposition of $I_\Gamma(\mathbb{B}_1)$, hence, by the same arguments as in (5.18)Theorem, we may assume that Γ is homogeneous.

Given a homogeneous Γ, we use a lower-order perturbation $\gamma \mapsto per\gamma$ (which is obtained by adding a constant term to each of the γ's). Since generically $per(\Gamma)$ is simple (in the sense of (5.8)), we may assume that our perturbed set $per(\Gamma)$ is simple. Also $K \in \mathbb{K}_1$ if and only if $per(K) \in per(\mathbb{K}_1)$ and hence $I_{per(\Gamma)}(per(\mathbb{B}_1))$ is indeed a lower order perturbation of the homogeneous ideal $I_\Gamma(\mathbb{B}_1)$. By (6.3)Proposition

$$\mathrm{codim} I_{per(\Gamma)}(per(\mathbb{B}_1)) = \#per(\mathbb{B}_1) = \#\mathbb{B}_1,$$

and application of (4.7)Theorem thus yields the desired result. ♠

This last result can be applied to various homogeneous and non-homogeneous ideals, provided that their generators can be factored into linear polynomials. A typical example is discussed at the end of this section.

Although in general the inequality in (6.6)Theorem may be strict, we now identify in the following special settings when equality is guaranteed to hold. For this purpose, we impose a (total) order on Γ. This order induces a partial ordering on $\mathbb{B}(\Gamma)$ as follows:

$$(\gamma_1, ..., \gamma_s) \leq (\widetilde{\gamma}_1, ..., \widetilde{\gamma}_s) \iff \gamma_j \leq \widetilde{\gamma}_j, \ j = 1, ..., s, \tag{6.7}$$

where the elements in the sequences in (6.7) are arranged in, say, an increasing order. We say that $\mathbb{B}_1 \subset \mathbb{B}(\Gamma)$ is an **order-closed** subset of $\mathbb{B}(\Gamma)$ if the condition

$$B_1 \in \mathbb{B}_1, B_2 \leq B_1 \implies B_2 \in \mathbb{B}_1 \tag{6.8}$$

holds with respect to all pairs of bases B_1, B_2.

(6.9) Theorem. *If* $\mathbb{B}_1$ *is an order-closed subset of* $\mathbb{B}(\Gamma)$, *then* $\mathrm{codim} I_\Gamma(\mathbb{B}_1) = \#\mathbb{B}_1$.

In view of (6.6)Theorem, we need only prove the inequality

$$\mathrm{codim} I_\Gamma(\mathbb{B}_1) \leq \#\mathbb{B}_1.$$

This inequality will be proved by introducing a basis $\mathcal{S}$ for Π/I_Γ together with a bijective map R from $\mathcal{S}$ to $\mathbb{B}(\Gamma)$. Then we will show that

$$R^{-1}(\mathbb{B}(\Gamma)\backslash\mathbb{B}_1) \subset I_\Gamma(\mathbb{B}_1)$$

in case $\mathbb{B}_1$ is an order-closed subset of $\mathbb{B}(\Gamma)$ and Γ is homogeneous. Proving the desired inequality by such an approach demands of course a very careful construction of $\mathcal{S}$ which in particular takes into account the order defined on Γ. The specific basis used here is borrowed from [7] and is introduced below. We refer to [7] for the proof that this indeed is a basis for Π/I_Γ.*

Given an ordered Γ and an element $B \in \mathbb{B}(\Gamma)$, define

$$\Gamma_B = \{\widetilde{\gamma} \in \Gamma\backslash B : \ x_{\widetilde{\gamma}} \notin \mathrm{span}\{x_\gamma\}_{\gamma \in B, \gamma < \widetilde{\gamma}}\}. \tag{6.10}$$

* The proof in [7] shows that the elements of that set are minimally total over $I_\Gamma\perp$ which is equivalent to the statement here.

The basis for Π/I_Γ is then

$$\{p_{\Gamma_B\uparrow} = \prod_{\gamma\in\Gamma_B} \gamma_\uparrow :\ B \in \mathbb{B}(\Gamma)\}. \tag{6.11}$$

Proof of the Theorem: We prove here the theorem only for homogeneous Γ. The extension to general Γ is done exactly as in the proof of (6.6)Theorem.

By the preceding arguments we need only show that if $B \in \mathbb{B}(\Gamma)\backslash\mathbb{B}_1$, then the polynomial p_{Γ_B} lies in $I_\Gamma(\mathbb{B}_1)$. This will be obtained the moment we verify that Γ_B intersects all the elements of $\mathbb{B}_1$, or, equivalently, that $\mathbb{B}_1 \cap \mathbb{B}(\Gamma\backslash\Gamma_B) = \emptyset$. Examination of the construction of Γ_B reveals that B is the unique minimal element of $\mathbb{B}(\Gamma\backslash\Gamma_B)$, and since $\mathbb{B}_1$ is order-closed and $B \notin \mathbb{B}_1$, we conclude that every basis $\widetilde{B} \in \mathbb{B}(\Gamma\backslash\Gamma_B)$ is excluded from $\mathbb{B}_1$. Consequently, the set Γ_B does intersect all elements of $\mathbb{B}_1$, and the desired result follows. ♠

The above theorem allows us to deduce, as in (5.24), that, whenever $\mathbb{B}_1$ is order-closed, one has

$$I_{\Gamma_\uparrow}(\mathbb{B}_{1\uparrow}) = I_\Gamma(\mathbb{B}_1)_\uparrow, \tag{6.12}$$

where

$$\mathbb{B}_{1\uparrow} := \{B_\uparrow :\ B \in \mathbb{B}_1\} \subset \mathbb{B}(\Gamma_\uparrow).$$

We can now use this last result to conclude:

(6.13) Theorem. *Let $\mathbb{B}_1$ be an order-closed subset of $\mathbb{B}(\Gamma)$. Then the local approximation order of*

$$H := I_\Gamma(\mathbb{B}_1)\perp$$

equals the least cardinality of the elements of $\mathbb{K}_1$.

Proof: By (2.14)Corollary, the local approximation order of H is determined by $H_\downarrow$. From (6.12) and (4.3)Theorem, we conclude that

$$H_\downarrow = I_{\Gamma_\uparrow}(\mathbb{B}_{1\uparrow})\perp. \tag{6.14}$$

Let d be the least cardinality of the elements of $\mathbb{K}_1$, which is the same as the least cardinality of the elements of $\mathbb{K}_{1\uparrow}$. This last set generates the ideal $I_{\Gamma_\uparrow}(\mathbb{B}_{1\uparrow})$. This means that each generator of this ideal annihilates $\Pi_{<d}$ but one of these generators does not annihilate Π_d. We may therefore apply (6.14) to conclude that d is the maximal integer satisfying $\Pi_{<d} \subset H_\downarrow$, and (2.14)Corollary thus yields the desired result. ♠

Remark. In case $\mathbb{B}_1$ is not order-closed, the above result need not be valid. (An explicit example follows by an application of (6.3)Proposition to a "simple" perturbation of the example before (6.6)Theorem). Yet, the sort of arguments used in the above proof show that the least cardinality of the elements of $\mathbb{K}_1$ always provides an *upper bound* for the local approximation power from $I_\Gamma(\mathbb{B}_1)\perp$.

In the rest of this section, we discuss an example that illustrates possible applications of (6.6)Theorem.

(6.15) Example. Assume that $\Gamma = \delta_1 \cup \delta_2 \cup ... \cup \delta_n$ is an arbitrary fixed partition of Γ. Define $\Delta := (\delta_j)_{j=1}^n$ (δ_j will be used here to denote the subset δ_j of Γ as well as the polynomial p_{δ_j}). In this way, we obtain a certain (multi)set of polynomials δ_j, each factorizable into linear factors. We construct ideals I_Δ analogous to the ideals I_Γ, i.e., investigate the case when the set Γ of linear polynomials is replaced by the set Δ of products of linear polynomials.

First, the role of the set $\mathbb{B}(\Gamma)$ of "bases" is now being played by the collection

$$\mathbb{D}(\Delta) := \{D \subset \Delta : \ \#D = s\} \tag{6.16}$$

of all subsets of Δ with cardinality s. Each $D \in \mathbb{D}(\Delta)$ gives rise to

$$\mathbb{B}_D := \{B \in \mathbb{B}(\Gamma) : \ B \cap \delta \neq \emptyset, \ \forall \delta \in D\};$$

in words, the elements of $\mathbb{B}_D$ are those "bases" in $\mathbb{B}(\Gamma)$ which are obtained by choosing one γ from each $\delta \in D$. As in the case of Γ we use

$$\mathbb{K}(\Delta) := \{K \subset \Delta : \ K \cap D \neq \emptyset, \ \forall D \in \mathbb{D}(\Delta)\},$$

and set I_Δ for the ideal generated by $\{p_K : \ K \in \mathbb{K}(\Delta)\}$.

Our claim is that

$$\mathrm{codim} I_\Delta \geq \sum_{D \in \mathbb{D}(\Delta)} \#\mathbb{B}_D. \tag{6.17}$$

The above claim follows directly from (6.6)Theorem by an appropriate choice of $\mathbb{B}_1 \subset \mathbb{B}(\Gamma)$. Indeed, we take here

$$\mathbb{B}_1 := \dot{\bigcup_{D \in \mathbb{B}(\Delta)}} \mathbb{B}_D, \tag{6.18}$$

and note that for every $K \in \mathbb{K}(\Delta)$, $\cup_{\delta \in K} \delta$ intersects all the "bases" from $\mathbb{B}_1$, and hence every generator of I_Δ is also a generator of $I_\Gamma(\mathbb{B}_1)$. We conclude that $I_\Delta \subset I_\Gamma(\mathbb{B}_1)$ and thus $\mathrm{codim} I_\Delta \geq \mathrm{codim} I_\Gamma(\mathbb{B}_1)$. Combining this last observation with (6.6)Theorem, we finally obtain

$$\mathrm{codim} I_\Delta \geq \#\mathbb{B}_1,$$

and (6.17) follows now from (6.18).

We note that the same argument supports more general statements when the set $\mathbb{D}(\Delta)$ is replaced by a subset $\mathbb{D}_1(\Delta)$ and a corresponding ideal $I_\Delta(\mathbb{D}_1(\Delta))$ is constructed. The bound for the codimension of this ideal will be the same as in (6.17), with $\mathbb{D}(\Delta)$ replaced by $\mathbb{D}_1(\Delta)$. The fact that only I_Δ was investigated here was merely for notational convenience.

As a special case of (6.17) one can choose Δ to consist of any bivariate homogeneous polynomials. Then, if some $\delta_1, \delta_2 \in \Delta$ share a common factor, I_Δ will have infinite codimension (since its variety will contain all the zeros of this common factor). Otherwise, one has $\#\mathbb{B}_D = \deg \delta_1 \deg \delta_2$ for every $D = (\delta_1, \delta_2)$. Similar statements can be made for homogeneous I_Δ in more than two variables.

Comparing the example with the results of [6], it is not clear, even in the special case when a matroid structure is imposed on Δ, whether the sufficient condition [6; Th. 3.2] can be applied to derive (6.17). However, in such a case (under further mild restrictions) [6; Th. 3.1] would guarantee the validity of the converse inequality.

13dec01 corrected the Hörmander-Leibniz formula (3.8) and changed π to Π throughout.

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